Special values of Motivic L-functions II
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- Talk 1
  - Some history
  - The example of number fields
  - Determinant functors
- Talk 2
  - General formulation of the Tamagawa number conjecture (Deligne,Beilinson,Bloch,Kato,Fontaine,Perrin-Riou,....)
  - Proofs of known cases: Iwasawa Theory and p-adic L-functions
  - Detailed proof for Dirichlet L-functions
- Talk 3
  - Zeta functions of arithmetic schemes
  - Special values in terms of Weil-Arakelov cohomology groups and (variants of) cyclic homology
- Talk 4
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  - Compatibility with the functional equation

Motives and motivic structures (over $\mathbb{Q}$)

$$X \to \mathrm{Spec}(\mathbb{Q}) \text{ smooth, projective variety,}$$

$$M_{gm}(X)^* =: h(X) \stackrel{?}{=} \bigoplus_{i\in\mathbb{Z}} h^i(X)[-i] \in \mathrm{Ob}\, DM_{gm}(\mathbb{Q})_{\mathbb{Q}} \quad \text{(def. by Voevodsky)}$$

$M = h^i(X)(j)$ for $i, j \in \mathbb{Z}$, more generally $M \in DM_{gm}(\mathbb{Q})^{\heartsuit}_{\mathbb{Q}}$ (heart of conjectural t-structure), leads to a

"Motivic structure":

$$M_l = H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(j) \quad \text{continuous rep'n of } G_{\mathbb{Q}}$$

$$M_B = H^i(X(\mathbb{C}), \mathbb{Q})(j) \quad \text{pure } \mathbb{Q}\text{-Hodge structure over } \mathbb{R}$$

$$M_{dR} = H^i_{dR}(X/\mathbb{Q})(j) \quad \text{filtered } \mathbb{Q}\text{-vector space}$$

Comparison isomorphisms:

$$M_l \cong M_{B,\mathbb{Q}_l}, \quad M_{B,\mathbb{C}} \cong M_{dR,\mathbb{C}}, \quad M_{l,B_{dR}} \cong M_{dR,B_{dR}}$$

Motivic L-functions

$$P_p(T) = \det(1 - \mathrm{Fr}_p^{-1} \cdot T | M_l^{I_p}) \overset{?}{\in} \mathbb{Q}[T]$$

$$L(M, s) = \prod_p P_p(p^{-s})^{-1}, \quad Re(s) >> 0$$

- $M = \mathbb{Q}(j)_F := h^0(\mathrm{Spec}(F))(j)$
  $L(\mathbb{Q}(j)_F, s) = \zeta_F(j + s)$ (Dedekind Zeta-Function)
- $M = h^0(\mathrm{Spec}(\mathbb{Q}(\sqrt{-1})))(0) = \mathbb{Q}(0) \oplus \mathbb{Q}(\epsilon)$
  $L(\mathbb{Q}(\epsilon), s) = L(\epsilon, s)$ (Dirichlet L-Function)
- $E : y^2 = x^3 - x$
  $L(h^1(E), s) = L(\phi, s)$ (Hecke L-Function for a character ϕ of $\mathbb{Q}(i)$)
  Here $\phi((\alpha)) = \alpha$ where $\alpha \equiv 1 \mod (1 + i)^3$
- $E : y^2 + y = x^3 - x$
  $L(h^1(E), s) = L(f, s)$ (f weight 2 cusp form on $X_0(37)$)

Meromorphic continuation

Conjecture: $L(M,s)$ has meromorphic continuation to all $s \in \mathbb{C}$ and satisfies

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(M^*, 1-s)$$

where

$$\Lambda(M,s) = L_\infty(M,s)L(M,s)$$

$$L_\infty(M,s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_{p} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

Known for

- $h^0(X) = h^0(\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$
- $h^1(X)$ for X/F an elliptic curve over $F = \mathbb{Q}$ or F real quadratic or cubic (holomorphic continuation) or F totally real or CM (meromorphic continuation)
- $h^1(X)$ for $X : z_0^n + z_1^n + z_2^n = 0$
- $\mathrm{Sym}^n h^1(E)$ for $E/\mathbb{Q}$ an elliptic curve
- $h^d(X)$, X Shimura variety of dimension d

Motivic Cohomology

$$H^0(M) := \mathrm{Hom}_{DM(\mathbb{Q})_{\mathbb{Q}}}(\mathbb{Q}(0), M) = CH^j(X)_{\mathbb{Q}}/hom \text{ for } M = h^{2j}(X)(j)$$

$$H^1(M) := \mathrm{Hom}_{DM(\mathbb{Q})_{\mathbb{Q}}}(\mathbb{Q}(0), M[1]) = \begin{cases} K^{(j)}_{2j-i-1}(X)_{\mathbb{Q}} \text{ for } M = h^i(X)(j) \\ CH^j(X)^0_{\mathbb{Q}} \text{ if } 2j-i-1=0 \end{cases}$$

$$H^0_f(M) := H^0(M)$$

$$H^1_f(M) := \text{image of } K^{(j)}_{2j-i-1}(\mathfrak{X})_{\mathbb{Q}} \text{ where } \mathfrak{X} \text{ is regular, proper over } \mathrm{Spec}(\mathbb{Z})$$

$M_{B,\mathbb{C}} \cong M_{dR,\mathbb{C}}$ induces $\alpha_M : M^+_{B,\mathbb{R}} \to (M_{dR}/M^0_{dR})_{\mathbb{R}}$.

Conjecture Mot$_\infty$: There exists an exact sequence

$$0 \to H^0_f(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \to H^1_f(M^*(1))^*_{\mathbb{R}} \xrightarrow{h} H^1_f(M)_{\mathbb{R}} \xrightarrow{r} \mathrm{coker}(\alpha_M) \to H^0_f(M^*(1))^*_{\mathbb{R}} \to 0$$

c = cycle class map, h = height pairing, and r = Beilinson regulator.

Vanishing order

Taylor expansion at $s = 0$

$$L(M, s) = L^*(M)s^{r(M)} + \cdots$$

Aim: Describe $L^*(M) \in \mathbb{R}^\times$ and $r(M) \in \mathbb{Z}$

Conjecture (Van): $r(M) = \dim_{\mathbb{Q}} H^1_f(M^*(1)) - \dim_{\mathbb{Q}} H^0_f(M^*(1))$

Known cases:

- F number field, $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$,

$$\dim_{\mathbb{Q}} H^1_f(M^*(1)) - \dim_{\mathbb{Q}} H^0_f(M^*(1)) = \begin{cases} K_{1-2j}(\mathcal{O}_F^\times)_{\mathbb{Q}} & j \leq 0 \\ -1 & j = 1 \\ 0 & j \geq 2 \end{cases}$$

- $M = h^1(E)(1)$, $E/\mathbb{Q}$ elliptic curve, $\mathrm{ord}_{s=1} L(E, s) \leq 1$, individual examples with $\mathrm{ord}_{s=1} L(E, s) = 2, 3$.

Rationality conjecture

Define **Fundamental Line**

$$\begin{aligned}\Xi(M) := {} & \det_{\mathbb{Q}}(H^0_f(M)) \otimes \det^{-1}_{\mathbb{Q}}(H^1_f(M)) \\ & \otimes \det_{\mathbb{Q}}(H^1_f(M^*(1))^*) \otimes \det^{-1}_{\mathbb{Q}}(H^0_f(M^*(1))^*) \\ & \otimes \det^{-1}_{\mathbb{Q}}(M_B^+) \otimes \det_{\mathbb{Q}}(M_{dR}/M^0_{dR}),\end{aligned}$$

Conjecture (Rat): $\vartheta_\infty(L^*(M)^{-1}) \in \Xi(M) \otimes 1$ where

$$\vartheta_\infty : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

is the isomorphism induced by Conjecture $\mathbf{Mot}_\infty$.

Known cases:

- F number field, $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$ (Borel)
- X/F Shimura curve over totally real F, A direct factor of $Jac(X)$, $M = h^1(A)(1)$, $\mathrm{ord}_{s=1} L(A, s) \leq 1$ (Gross-Zagier-Zhang formula)

An example with $\Xi(M) = \mathbb{Q}$

F totally real, $j < 0$ odd, $M = h^0(\mathrm{Spec}(F))(j)$

$$\Xi(M) = \mathbb{Q}$$

since all spaces in the definition of $\Xi(M)$ are zero!

For $F = \mathbb{Q}$

$$\zeta(1-n) = -\frac{B_n}{n} \quad \text{where} \quad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

For F totally real $\zeta_F(j) \in \mathbb{Q}$ for $j \leq 0$ by the **Klingen-Siegel** theorem.

Galois cohomology

Fix prime l. For each prime p define a complex $R\Gamma_f(\mathbb{Q}_p, M_l)$

$$= \begin{cases} M_l^{I_p} \xrightarrow{1-\mathrm{Fr}_p} M_l^{I_p} & l \neq p \\ D_{cris}(M_l) \xrightarrow{(1-\mathrm{Fr}_p,\pi)} D_{cris}(M_l) \oplus D_{dR}(M_l)/D_{dR}^0(M_l) & l = p \end{cases}$$

There is a distinguished triangle of $\mathbb{Q}_l$-vector spaces.

$$R\Gamma_f(\mathbb{Q}_p, M_l) \to R\Gamma(\mathbb{Q}_p, M_l) \to R\Gamma_{/f}(\mathbb{Q}_p, M_l).$$

Let S be a finite set of primes containing l, ∞ and primes of bad reduction. There are distinguished triangles

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \to \bigoplus_{p\in S} R\Gamma(\mathbb{Q}_p, M_l)$$

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$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma_f(\mathbb{Q}, M_l) \to \bigoplus_{p\in S} R\Gamma_f(\mathbb{Q}_p, M_l) \qquad (1)$$

Galois cohomology and motivic cohomology

Conjecture $\mathbf{Mot}_l$: a) The cycle class map induces an isomorphism $H^0_f(M)_{\mathbb{Q}_l} \cong H^0_f(\mathbb{Q}, M_l)$ (Tate conjecture).
b) The Chern class maps induce an isomorphism $H^1_f(M)_{\mathbb{Q}_l} \cong H^1_f(\mathbb{Q}, M_l)$ (Bloch-Kato).

Poitou-Tate duality gives an isomorphism

$$H^i_f(\mathbb{Q}, M_l) \cong H^{3-i}_f(\mathbb{Q}, M_l^*(1))^*$$

for all i. Hence Conjecture $\mathbf{Mot}_l$ computes the cohomology of $R\Gamma_f(\mathbb{Q}, M_l)$ in all degrees.

Integrality Conjecture

The exact triangle (1) and conjecture **Mot**$_l$ induce an isomorphism

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{\mathbb{Q}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l)$$

Let $T_l \subset M_l$ be any $G_{\mathbb{Q}}$-stable $\mathbb{Z}_l$-lattice.

Conjecture (Int):

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_\infty(L^*(M)^{-1}) = \det_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$$

This conjecture (for all l) determines $L^*(M) \in \mathbb{R}^\times$ up to sign. It is independent of the choice of S and T_l.

Known cases:

- $M = h^0(\mathrm{Spec}(F))(j)$, $j = 0, 1$ (Analytic class number formula)
- $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$, $F/\mathbb{Q}$ abelian
- $M = h^0(\mathrm{Spec}(F))(j)$, $j \in \mathbb{Z}$, or F/K abelian, K imaginary quadratic, $l > 3$ split in K (Johnson-Leung)
- $M = h^1(E)(1)$, $\mathrm{ord}_{s=1} L(E, s) = 0$, $l \notin S$, S finite, $E/\mathbb{Q}$ CM elliptic curve (Rubin), $E/\mathbb{Q}$ semistable elliptic curve (Kato, Skinner-Urban, Wan)

The equivariant refinement

Let A be a finite-dimensional semisimple $\mathbb{Q}$-algebra, acting on M.
Examples.

- X abelian variety, $M = h^1(X)$, $A = \mathrm{End}(X)_{\mathbb{Q}}$
- X variety with action of a finite group G, e.g. $X = X' \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(F)$, $F/\mathbb{Q}$ Galois with group G, $M = h^i(X)(j)$, $A = \mathbb{Q}[G]$.

For simplicity assume A **commutative**, so

$$A \cong E_1 \times \cdots \times E_r, \quad (E_i \text{ number fields})$$

Define $L({}_AM, s)$, $\Xi({}_AM)$, ${}_A\vartheta_\infty$, ${}_A\vartheta_l$ as above using the determinant functor over $A, A_{\mathbb{R}}, A_l := A \otimes \mathbb{Q}_l$.

- $L({}_AM, s) \in A_{\mathbb{C}} \cong \prod_\tau \mathbb{C}$
- $r({}_AM) \in H^0(\mathrm{Spec}(A_{\mathbb{C}}), \mathbb{Z}) \cong \prod_\tau \mathbb{Z}$
- $L^*({}_AM) \in (A_{\mathbb{R}})^\times$

The equivariant refinement, ctd.

$$_A\vartheta_\infty : A_{\mathbb{R}} \cong \Xi(_AM) \otimes_{\mathbb{Q}} \mathbb{R}$$

$$_A\vartheta_l : \Xi(_AM) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{A_l} R\Gamma_{c,\mathrm{et}}(\mathbb{Z}[\frac{1}{S}], M_l)$$

Equivariant Tamagawa number conjecture - ETNC

Van $r(_AM) = \dim_A H^1_f(M^*(1)) - \dim_A H^0_f(M^*(1))$

Rat $_A\vartheta_\infty(L^*(_AM)^{-1}) \in \Xi(_AM) \otimes 1$

Int $\mathfrak{A}_l \cdot {_A\vartheta_l}{_A\vartheta_\infty}(L^*(_AM)^{-1}) = \det_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$

Here $\mathfrak{A} \subset A$ is a $\mathbb{Z}$-order so that there is a **projective** $G_{\mathbb{Q}}$-stable $\mathfrak{A}_l := \mathfrak{A} \otimes \mathbb{Z}_l$-lattice $T_l \subseteq V_l$.

Example. F/K Galois with group G, $\mathfrak{A} = \mathbb{Z}[G]$, $M = h^0(\mathrm{Spec}(F))(j)$
Conj. **Int** known if $F/\mathbb{Q}$ abelian for all j. In general **Rat** not even known for $j = 0, 1$! (Stark conjectures)

Non-commutative coefficients

For any ring R

$$\tau_{\leq 1} K(R) \cong \mathcal{P}(R)$$

where $\mathcal{P}(R)$ is a **Picard category** (groupoid with $\otimes$). Universal Determinant functor

$$\mathcal{D}^{perf}(R)^{\cong} \to K(R) \to \tau_{\leq 1} K(R) \cong \mathcal{P}(R)$$

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If R is commutative semilocal then

$$\pi_0\mathcal{P}(R) = K_0(R) = H^0(\mathrm{Spec}(R), \mathbb{Z}); \quad \pi_1\mathcal{P}(R) = K_1(R) = R^\times$$

Hence: universal determinant functor = usual graded determinant functor

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$R = A, A \otimes \mathbb{Q}_l$, $\mathfrak{A} \otimes \mathbb{Z}_l$ semilocal
If A is non-commutative use universal determinant functor.

Proven cases of the weak TNC

One has the following situation:

- Conjecture **Mot**$_\infty$ reduces to $H^1_f(M)_\mathbb{R} \cong H^1_\mathcal{D}(M) := \mathrm{coker}(\alpha_M)$.
- $\dim_{A_\mathbb{R}} H^1_\mathcal{D}(M) = 1$.
- There is $\xi \in H^1_f(M)$ with $A_\mathbb{R} \cdot r(\xi) = H^1_\mathcal{D}(M)$.

Main example. f elliptic modular form of weight $k \geq 2$, $M = M(f)(j)$, $j \leq 0$.

- Weak form of **Rat** is known (Bloch-Beilinson)
- **Int** follows from the main conjecture of Kato/Skinner/Urban if one also assumes $A_{\mathbb{Q}_l} \cdot r_l(\xi) = H^1_f(M_l)$ (Gealy).

Dirichlet L-Functions

$$F = F_m := \mathbb{Q}(\zeta_m); \quad M = h^0(\mathrm{Spec}(F_m))(0)$$

$$G = G_m := \mathrm{Gal}(F_m/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

$$A = \mathbb{Q}[G_m] \cong \prod_{\chi \in \hat{G}_{rat}} \mathbb{Q}(\chi)$$

$$L({}_AM, s) = (L(\eta, s))_{\eta \in \hat{G}} \in \prod_{\eta \in \hat{G}} \mathbb{C} \cong A_{\mathbb{C}}$$

Note:

$$\zeta_{F_m}(s) = \prod_{\eta \in \hat{G}} L(\eta, s) \in \mathbb{C}$$

$$\mathrm{ord}_{s=0} L(\eta, s) = \begin{cases} 0 & \eta = 1 \text{ or } \eta(-1) = -1 \\ 1 & \eta \neq 1 \text{ and } \eta(-1) = 1 \end{cases}$$
$$= \dim_{\mathbb{Q}(\eta)}(\mathcal{O}_F^\times \otimes_{\mathbb{Z}[G]} \mathbb{Q}(\eta))$$

Leading Taylor coefficient at $s = 0$

$$L(\eta, 0) = -\sum_{a=1}^{f_\eta} \left(\frac{a}{f_\eta} - \frac{1}{2} \right) \eta(a)$$

$$\frac{d}{ds} L(\eta, s)\bigg|_{s=0} = -\frac{1}{2} \sum_{a=1}^{f_\eta} \log |1 - e^{2\pi i a / f_\eta}| \eta(a)$$

$$\begin{aligned} &\Xi({}_A M)^{\#} \\ &= \prod_{\substack{\chi \neq 1 \\ \text{even}}} (\mathcal{O}_{F_m}^{\times} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi))^{-1} \otimes_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \\ &\quad \times \prod_{\text{other } \chi} \mathbb{Q}(\chi) \end{aligned}$$

$${}_A\vartheta_\infty(L^*({}_A M)^{-1})_\chi =$$

$$\begin{cases} 2 \cdot [F_m : F_{f_\chi}][1 - \zeta_{f_\chi}]^{-1} \otimes \sigma_m & \chi \neq 1 \text{ even} \\ \left(L(\chi, 0)^{\#} \right)^{-1} & \text{else.} \end{cases}$$

Iwasawa-Theory

Let l be a prime, $m \geq 1$

$$\Xi({}_A M)^{\#} \otimes \mathbb{Q}_l \xrightarrow{{}_A\vartheta_l} \det_{A_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l)^{\#}$$
$$\cong \det_{A_l} \Delta(F_m) \otimes \mathbb{Q}_l$$

$$\Delta(F_m) := R\operatorname{Hom}_{\mathbb{Z}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l), \mathbb{Z}_l)[-3]$$

Iwasawa-algebra

$$\Lambda = \varprojlim_n \mathbb{Z}_l[G_{ml^n}] \cong \mathbb{Z}_l[G_{\ell m_0}][[T]]$$

where

$$m = m_0 l^{\mathrm{ord}_l(m)}; \quad \ell = \begin{cases} l & l \neq 2 \\ 4 & l = 2. \end{cases}$$

Define perfect complex of Λ-modules

$$\Delta^{\infty} = \varprojlim_n \Delta(L_{m_0 l^n})$$

Iwasawa-Theory, ctd.

Define Elements

$$\eta_{m_0} := (1-\zeta_{\ell m_0 l^n})_{n\geq 0} \in \varprojlim_n \mathcal{O}_{F_{m_0 l^n}}[\frac{1}{ml}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_l = H^1(\Delta^\infty)$$

$$\sigma := (\sigma_{\ell m_0 l^n})_{n\geq 0} \in H^2(\Delta^\infty)$$

$$\theta_{m_0} := (g_{\ell m_0 l^n})_{n\geq 0} \in (\gamma - \chi_{\text{cyclo}}(\gamma))^{-1}\Lambda \subset Q(\Lambda)$$

where

$$g_k = -\sum_{0<a<k,(a,k)=1} \left(\frac{a}{k}-\frac{1}{2}\right)\tau_{a,k}^{-1} \in \mathbb{Q}[G_k]$$

and $\tau_{a,k} \in G_k$ is defined by $\tau_{a,k}(\zeta_k) = \zeta_k^a$.

$$0 \to P^\infty \to H^2(\Delta) \to X^\infty \to 0$$

$$P^\infty = \varprojlim_n \text{Pic}(\mathcal{O}_{F_{m_0 l^n}}[1/ml]) \otimes_{\mathbb{Z}} \mathbb{Z}_l, \quad X^\infty = \varprojlim_n X_{\{v|lm_0\infty\}}(F_{m_0 l^n}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

Iwasawa-Theory, ctd.

Total quotient ring

$$Q(\Lambda) \cong \prod_{\psi \in \hat{G}_{\ell m_0}^{\mathbb{Q}_l}} Q(\psi)$$

l-adic L-Functions

$$\mathcal{L} := \theta_{m_0}^{-1} + 2 \cdot \eta_{m_0}^{-1} \otimes \sigma \in \det_{Q(\Lambda)}(\Delta^{\infty} \otimes_{\Lambda} Q(\Lambda))$$

Theorem(Main Conjecture). One has an equality of invertible Λ-submodules

$$\Lambda \cdot \mathcal{L} = \det_{\Lambda} \Delta^{\infty}$$

of $\det_{Q(\Lambda)}(\Delta^{\infty} \otimes_{\Lambda} Q(\Lambda))$.

Iwasawa-Theory, ctd.

Since Λ is Cohen-Macaulay (even complete intersection) it suffices to show

$$\Lambda_{\mathfrak{q}} \cdot \mathcal{L} = \det_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^{\infty}$$

for all **height one** prime ideals $\mathfrak{q}$.
If $I \notin \mathfrak{q}$ then $\Lambda_{\mathfrak{q}}$ is a d.v.r. with fraction field $Q(\psi_{\mathfrak{q}})$. Main conjecture reduces to

$$\begin{aligned} &\mathrm{Fit}_{\mathfrak{q}}(P_{\mathfrak{q}}^{\infty}) \sim \mathrm{Fit}_{\mathfrak{q}}(H^1(\Delta)_{\mathfrak{q}}/\Lambda_{\mathfrak{q}} \cdot \eta_{m_0}) && \psi_{\mathfrak{q}} \text{ even} \\ &\mathrm{Fit}_{\mathfrak{q}}(P_{\mathfrak{q}}^{\infty}) \sim \theta_{m_0} && \psi_{\mathfrak{q}} \text{ odd} \end{aligned}$$

which is the classical Iwasawa main conjecture (Theorem of Mazur-Wiles)

For **odd** $I \in \mathfrak{q}$ main conjecture follows from $\mu = 0$ (Ferrero-Washington)

Proof for $l = 2$

For $l = 2$ and $\mathfrak{q}$ a prime ideal of height one of Λ with $2 \in \mathfrak{q}$ the $\Lambda_{\mathfrak{q}}$-module

$$H^1(\Delta^\infty)_{\mathfrak{q}} \cong H^2(\Delta^\infty)_{\mathfrak{q}} \cong \Lambda_{\mathfrak{q}}/(c-1)$$

is **not** of finite projective dimension ($c \in \Lambda$ complex conjugation). The determinant $\det_{\Lambda_{\mathfrak{q}}} \Delta^\infty_{\mathfrak{q}}$ **cannot** be computed by passing to cohomology. One needs to construct $\Delta^\infty_{\mathfrak{q}}$ explicitly, using results of Coleman, Leopoldt et al. The proof of the main conjecture for such $\mathfrak{q}$ reduces to a "mod 2 congruence" between

$$(\gamma - \chi_{\text{cyclo}}(\gamma))g_m$$

und

$$(1-\zeta_m)^{\gamma - \chi_{\text{cyclo}}(\gamma)}$$

expressed by the following Lemma.

Lemma Let $M \equiv 1 \mod 4$ be an integer and $0 < x < 1$. The sign of the real number

$$\frac{1 - e^{2\pi i x M}}{(1 - e^{2\pi i x})^M}$$

is $(-1)^{\lfloor xM \rfloor}$.

m_0 odd, $M = 1 + 4m_0 = \chi_{\text{cyclo}}(\gamma)$

$$(1 - \zeta_m^a)^{\gamma - \chi_{\text{cyclo}}(\gamma)} = \frac{1 - e^{2\pi i \frac{a}{m} M}}{(1 - e^{2\pi i \frac{a}{m}})^M}$$

$$(\gamma - \chi_{\text{cyclo}}(\gamma)) g_m \equiv \sum_{\substack{0<a<m \\ (a,m)=1}} \left\lfloor \frac{aM}{m} \right\rfloor \tau_{a,m}^{-1} \mod 2$$

Descent to F_m

For $n \in \mathbb{Z}$ there is a homomorphism $\kappa^n : \Lambda \to \mathbb{Z}[G_m]$ and an isomorphism

$$\Delta^\infty \otimes^L_{\Lambda,\kappa^n} \mathbb{Z}_l[G_m] \cong R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], T_l(n))$$

For $n \leq 0$ one can compute the image of $\mathcal{L}$ in

$$\det_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{ml}], M_l(n)) \cong \Xi({}_A M(n)) \otimes \mathbb{Q}_l$$

in terms of Beilinson-Soule elements in $K_{1-2n}(F_m)$, verifying ETNC.
For $n = 0$ one needs theorems of Ferrero-Greenberg and Solomon to handle trivial zeros of $\mathcal{L}$.
To show ETNC for $n > 0$ one proves compatibility of ETNC with the functional equation.

Elliptic curves over $\mathbb{Q}$

Theorem

(Skinner-Urban, Kato) f elliptic modular form of weight $k = 2$ and level N, p a prime of good ordinary reduction,

- $\overline{\rho}_f$ *irreducible*
- *For some $p \neq q \mid N$ $\overline{\rho}_f$ is ramified at q*

$$\mathrm{char}(\mathrm{Sel}^{\Sigma}(T_f)) \sim \mathcal{L}^{\Sigma}_{alg}(f)$$

Theorem

(X. Wan, Li Cai, Chao Li, Shuai Zhai) The full BSD formula

$$\frac{L(E,1)}{\Omega_E} = \frac{\#\text{Ш}(E/\mathbb{Q})}{\#E(\mathbb{Q})^2} \prod_{\ell \mid N} c_\ell$$

holds for certain infinite families of $E/\mathbb{Q}$ with $L(E,1) \neq 0$. Example: An infinite family of quadratic twists of

$$46A1 : y^2 + xy = x^3 - x^2 - 10x - 12$$

Adjoint motives of modular forms

Theorem

(Diamond-Flach-Guo) f elliptic modular form of weight $k \geq 2$, level N, coefficients in E, Σ set of primes λ of E such that

- *$\lambda \mid Nk!$ or*
- *$\overline{\rho}_f$ restricted to $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ is not abs. irr.*

If $\lambda \notin \Sigma$ the TNC holds for $L(Ad(f), 0)$ and $L(Ad(f), 1)$

Proof uses Taylor-Wiles method and $R = T$ theorems.

Theorem

(Tilouine-Urban) Under similar assumptions TNC holds for $L(Ad(f) \otimes \alpha, 0)$ where α is a Dirichlet character corresponding to a real quadratic field F.

Proof uses $R_F = T_F$ and relations between periods of f and f_F.