Zeta functions of arithmetic schemes
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Joint work with B. Morin, Bordeaux

- $\mathcal{X}$ is a regular connected scheme of dimension d, proper over Spec($\mathbb{Z}$)
- Zeta function

$$\zeta(\mathcal{X}, s) = \prod_{x \in \mathcal{X} \text{ closed}} \frac{1}{1 - Nx^{-s}}$$

  Converges for $\Re(s) > d$

- Aim: For any $n \in \mathbb{Z}$ describe

$$\mathrm{ord}_{s=n}\, \zeta(\mathcal{X}, s)$$

  and

$$\zeta^*(\mathcal{X}, n) \in \mathbb{R}^\times$$

Weil-étale cohomology
(Lichtenbaum)

- $\mathcal{X} \to \mathrm{Spec}(\mathbb{F}_p)$ smooth, proper
- $\mathbb{Z}(n)$ on $\mathcal{X}_{et}$ (Higher Chow or Suslin-Voevodsky complex)
  $\mathbb{Z}(0) = \mathbb{Z}, \quad \mathbb{Z}(1) = \mathbb{G}_m[-1], \ldots$
- $W_{\mathbb{F}_p} \cong \mathbb{Z} \subseteq \hat{\mathbb{Z}} \cong G_{\mathbb{F}_p}$
- $\mathcal{X} = \mathrm{Spec}(\mathbb{F}_p)$, $n = 0$

$$H^i(G_{\mathbb{F}_p}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Q}/\mathbb{Z} & i = 2 \end{cases} \qquad H^i(W_{\mathbb{F}_p}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{else} \end{cases}$$

- $\mathcal{X}$ a smooth, proper curve, $n = 1$

$$H^i(\mathcal{X}_{et}, \mathbb{Z}(1)) = \begin{cases} \mathcal{O}(\mathcal{X})^\times & i = 1 \\ \mathrm{Pic}(\mathcal{X}) = \text{finite} \oplus \mathbb{Z} & i = 2 \\ 0 & i = 3 \\ \mathbb{Q}/\mathbb{Z} & i = 4 \end{cases}$$

$$H^i(\mathcal{X}_W, \mathbb{Z}(1)) = \begin{cases} \mathcal{O}(\mathcal{X})^\times & i = 1 \\ \mathrm{Pic}(\mathcal{X}) = \text{finite} \oplus \mathbb{Z} & i = 2 \\ \mathbb{Z} & i = 3 \end{cases}$$

Weil-étale cohomology
(Special values of $\zeta(\mathcal{X}, s)$)

- **Conjecture:** $R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) := R\Gamma(W_{\mathbb{F}_p}, R\Gamma(X_{\overline{\mathbb{F}}_p}, \mathbb{Z}(n))$ is a perfect complex of abelian groups.
- Known for $d \leq 1$
- **Theorem:** The conjecture implies (Milne, Lichtenbaum, Geisser)
  - There is a long exact sequence (concentrated in degrees $2n, 2n+1$)

$$\cdots \to H^i(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\cup\theta} H^{i+1}(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}} \to \cdots$$

  - $$\mathrm{ord}_{s=n}\, \zeta(\mathcal{X}, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i(\mathcal{X}_W, \mathbb{Z}(n))_{\mathbb{R}}$$

  - $$\zeta^*(\mathcal{X}, n) = \pm\chi(R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)), \cup\theta) \cdot p^{\chi(\mathcal{X}, \mathcal{O}, n)}$$

    where

$$\chi(\mathcal{X}, \mathcal{O}, n) = \sum_{i \leq n, j} (-1)^{i+j} (n-i) \dim_{\mathbb{F}_p} H^j(\mathcal{X}, \Omega^i)$$

Proofs

- Grothendieck's formula: $l \neq p$ prime

$$\zeta(\mathcal{X},s) = Z(\mathcal{X},p^{-s})$$

$$Z(\mathcal{X},T) = \prod_{i=0}^{2\dim(\mathcal{X})} \det(1-\mathrm{Frob}^{-1}\cdot T|H^i(\mathcal{X}_{\overline{\mathbb{F}}_p},\mathbb{Q}_l))^{(-1)^{i+1}}$$

- $\mathbb{Z}(n)/l^\nu \cong \mu_{l^\nu}^{\otimes n}$

$$R\Gamma(G_{\mathbb{F}_p}, R\Gamma(\mathcal{X}_{\overline{\mathbb{F}}_p},\mathbb{Z}_l(n))) \cong R\Gamma(\mathcal{X}_W,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Z}_l$$

- For $l = p$ one has

$$Z(\mathcal{X},T) = \prod_{i=0}^{2\dim(\mathcal{X})} \det(1-\mathrm{Frob}\cdot T|H^i_{cris}(\mathcal{X}/\mathbb{F}_p))^{(-1)^{i+1}}$$

Higher Chow groups

$\Delta^m := \mathrm{Spec}(\mathbb{Z}[t_0,\ldots,t_m]/(t_0+\cdots t_m-1)) \simeq \mathbb{A}^m$ (algebraic m-simplex)

$$\begin{aligned} z^n(Y,m) :=& \text{free abelian group on codimension } n \text{ points on } Y\times\Delta^m \\ & \text{intersecting all faces } Y\times\Delta^i \subseteq Y\times\Delta^m \text{ properly} \\ z^n(Y,\bullet) :=& \text{corresponding simplicial abelian group (or homological complex)} \\ CH^n(Y,m) :=& H_m(z^n(Y,\bullet)) \quad \text{Higher Chow groups} \end{aligned}$$

For regular Y and $n\geq 0$ define a cohomological complex $\mathbb{Z}(n)$ on Y_{et} by

$$\mathbb{Z}(n)(U) := z^n(U, 2n-\bullet)$$

$$\mathbb{Z}(n)/\ell^\nu \simeq \mu_{\ell^\nu}^{\otimes n} \quad \text{if } \ell \text{ is invertible on } Y$$

$$H^i(Y_{et},\mathbb{Z}(n))_{\mathbb{Q}} \simeq CH^n(Y,2n-i)_{\mathbb{Q}} \simeq K_{2n-i}(Y)_{\mathbb{Q}}^{(n)}$$

Deligne cohomology

$\mathcal{X}$ regular of dimension d, $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ flat, proper

$\mathcal{X}(\mathbb{C})$ compact complex manifold with $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action

Deligne complex in the analytic topology

$$\mathbb{Z}(n)_{\mathcal{D}} := (2\pi i)^n \mathbb{Z} \to \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{n-1}$$

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{Z}(n)) := R\Gamma(G_{\mathbb{R}}, R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)_{\mathcal{D}}))$$

$$\mathbb{R}(n)_{\mathcal{D}} := (2\pi i)^n \mathbb{R} \to \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{n-1}$$

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) := R\Gamma(G_{\mathbb{R}}, R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)_{\mathcal{D}}))$$

Weil-Arakelov cohomology
(Assumptions)

- $\mathcal{X}$ regular of dimension d, $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ flat, proper
- $\mathbb{Z}(n)$ on $\mathcal{X}_{et}$ (Higher Chow complex) For $n < 0$ define $\mathbb{Z}(n)$ by pushforward under $f : \mathbb{P}^N_{\mathcal{X}} \to \mathcal{X}$

$$Rf_{et,*}\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}(-1)[-2] \oplus \cdots \oplus \mathbb{Z}(-N)[-2N].$$

- **Assumptions:**

  FG $H^i(\mathcal{X}_{et}, \mathbb{Z}(n))$ is finitely generated for $i \leq 2n+1$

  B Beilinson conjectures. There is a perfect duality for all $i, n \in \mathbb{Z}$

$$H^i_c(\mathcal{X}, \mathbb{R}(n)) \times H^{2d-i}(\mathcal{X}, \mathbb{R}(d-n)) \to H^{2d}_c(\mathcal{X}, \mathbb{R}(d)) \to \mathbb{R}$$

  where (with Б the Beilinson regulator)

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma(\mathcal{X}, \mathbb{R}(n)) \xrightarrow{\text{Б}} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n))$$

  Known for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$

  AV Artin-Verdier duality. There is a perfect duality for any $m, i, n \in \mathbb{Z}$

$$\hat{H}^i_c(\mathcal{X}_{et}, \mathbb{Z}(n)/m) \times H^{2d+1-i}(\mathcal{X}_{et}, \mathbb{Z}(d-n)/m) \to \hat{H}^{2d+1}_c(\mathcal{X}_{et}, \mathbb{Z}(d)/m) \to \mathbb{Z}/m$$

  Known for $d \leq 2$ or $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ smooth, or $n \geq d$ or $n \leq 0$.

Weil-Arakelov cohomology ($\mathbb{Z}(n)$-coefficients)

- If $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is flat then $\mathcal{X}$ is not "compact". One has a diagram with exact rows and columns

$$\begin{array}{ccccc}
R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar,\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{Z}(n)) \\
\| & & \uparrow & & \uparrow \\
R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\overline{\mathcal{X}},\mathbb{Z}(n)) & \to & R\Gamma_{ar}(\mathcal{X}_\infty,\mathbb{Z}(n)) \\
 & & \uparrow & & \uparrow \\
 & & R\Gamma_{ar,\mathcal{X}_\infty}(\overline{\mathcal{X}},\mathbb{Z}(n)) & = & R\Gamma_{ar,\mathcal{X}_\infty}(\overline{\mathcal{X}},\mathbb{Z}(n))
\end{array}$$

in D^b(l.c.a.grps). In general, only $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))$ is a perfect complex of abelian groups.

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$$H^i_{ar}(\mathrm{Spec}(\mathcal{O}_F),\mathbb{Z}(1)) = \begin{cases} \mathcal{O}_F^\times & i=1 \\ \mathrm{Pic}(\mathcal{O}_F) \oplus (\bigoplus_{v|\infty} \mathbb{Z})^{\Sigma=0} & i=2 \end{cases}$$

$$H^i_{ar}(\overline{\mathrm{Spec}(\mathcal{O}_F)},\mathbb{Z}(1)) = \begin{cases} \mu_F^\times & i=1 \\ \mathrm{Pic}(\mathcal{O}_F) \oplus (\bigoplus_{v|\infty} \mathbb{R})/\log(\mathcal{O}_F^\times) & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

Weil-Arakelov cohomology ($\tilde{\mathbb{R}}(n)$- and $\tilde{\mathbb{R}}/\mathbb{Z}(n)$-coefficients)

- For $\mathcal{Y} = \mathcal{X}, \overline{\mathcal{X}}, \mathcal{X}_\infty$ there are exact triangles in D^b(l.c.a.grps)

$$R\Gamma_{ar,?}(\mathcal{Y}, \mathbb{Z}(n)) \to R\Gamma_{ar,?}(\mathcal{Y}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{ar,?}(\mathcal{Y}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \to$$

- $H^{2n}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) \cong CH(\overline{\mathcal{X}})_{\mathbb{R}}$ (Gillet-Soulé Arakelov Chow group)
- **Proposition**
  - There are dualities of finite-dimensional $\mathbb{R}$-vector spaces

$$H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{R}}(d-n)) \to H^{2d+1}_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(d)) \to \mathbb{R}$$

and

$$H^i_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(d-n)) \to H^{2d+1}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(d)) \to \mathbb{R}$$

  - There are Pontryagin dualities

$$R\,\mathrm{Hom}(R\Gamma_{ar,c}(\mathcal{X}, \mathbb{Z}(n)), \tilde{\mathbb{R}}/\mathbb{Z}) \cong R\Gamma(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(d-n))[-2d-1]$$

and

$$H^i_{\mathrm{ar}}(\overline{\mathcal{X}}, \mathbb{Z}(n)) \times H^{2d+1-i}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}/\mathbb{Z}(d-n)) \to H^{2d+1}_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$$

Weil-Arakelov cohomology (Special values of $\zeta(\mathcal{X}, s)$)

For any $n \in \mathbb{Z}$ the exact triangle

$$R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \to \qquad (1)$$

has the following properties

- For all $i \in \mathbb{Z}$ the groups $H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n))$ are finite dimensional vector spaces over $\mathbb{R}$ and there is an exact sequence

$$\cdots \xrightarrow{\cup\theta} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} H^{i+1}_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} \cdots \qquad (2)$$

  In particular, the complex $R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n))$ has vanishing Euler characteristic:

$$\sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)) = 0.$$

- For all $i \in \mathbb{Z}$ the groups $H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))$ are compact, commutative Lie groups, i.e. isomorphic to

$$S^1 \times \cdots \times S^1 \times \text{finite}.$$

Conjectural relation to $\zeta(\mathcal{X}, s)$

- The function $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to $s = n$ and

$$\mathrm{ord}_{s=n}\,\zeta(\mathcal{X}, s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}(n)).$$

- If $\zeta^*(\mathcal{X}, n) \in \mathbb{R}$ denotes the leading Taylor-coefficient of $\zeta(\mathcal{X}, n)$ at $s = n$ then

$$|\zeta^*(\mathcal{X}, n)|^{-1} = \prod_{i\in\mathbb{Z}} \left(\mathrm{vol}(H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)))\right)^{(-1)^i}. \qquad (3)$$

Definition of the volume

If G is a locally compact abelian group, define its **tangent space** $T_\infty G$ by

$$T_\infty G := \mathrm{Hom}_{cts}(\mathrm{Hom}_{cts}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R})$$

which comes with an exponential map

$$\exp : T_\infty G \to \mathrm{Hom}_{cts}(\mathrm{Hom}_{cts}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}) = G.$$

Examples:

G	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{R}/\mathbb{Z}$
$T_\infty G$	0	$\mathbb{R}$	$\mathbb{R}$

If G is a compact, commutative Lie group one has an exact triangle

$$L \to L \otimes_{\mathbb{Z}} \mathbb{R} \simeq T_\infty G \xrightarrow{\exp} G \to$$

where L is a perfect complex of abelian groups. A **volume form** is a nonzero section $v \in det_{\mathbb{R}} T_\infty G$. The **volume** $\mathrm{vol}(G) \in \mathbb{R}^{>0}$ satisfies

$$\det_{\mathbb{Z}} L = \mathbb{Z} \cdot \mathrm{vol}(G) \cdot v$$

Weil-étale cohomology

T_∞ extends to an exact functor

$$T_\infty : D^b(l.c.a.) \to D^b(\mathbb{R})$$

Weil-étale cohomology is the perfect complex of abelian groups $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ defined as the mapping fibre of the exponential map

$$R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \to T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \xrightarrow{\exp} R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))$$

Given a **volume form**

$$v \in det_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n)) \simeq \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_{\mathbb{R}}$$

the volume

$$\prod_{i \in \mathbb{Z}} \left(\mathrm{vol}(H^i_{\mathrm{ar},c}(\mathcal{X}, \tilde{\mathbb{R}}/\mathbb{Z}(n))) \right)^{(-1)^i}$$

in (3) is the unique $\mu \in \mathbb{R}^{>0}$ with

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \mathbb{Z} \cdot \mu \cdot v$$

Definition of the volume form

Applying T_∞ to (1) we get an exact triangle in $D^b(\mathbb{R})$

$$T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar},c}(\mathcal{X},\tilde{\mathbb{R}}(n)) \to R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}} \to \qquad (4)$$

Taking determinants of (4) gives an isomorphism

$$\begin{aligned}
&\det{}_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}}\\
\cong&\det{}_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(\mathcal{X},\tilde{\mathbb{R}}(n)) \otimes_{\mathbb{R}} \det{}_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X},\mathbb{Z}(n))[-1]\\
\cong&\det{}_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X},\mathbb{Z}(n))[-1]
\end{aligned}$$

where the trivialization $\det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(\mathcal{X},\tilde{\mathbb{R}}(n)) \cong \mathbb{R}$ is induced by the exact sequence (2). Applying T_∞ to the defining triangle

$$R\Gamma_{\mathrm{ar},c}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar}}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar},\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{Z}(n))$$

gives isomorphisms

$$\begin{aligned}
\det{}_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},c}(\mathcal{X},\mathbb{Z}(n)) \cong&\det{}_{\mathbb{R}} T_\infty R\Gamma_{\mathrm{ar},\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{Z}(n))[-1]\\
\cong&\det{}_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}),\Omega^{<n}_{hol})^{G_{\mathbb{R}}}[-2]\\
\cong&\det{}_{\mathbb{R}} R\Gamma(\mathcal{X}_{\mathbb{Q},Zar},\Omega^{<n}_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}})_{\mathbb{R}}[-2]
\end{aligned}$$

Definition of the volume form (ctd)

A natural way to define a volume form $v \in \det_{\mathbb{R}} R\Gamma(\mathcal{X}_{\mathbb{Q},Zar}, \Omega^{<n}_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}})_{\mathbb{R}}$ would be via a perfect complex of abelian groups P so that

$$P_{\mathbb{Q}} \simeq R\Gamma(\mathcal{X}_{\mathbb{Q},Zar}, \Omega^{<n}_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}}); \qquad \mathbb{Z} \cdot v = \det_{\mathbb{Z}} P$$

Possible choices for P:

- $R\Gamma(\mathcal{X}_{Zar}, \Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$ (**naive de Rham cohomology** modulo Fil^n) Clearly wrong
- $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$ (**derived de Rham cohomology** modulo Fil^n as defined by Illusie) The special value conjecture becomes

  $$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \zeta^*(\mathcal{X}, n) \cdot C(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})[-1]$$

  for a certain correction factor $C(\mathcal{X}, n) \in \mathbb{Q}^{\times}$.
- $R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$ (**motivic weight n graded piece of $TC^+(\mathcal{X})$** as defined by Morin) The special value conjecture becomes

  $$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) = \zeta^*(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})[-1]$$

Concerning the correction factor $C(\mathcal{X}, n)$

Definition of $C(\mathcal{X}, n)$ is forced by compatibility with Tamagawa number conjecture and involves p-adic Hodge theory.

Theorem

a) *One has* $C(\mathcal{X}, n) = 1$ *if* $n \leq 0$ *(trivial).*

b) *One has* $C(\mathcal{X}, 1) = 1$ *(nontrivial)*

c) *One has* $C(\mathcal{X}, n) = 1$ *if* $\mathcal{X} \to \mathrm{Spec}(\mathbb{F}_p)$ *is smooth, proper over a finite field (Thm of Morin)*

d) *For* $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$*, all* $F_v/\mathbb{Q}_p$ *abelian and* $n \geq 1$ *one has*

$$C(\mathcal{X}, n) = (n-1)!^{-[F:\mathbb{Q}]}$$

In general we expect $C(\mathcal{X}, n) = C_\infty(\mathcal{X}, n)^{-1}$ where

$$C_\infty(\mathcal{X}, n) := \prod_{i \leq n-1;\, j} (n-1-i)!^{(-1)^{i+j} \dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}}, \Omega^i)}$$

since

$$\det{}_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) = C_\infty(\mathcal{X}, n) \cdot \det{}_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}).$$

An integral fundamental line

For $\mathcal{X}$ regular, proper over Spec($\mathbb{Z}$) and $n \in \mathbb{Z}$ define

$$\Delta(\mathcal{X}/\mathbb{Z}, n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$$

The Beilinson regulator, Arakelov intersection pairing and Period isomorphism induce

$$\lambda_\infty : \mathbb{R} \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, n) \otimes_{\mathbb{Z}} \mathbb{R}$$

The special value conjecture says

$$\lambda_\infty(\zeta^*(\mathcal{X}, n)^{-1} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z}, n)$$

If $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ is smooth proper over a number ring then

$$\Delta(\mathcal{X}/\mathbb{Z}, n) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigotimes_{i=0}^{2d-2} \Xi(h^i(\mathcal{X}_{\mathbb{Q}})(n))^{(-1)^i}$$

is the fundamental line of Fontaine and Perrin-Riou for the motive

$$h(\mathcal{X}_{\mathbb{Q}})(n) = \bigoplus_{i=0}^{2d-2} h^i(\mathcal{X}_{\mathbb{Q}})(n)[-i]$$

of the generic fibre of $\mathcal{X}$. Moreover $\lambda_\infty = \bigotimes_i \vartheta_\infty^{(-1)^i}$.

The definition of $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$

Key assumption (known for $d \leq 1 \ldots$)

$$H^i(\mathcal{X}_{\mathrm{et}}, \mathbb{Z}(n)) \text{ is finitely generated for } i \leq 2n+1$$

Artin-Verdier duality for $\mathbb{Z}(n)/m$ on $\overline{\mathcal{X}}_{\mathrm{et}}$ implies that

$$H^i(\overline{\mathcal{X}}_{\mathrm{et}}, \mathbb{Z}(n)) \cong \mathrm{Hom}_{\mathbb{Z}}(H^{2d+2-i}(\overline{\mathcal{X}}_{\mathrm{et}}, \mathbb{Z}(d-n)), \mathbb{Q}/\mathbb{Z})$$

is cofinitely generated for $i \geq 2n+1$. Define a perfect complex of abelian groups $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))$ as the mapping cone

$$R\mathrm{Hom}(R\Gamma(\mathcal{X}, \mathbb{Z}(d-n)), \mathbb{Q}[-2d-2]) \to R\Gamma(\overline{\mathcal{X}}_{et}, \mathbb{Z}(n)) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))$$

and a perfect complex $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ as a mapping fibre

$$R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))$$

also involving Betti cohomology of $\mathcal{X}(\mathbb{C})$.

Compatibility with the Tamagawa number conjecture

Theorem

If $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ is smooth proper over a number ring then

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1} \cdot \frac{C(\mathcal{X},n)}{C_\infty(\mathcal{X},n)} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z},n)$$

is equivalent to the Tamagawa number conjecture for $h(\mathcal{X}_\mathbb{Q})(n)$.

Recall: $\frac{C(\mathcal{X},n)}{C_\infty(\mathcal{X},n)} = 1$ for $n \le 1$ or $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$, all $F_v/\mathbb{Q}_p$ abelian.

Corollary

Our conjecture

$$\lambda_\infty(\zeta^*(\mathcal{X},n)^{-1} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z},n)$$

holds true for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ and any $n \in \mathbb{Z}$ if $F/\mathbb{Q}$ is abelian.

Follows from proof of TNC for Dirichlet L-functions.

The example $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$

F number field with r_1 real and r_2 complex places.

$$\zeta(\mathcal{X}, s) = \zeta_F(s) \qquad \text{Dedekind Zeta function}$$

All assumptions going into the definition of our groups are satisfied, in particular for $i = 1, 2$

$$H^i(\mathcal{X}_{\mathrm{et}}, \mathbb{Z}(n)) \sim_2 K_{2n-i}(\mathcal{O}_F)$$

is finitely generated. Note

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K_3(\mathbb{Z}) \to H^1(\mathrm{Spec}(\mathbb{Z})_{et}, \mathbb{Z}(2)) \to 0$$

The conjectures on the vanishing order hold true (Borel 1975)

$$\mathrm{ord}_{s=n}\, \zeta_F(s) = \begin{cases} r_2 & n < 0 \text{ odd} \\ r_1 + r_2 & n < 0 \text{ even} \\ r_1 + r_2 - 1 & n = 0 \\ -1 & n = 1 \\ 0 & n > 1 \end{cases}$$

The Beilinson regulator map

$$H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n)) \xrightarrow{r_n} H^1_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(n)) \cong \prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{R})$$

induces isomorphisms

$$r_{n,\mathbb{R}}: H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))_{\mathbb{R}} \xrightarrow{\sim} \prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{R})$$

for $n>1$ and

$$r_{1,\mathbb{R}}: H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(1))_{\mathbb{R}} \cong \Big(\prod_{v|\infty}\mathbb{R}\Big)^{\Sigma=0}$$

for $n=1$. For $n\geq 1$ we set

$$\begin{aligned} h_n &:=|H^2(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))| \sim_2 |K_{2n-2}(\mathcal{O}_F)| \\ w_n &:=|H^1(\mathcal{X}_{\mathrm{et}},\mathbb{Z}(n))_{tor}| \sim_2 |K_{2n-1}(\mathcal{O}_F)_{tor}| \\ R_n &:=\mathrm{vol}(\mathrm{coker}(r_n)) \end{aligned}$$

where the volume is taken with respect to the $\mathbb{Z}$-structure $\prod_{v|\infty} H^0(F_v,(2\pi i)^{n-1}\mathbb{Z})$, resp. $(\prod_{v|\infty}\mathbb{Z})^{\Sigma=0}$, of the target.

Our conjecture is equivalent to

$$\zeta_F^*(n) = \pm \frac{h_{1-n} \cdot R_{1-n}}{w_{1-n}} \tag{5}$$

for $n \leq 0$ and to

$$\begin{aligned} \zeta_F^*(n) = \zeta_F(n) = \\ = (n-1)!^{-[F:\mathbb{Q}]} \cdot \frac{2^{r_1 \cdot (-1)^{n-1}} (2\pi)^{[F:\mathbb{Q}] \cdot n - r_2 - r_1 \cdot (((-1)^n - 1)/2)} h_n R_n}{|D_F|^{n-1} \sqrt{|D_F|} \cdot w_n} \end{aligned} \tag{6}$$

for $n \geq 1$.

Proposition

Equations (5) and (6) hold for $n = 0, 1$ if F is arbitrary and for any $n \in \mathbb{Z}$ if $F/\mathbb{Q}$ is abelian.

This follows from known cases of the Tamagawa number conjecture.

Cyclic Homology (Additive K-theory)

If k is a commutative ring and A/k a k-algebra define (derived) Hochschild homology

$$HH(A/k) := A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_k A} A$$

and (derived) cyclic homology

$$HC(A/k) := HH(A/k)_{hS^1}$$

Theorem

(Majadas, Antieau) There is a (motivic) filtration $F^{\star}_{Mot} HC(\mathcal{X}/\mathbb{Z})$ on $HC(\mathcal{X}/\mathbb{Z})$ so that for all $n \in \mathbb{Z}$

$$\mathrm{gr}^n_{Mot} HC(\mathcal{X}/\mathbb{Z}) \simeq R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})[2n-2]$$

Compare with the motivic filtration on K-theory

$$\mathrm{gr}^n_{Mot} K(\mathcal{X}) \simeq R\Gamma(\mathcal{X}_{Zar}, \mathbb{Z}(n))[2n]$$

Additive K-theory of $\mathrm{Spec}(\mathcal{O}_F)$

For $n \geq 1$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ one has

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{O}_F/\mathbb{Z}}) \cong \left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n) \right)$$

where $\Omega_{\mathcal{O}_F/\mathbb{Z}}(n)$ is a certain finite abelian group of order $|D_F|^{n-1}$.

$$K^{add}_{2n-1}(\mathcal{O}_F) := HC_{2n-2}(\mathcal{O}_F) = \ker \left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n) \right)$$

$$K^{add}_{2n-2}(\mathcal{O}_F) := HC_{2n-3}(\mathcal{O}_F) = \mathrm{coker} \left(\mathcal{O}_F \xrightarrow{d(n)} \Omega_{\mathcal{O}_F/\mathbb{Z}}(n) \right)$$

One has

$$|D_F|^{n-1}\sqrt{|D_F|} = h^{add}_n \cdot R^{add}_n$$

where $R^{add}_n := \mathrm{covol}(K^{add}_{2n-1}(\mathcal{O}_F))$ and $h^{add}_n := |K^{add}_{2n-2}(\mathcal{O}_F)|$.

Improved additive K-theory of Spec($\mathcal{O}_F$) ($TC^+(\mathcal{O}_F)$)

How to explain $C(\mathrm{Spec}(\mathcal{O}_F), n) = (n-1)!^{-[F:\mathbb{Q}]}$?

Topological Hochschild homology (Bökstedt,...)

$$THH(\mathcal{X}) := HH(\mathcal{X}/\mathbb{S})$$

where $\mathbb{S}$ is the sphere spectrum.

Definition

Topological positive cyclic homology

$$TC^+(\mathcal{X}) := THH(\mathcal{X})_{hS^1}$$

Theorem

(Madsen, Lindenstrauss, 2000)

$$THH_i(\mathcal{O}_F) = \begin{cases} \mathcal{O}_F & i = 0 \\ \mathcal{D}_F^{-1}/j \cdot \mathcal{O}_F & i = 2j-1 \\ 0 & else \end{cases}$$

Improved additive K-theory of $\mathrm{Spec}(\mathcal{O}_F)$ **(**$TC^+(\mathcal{O}_F)$**)**

The spectral sequence

$$H_i(BS^1, THH_j(\mathcal{O}_F)) \Rightarrow TC^+_{i+j}(\mathcal{O}_F)$$

shows that $TC^+_{2n-3}(\mathcal{O}_F)$ is finite and $TC^+_{2n-2}(\mathcal{O}_F) \subseteq \mathcal{O}_F$ is a sublattice so that

$$(n-1)!^{[F:\mathbb{Q}]}|D_F|^{n-1}\sqrt{|D_F|} = h_n^{add} \cdot R_n^{add}$$

where $R_n^{add} := \mathrm{covol}(TC^+_{2n-2}(\mathcal{O}_F))$ and $h_n^{add} := |TC^+_{2n-3}(\mathcal{O}_F)|$.

Hence for $n \geq 1$ we have

$$\zeta_F^*(n) = \zeta_F(n) = \frac{2^{r_1 \cdot (-1)^{n-1}}(2\pi)^{[F:\mathbb{Q}]\cdot n - r_2 - r_1 \cdot (((-1)^n - 1)/2)} h_n \cdot R_n}{h_n^{add} \cdot R_n^{add} \cdot w_n}$$

without any correction factor!

The motivic filtration on $TC^+(\mathcal{X})$

For $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ we have

$$F^n_{Mot} TC^+(\mathcal{O}_F) := \tau_{\geq 2n-3} TC^+(\mathcal{O}_F)$$

but the motivic filtration in general is more complicated.

Theorem

(Morin, Bhatt-Lurie) There is a (motivic) filtration $F^\star_{Mot} TC^+(\mathcal{X})$ on $TC^+(\mathcal{X})$ so that for all $n \in \mathbb{Z}$

$$\mathrm{gr}^n_{Mot} TC(\mathcal{X})^+ =: R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})[2n-2]$$

satisfies

$$\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) = C_\infty(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}).$$

where

$$C_\infty(\mathcal{X}, n) := \prod_{i \leq n-1; j} (n-1-i)!^{(-1)^{i+j} \dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}}, \Omega^i)}$$

Compatibility with the Birch and Swinnerton-Dyer conjecture

Assume $\mathcal{X}$ is regular, connected, proper, flat of dimension $d = 2$. Then

$$f : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F) =: S; \quad f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$$

for a unique number field F and

$$\mathcal{X}_F \to \mathrm{Spec}(F)$$

is a smooth, projective, geometrically connected curve. Moreover

$$\zeta(\mathcal{X}, s) = \frac{\zeta(H^0, s)\zeta(H^2, s)}{\zeta(H^1, s)} = \frac{\zeta_F(s)\zeta_F(s-1)}{\zeta(H^1, s)}$$

where $\zeta(H^i, s)$ should be viewed as the Zeta function of a relative H^i of f in the sense of a motivic (i.e. perverse) t-structure

Compatibility with BSD: The Zeta function of H^1

For each finite place v of F set

$$C_v := \text{set of irreducible components of the fibre } \mathcal{X}_{\kappa(v)}$$
$$r_{v,i} := [\kappa(v)_i : \kappa(v)]$$

where $\kappa(v)_i$ is the constant field of the component $i \in C_v$. Then

$$\zeta(H^1, s) = L(J, s) \cdot \prod_{v \text{ finite}} \left(\frac{1}{1 - Nv^{-(s-1)}} \prod_{i \in C_v} (1 - Nv^{-(s-1) r_{v,i}}) \right)$$

where $L(J, s)$ is the Hasse-Weil L-function of $J := \mathrm{Jac}(\mathcal{X}_F)$.

Want to describe $\zeta(H^1, s)$ at $s = n = 1$. Recall $\mathbb{Z}(1) = \mathbb{G}_m[-1]$. Need motivic decomposition of $Rf_* \mathbb{G}_m$.

Compatibility with BSD: The motivic complex of H^1

One has $R^i f_* \mathbb{G}_m = 0$ for $i \geq 2$ (Grothendieck) and

$$P = \mathrm{Pic}_{\mathcal{X}/S} := R^1 f_* \mathbb{G}_m$$

is the relative Picard functor (étale sheafification of $U \mapsto \mathrm{Pic}(\mathcal{X} \times_S U)$). One has a truncation triangle

$$\mathbb{G}_m = f_* \mathbb{G}_m \to Rf_* \mathbb{G}_m \to P[-1] \to$$

and we define a complex of étale sheaves P^0 on S by the exact triangle

$$P^0 \to P \xrightarrow{\deg} \mathbb{Z} \to$$

The complex P^0 serves as a substitute for the relative H^1-motive and one has $P^0|_{\mathrm{Spec}(F)_{\mathrm{et}}} = J$.

Compatibility with BSD: The main theorem

Theorem

a) One has

$$\mathrm{ord}_{s=1}\,\zeta(H^1,s)=\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Pic}^0(\mathcal{X})\;\Leftrightarrow\;\mathrm{ord}_{s=1}\,L(J,s)=\mathrm{rank}_{\mathbb{Z}}\,J(F).$$

b) The following statements are equivalent

$$\lambda_\infty(\zeta^*(H^1,1)\cdot\mathbb{Z})=\det_{\mathbb{Z}}R\Gamma_{W,c}(S,P^0)\otimes_{\mathbb{Z}}\det_{\mathbb{Z}}^{-1}H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})$$

$$\zeta^*(H^1,1)=\frac{\mathrm{vol}\left(H^0_{\mathrm{ar},c}(S,P^0\otimes\tilde{\mathbb{R}}/\mathbb{Z})\right)}{\mathrm{vol}\left(H^1_{\mathrm{ar},c}(S,P^0\otimes\tilde{\mathbb{R}}/\mathbb{Z})\right)}$$

$$\zeta^*(H^1,1)=\frac{\#\mathrm{Br}(\overline{\mathcal{X}})\cdot\delta^2\cdot\Omega(\mathcal{X})\cdot R(\mathcal{X})}{(\#(\mathrm{Pic}^0(\mathcal{X})_{tor}/\,\mathrm{Pic}(\mathcal{O}_F)))^2}\cdot\prod_{v\ real}\frac{\#\Phi_v}{\delta'_v\delta_v}$$

and all these statements are equivalent to the BSD formula

$$L^*(J,1)=\frac{\#\text{Ш}(J)\cdot\Omega(\mathcal{J})\cdot R(J(F))}{(\#J(F)_{tor})^2}\cdot\prod_v\#\Phi_v.$$

Compatibility with BSD: $Ш(J)$ vs $\mathrm{Br}(\mathcal{X})$

Define the local and global index

$$\delta_v := \#\,\mathrm{coker}\left(\mathrm{Pic}(\mathcal{X}_{F_v}) \xrightarrow{\deg} \mathbb{Z}\right), \quad \delta := \#\,\mathrm{coker}\left(\mathrm{Pic}(\mathcal{X}_F) \xrightarrow{\deg} \mathbb{Z}\right)$$

and the period

$$\delta'_v := \#\,\mathrm{coker}\left(P(F_v) \xrightarrow{\deg} \mathbb{Z}\right) \quad \alpha := \#\,\mathrm{coker}\left(\mathrm{Pic}^0(\mathcal{X}_F) \to J(F)\right)$$

Then $\delta_v/\delta'_v \in \{1,2\}$ for all places v.

Proposition

(Geisser, F.) If $\mathrm{Br}(\mathcal{X}) \simeq H^2(\mathcal{X}_{et}, \mathbb{G}_m)$ is finite then

$$\#\,\mathrm{Br}(\overline{\mathcal{X}}) \cdot \delta^2 = \frac{\prod_v \delta'_v \delta_v}{\alpha^2} \cdot \#Ш(J_F) \tag{7}$$

where the product is over all places v of F and

$$\mathrm{Br}(\overline{\mathcal{X}}) := \ker\left(\mathrm{Br}(\mathcal{X}) \to \bigoplus_{v \text{ real}} \mathrm{Br}(\mathcal{X}_{F_v})\right)$$

One shows that $\#\,\mathrm{Br}(\overline{\mathcal{X}})$ is a square if it is finite.

Compatibility with BSD: $R(J(F))$ vs $R(\mathcal{X})$

$R(J(F))$:=regulator of the Neron-Tate height pairing on $J(F)$

$R(\mathcal{X})$:=regulator of the Arakelov intersection pairing on $\mathrm{Pic}^0(\mathcal{X})$

Proposition

$$\frac{R(\mathcal{X})}{(\#(\mathrm{Pic}^0(\mathcal{X})_{tor}/\,\mathrm{Pic}(\mathcal{O}_F)))^2 \cdot \alpha^2} = \prod_{v\ bad}\left(\frac{\#\Phi_v}{\delta'_v\delta_v}(\log Nv)^{\#C_v-1}\prod_{i\in C_v} r_{v,i}\right)\cdot\frac{R(J(F))}{(\#J(F)_{tor})^2}$$

Proof uses results of Bosch and Liu on component groups of Neron Models.

Compatibility with BSD: $\Omega(\mathcal{J})$ vs $\Omega(\mathcal{X})$

Let

$$\mathcal{J} \to \mathsf{Spec}(\mathcal{O}_F)$$

be the Neron model of J.
Let $\Omega(\mathcal{X})$, $\Omega(\mathcal{J}) \in \mathbb{R}^\times$ be such that

$$\det_{\mathbb{Z}} H^1(\mathcal{X}(\mathbb{C}), (2\pi i)\mathbb{Z})^{G_{\mathbb{R}}} = \Omega(\mathcal{X}) \cdot \det_{\mathbb{Z}} H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

$$\det_{\mathbb{Z}} H^1(J(\mathbb{C}), (2\pi i)\mathbb{Z})^{G_{\mathbb{R}}} = \Omega(\mathcal{J}) \cdot \det_{\mathbb{Z}} \mathrm{Lie}(\mathcal{J})$$

under the Deligne period isomorphism.

Proposition

$$\Omega(\mathcal{X}) = \pm\Omega(\mathcal{J})$$

Proof uses results of Liu, Lorenzini and Raynaud on tangent spaces of Neron models.

Compatibility with BSD: Some proven cases

Theorem

(Rubin, Burungale, F.) Let E/F be an elliptic curve with CM by $\mathcal{O}_K$ for an imaginary quadratic field K and such that $F(E_{tors})/K$ is abelian. If $L(E,1) \neq 0$ then $E(F)$ and $Ш(E/F)$ are finite and the BSD formula holds true.

Theorem

(Yongxiong Li, Yu Liu, Ye Tian) Let $p \equiv 5 \mod 8$ be a prime number and $E/\mathbb{Q}$ the elliptic curve

$$y^2 = x^3 - p^2 x.$$

Then $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s) = 1$, $Ш(E/\mathbb{Q})$ is finite and the BSD formula holds true.

Corollary

Let X/F be a genus 1 curve which is a torsor for E/F as above and $\mathcal{X}/\mathcal{O}_F$ a proper, regular model of X. Then our conjecture on $\zeta(\mathcal{X},s)$ at $s=1$ holds true.

Compatibility with the functional equation

Let $\mathcal{X}$ be regular of dimension d, proper and flat over $\mathrm{Spec}(\mathbb{Z})$. Define the completed Zeta-function

$$\zeta(\overline{\mathcal{X}}, s) = \zeta(\mathcal{X}_\infty, s)\zeta(\mathcal{X}, s)$$

where

$$\zeta(\mathcal{X}_\infty, s) = \prod_{i=0}^{2d-2} L_\infty(h^i(X), s)^{(-1)^i} \tag{8}$$

Here $h^i(X)$ is the $\mathbb{R}$-Hodge structure on $H^i(\mathcal{X}(\mathbb{C}), \mathbb{R})$. For simple $\mathbb{R}$-Hodge structures we have

M	$\dim_{\mathbb{R}} M$	condition on $p, q \in \mathbb{Z}$	$L_\infty(M, s)$
$M_{p,q}$	2	$p < q$	$\Gamma_{\mathbb{C}}(s-p)$
$M_{p,+}$	1	$c = (-1)^p$	$\Gamma_{\mathbb{R}}(s-p)$
$M_{p,-}$	1	$c = (-1)^{p+1}$	$\Gamma_{\mathbb{R}}(s-p+1)$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2); \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

Compatibility with FE: Main Theorem

Theorem

Assume $\zeta(\mathcal{X}, s)$ satisfies the functional equation

$$A(\mathcal{X})^{(d-s)/2}\zeta(\overline{\mathcal{X}}, d-s) = A(\mathcal{X})^{s/2}\zeta(\overline{\mathcal{X}}, s)$$

where $A(\mathcal{X})$ is the Bloch conductor of $\mathcal{X}$. Then for any $n \in \mathbb{Z}$

$$\lambda_\infty(\zeta^*(\mathcal{X}, n)^{-1} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z}, n)$$

if and only if

$$\lambda_\infty(\zeta^*(\mathcal{X}, d-n)^{-1} \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z}, d-n).$$

$A(\mathcal{X})$ is defined in terms of $\Omega_{\mathcal{X}/\mathbb{Z}}$. Example: $A(\mathrm{Spec}(\mathcal{O}_F)) = |D_F|$.

Note: Compatibility with FE is not in general known for TNC.

Compatibility with FE: Proof

Defining

$$\begin{aligned}\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \quad := \quad & \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}) \\ & \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d-n}_{\mathcal{X}/\mathbb{Z}})\end{aligned}$$

one has

$$\Delta(\mathcal{X}/\mathbb{Z},n)\otimes \Xi_\infty(\mathcal{X}/\mathbb{Z},n) \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z},d-n)$$

and a canonical trivialization and period $x_\infty \in \mathbb{R}^\times$

$$\xi_\infty : \mathbb{R} \xrightarrow{\sim} \Xi_\infty(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{R}; \qquad \xi_\infty(\mathbb{Z}\cdot x_\infty^{-1}) = \Xi_\infty(\mathcal{X}/\mathbb{Z},n)$$

Here

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{R} \simeq R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^+ := R\Gamma(\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{R})^+$$

is a certain $\mathbb{Z}$-lattice in the Betti plus space.

Compatibility with FE: Proof

ξ_∞ is induced by

$$
\begin{aligned}
& \left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n))\right)_{\mathbb{R}} \\
\xrightarrow{\sim}\ & \det_{\mathbb{R}} \left(R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^+ \oplus R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1))^+\right) \\
\xrightarrow{\sim}\ & \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})^+ \\
\xrightarrow{\sim}\ & \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C})^+ \\
\xrightarrow{\sim}\ & \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R}) \simeq \left(\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d}_{\mathcal{X}/\mathbb{Z}})\right)_{\mathbb{R}} \\
\xrightarrow{\lambda_{dR}^{-1}}\ & \left(\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}}) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d-n}_{\mathcal{X}/\mathbb{Z}})\right)_{\mathbb{R}}
\end{aligned}
$$

Need to show

$$
x_\infty = \pm A(\mathcal{X})^{n-d/2} \cdot \frac{\zeta^*(\mathcal{X}_\infty, n)}{\zeta^*(\mathcal{X}_\infty, d-n)} \cdot \frac{C_\infty(\mathcal{X}, d-n)}{C_\infty(\mathcal{X}, n)}.
$$

or equivalently

$$
x_\infty = \pm A(\mathcal{X})^{n-d/2} \cdot 2^{d_+(\mathcal{X},n)-d_-(\mathcal{X},n)} \cdot (2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)}
$$

Compatibility with FE: Proof

- Verdier duality on the locally compact space $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_\mathbb{R}$ gives

$$\lambda_B\left(\det_\mathbb{Z} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes\det_\mathbb{Z}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(d-n))\right)$$
$$=\det_\mathbb{Z} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))\cdot 2^{d_-(\mathcal{X},n)-d_+(\mathcal{X},n)}$$

- Comparing Poincaré duality for both sides gives

$$\det_\mathbb{Z} R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)) = (2\pi)^{d_-(\mathcal{X},n)+t_H(\mathcal{X},n)}\cdot A(\mathcal{X})^{\frac{d}{2}}\cdot\det_\mathbb{Z}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d}_{\mathcal{X}/\mathbb{Z}})$$

- A result of Takeshi Saito implies

$$\lambda_{dR}\left(\det_\mathbb{Z}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})\otimes\det_\mathbb{Z} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d-n}_{\mathcal{X}/\mathbb{Z}})\right)$$
$$= A(\mathcal{X})^{d-n}\cdot\det_\mathbb{Z}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega^{<d}_{\mathcal{X}/\mathbb{Z}})$$

Compatibility with FE: Proof

Theorem

(T. Saito) For any $r \in \mathbb{Z}$ define $C^r_{\mathcal{X}/\mathbb{Z}} \in D^b(\mathrm{Coh}(\mathcal{X}))$ by the exact triangle

$$L\wedge^r \Omega_{\mathcal{X}/\mathbb{Z}} \to \underline{R\,\mathrm{Hom}}(L\wedge^{d-1-r}\Omega_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}}) \to C^r_{\mathcal{X}/\mathbb{Z}}$$

Then $R\Gamma(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}})$ has finite cohomology and

$$\prod_{i\in\mathbb{Z}} \left(\# H^i(\mathcal{X}, C^r_{\mathcal{X}/\mathbb{Z}})\right)^{(-1)^i} = A(\mathcal{X})^{(-1)^r}.$$