Special values of Motivic L-functions I
Bengaluru, August 9, 2022

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- Talk 1
  - Some history
  - The example of number fields
  - Determinant functors
- Talk 2
  - General formulation of the Tamagawa number conjecture (Deligne,Beilinson,Bloch,Kato,Fontaine,Perrin-Riou,....)
  - Proofs of known cases: Iwasawa Theory and p-adic L-functions
  - Detailed proof for Dirichlet L-functions
- Talk 3
  - Zeta functions of arithmetic schemes
  - Special values in terms of Weil-Arakelov cohomology, or an integral fundamental line. The example $\mathrm{Spec}(\mathcal{O}_F)$
- Talk 4
  - Compatibility with the Conjecture of Birch and Swinnerton-Dyer
  - Compatibility with the functional equation

Arithmetic schemes vs. motives over $\mathbb{Q}$

$$\mathfrak{X} = \operatorname{Spec} \mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$$

Zeta function

$$\zeta(\mathfrak{X}, s) = \prod_{x \in \mathfrak{X} \text{ closed}} \frac{1}{1 - Nx^{-s}}$$

L-functions : "Decompose under correspondences"

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L-functions : "Decompose under correspondences"

$X \to \operatorname{Spec}(\mathbb{Q})$ smooth, projective
Motive: Direct Summand M of $h(X)(n) = \bigoplus h^i(X)(n)[-i]$
For $M = h^i(X)(n)$ one has realisations

$$M_B = H^i(X(\mathbb{C}), \mathbb{Q}),\ M_{dR} = H^i_{dR}(X/\mathbb{Q}),\ M_l = H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(n)$$

and motivic cohomology

$$H^0(M), H^1(M)$$

and an L-function $L(M, s)$

L-functions

$$\begin{tikzcd}
\text{Motivic L-functions (Arithmetic geometry)} \arrow[rr, "\text{Reciprocity Law}", "\text{Langlands Correspondene}"'] \arrow[dr] & & \text{Automorphic L-functions (Representation Theory)} \arrow[dl, leftrightarrow, "?"] \\
& \text{Selberg class (Analysis)} &
\end{tikzcd}$$

L-functions

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\begin{array}{l}\text{Motivic L-functions}\\ \text{(Arithmetic geometry)}\end{array} \arrow[rr, "\text{Reciprocity Law}", "\text{Langlands Correspondene}"'] \arrow[dr] & & \begin{array}{l}\text{Automorphic L-functions}\\ \text{(Representation Theory)}\end{array} \arrow[dl, leftrightarrow, "?"] \\
& \begin{array}{l}\text{Selberg class}\\ \text{(Analysis)}\end{array} &
\end{tikzcd}
$$

Basic example: **Dirichlet L-functions**

$$
\begin{tikzcd}
\text{Cyclotomic fields} \arrow[rr] \arrow[dr] & & \mathrm{GL}_1/\mathbb{Q} \arrow[dl, leftrightarrow] \\
& \text{Degree one} &
\end{tikzcd}
$$

Leibniz 1673

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

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Euler 1734

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \pi^{2k} \cdot \text{rational number}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} =?$$

Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(s) = ?, \quad s = 2, 3, 4, 5, \ldots.$$

Dirichlet L-function

$$L(\eta, s) = \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s}$$

$$\eta : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times \qquad \text{character}$$

example: $m = 4$

$$\epsilon : (\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \{\pm 1\}$$

Generalisation to number fields F

$$\zeta_F(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}} \quad \text{(Dedekind Zeta)}$$

$$= \zeta(\mathrm{Spec}(\mathcal{O}_F), s) = L(h^0(\mathrm{Spec}(F)), s)$$

Analytic class number formula

$$\operatorname*{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} \cdot h \cdot R}{w \cdot \sqrt{|D|}}$$

$$\mathcal{O}_F^\times \xrightarrow{\log|\ |_v} \Big(\bigoplus_{v|\infty} \mathbb{R}\Big)^{\Sigma=0}, \quad R = \mathrm{covol}(\mathcal{O}_F^\times)$$

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

$$h = \#\mathrm{Pic}(\mathcal{O}_F), \quad w = \#\mathcal{O}_{F,\mathrm{tors}}^\times, \quad D = \mathrm{disc}(\mathcal{O}_F/\mathbb{Z})$$

Examples

$$F = \mathbb{Q}, \;\; r_2 = 0, \;\; r_1 = 1$$

$$h = R = D = 1, \;\; w = 2$$

$$\operatorname*{Res}_{s=1} \zeta(s) = \frac{2^1 \cdot 1 \cdot 1}{2 \cdot 1} = 1$$

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$$F = \mathbb{Q}(\zeta_m), \quad G := \mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

$$\zeta_F(s) = \prod_{\eta \in \hat{G}} L(\eta, s)$$

$$m = 4, \quad \zeta_{\mathbb{Q}(i)}(s) = \zeta(s) L(\epsilon, s)$$

$$\operatorname*{Res}_{s=1} \zeta_{\mathbb{Q}(i)}(s) = L(\epsilon, 1) = \frac{2\pi \cdot 1 \cdot 1}{4\sqrt{|-4|}} = \frac{\pi}{4}$$

Generalisation to s=2,3,4,...

$\operatorname*{Res}_{s=1} \zeta_F(s)$ involves $K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_F)$ and $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$
$\zeta_F(m)$ involves $K_{2m-2}(\mathcal{O}_F)$ and $K_{2m-1}(\mathcal{O}_F)$

Theorem (Lichtenbaum,Quillen,Borel, 1973-77)
a) $K_n(\mathcal{O}_F)$ is a finitely generated abelian group
b)

$$K_n(\mathcal{O}_F) \otimes \mathbb{Q} = \begin{cases} 0 & n \equiv 0 \mod 2 \\ \mathbb{Q}^{r_1+r_2-1} & n = 1 \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \mod 4 \\ \mathbb{Q}^{r_2} & n \equiv 3 \mod 4 \end{cases}$$

c) $d_m = r_1 + r_2$ (resp. r_2) for $m \equiv 1$ (resp. 0) $\mod 2$

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \frac{\pi^{md-d_m} R_{2m-1}}{\sqrt{|D|}}$$

Computation of $K_n(\mathbb{Z})$

Theorem (Suslin,Voevodsky,Rost et al 1997-2008)

n	0	1	2	3	4	5	6	7
$K_n(\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/48$	0	$\mathbb{Z}$	0	$\mathbb{Z}/240$

8	9	10	11	12	13	14	15
0?	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/1008$	0?	$\mathbb{Z}$	0	$\mathbb{Z}/480$

For $n < 20000$ one knows

n	$8a$	$8a+1$	$8a+2$	$8a+3$
$K_n(\mathbb{Z})$	0?	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2c_{2a+1}$	$\mathbb{Z}/2w_{4a+2}$

$8a+4$	$8a+5$	$8a+6$	$8a+7$
0?	$\mathbb{Z}$	$\mathbb{Z}/c_{2a+2}$	$\mathbb{Z}/w_{4a+4}$

$$c_k = \text{Numerator of } \frac{B_k}{4k}$$

$$w_k = \max\{m \mid k\cdot(\mathbb{Z}/m\mathbb{Z})^\times = 0\}, \quad \text{Example: } w_2 = 24$$

K-Theory vs. Cohomology

$$\zeta(2) = (2\pi)^2 \frac{2}{48} = (2\pi)^2 \frac{\# K_2(\mathbb{Z})}{\# K_3(\mathbb{Z})}$$

$$\zeta(m) \stackrel{?}{=} (2\pi)^{dm-d_m} \frac{\# K_{2m-2}(\mathbb{Z})}{\# K_{2m-1}(\mathbb{Z})_{\mathrm{tor}}} \cdot \frac{R_{2m-1}}{\sqrt{|D|}} \cdot 2^?$$

This is proven using étale cohomology. For primes l and $i = 1, 2$ one has

$$\textbf{Chern class maps:} \quad K_{2m-i}(\mathbb{Z}) \to H^i_{et}(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(m))$$

inducing isomorphisms

$$K_{2m-i}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \cong H^i_{et}(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(m))$$

for $m \geq 2$ and all l if $i = 2$, odd l if $i = 1$. But

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K_3(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to H^1_{et}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(2)) \to 0$$

$$\textbf{Note}: \quad H^1_{et}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(2)) \cong H^0_{et}(\mathbb{Z}[\frac{1}{2}], \mathbb{Q}_2/\mathbb{Z}_2(2)) \cong \mathbb{Z}/8\mathbb{Z}$$

Tamagawa number conjecture for $M = \mathbb{Q}(2)_F$ and F totally real.

Fundamental line

$$\Xi(M) = \det_{\mathbb{Q}} M_{dR}/M^0_{dR} \otimes \det_{\mathbb{Q}}^{-1} M_B^+ = \det_{\mathbb{Q}} F \otimes \det_{\mathbb{Q}}^{-1} \left(\bigoplus_{v|\infty} (2\pi i)^2 \mathbb{Q} \right)$$

There is a period isomorphism

$$\vartheta_\infty : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

and an isomorphism

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{\mathbb{Q}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Q}_l(2))$$

for all primes l. The conjecture says

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_\infty^{-1}(\zeta_F(2)) = \det_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2))$$

$$R\Gamma_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to R\Gamma(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to R\Gamma(\mathbb{Q}_l, \mathbb{Z}_l(2)) \oplus R\Gamma(\mathbb{R}, \mathbb{Z}_l(2))$$

$M = \mathbb{Q}(2)$, $F = \mathbb{Q}$ **ctd**

$$\vartheta_\infty((2\pi)^2) = \det_{\mathbb{Z}} \mathcal{O}_F \otimes \det_{\mathbb{Z}}^{-1} H^0(\mathbb{R}, (2\pi i)^2\mathbb{Z})$$

$$\exp : F_{\mathbb{Q}_l} \cong H^1(\mathbb{Q}_l, \mathbb{Q}_l(2))$$
$$\exp(\mathcal{O}_F) = (1 - l^{-2})H^1(\mathbb{Q}_l, \mathbb{Z}_l(2)) \cdot \#(H^2(\mathbb{Q}_l, \mathbb{Z}_l(2)))^{-1}$$
ϑ_l also involves $(1 - l^{-2})$

$$0 \to H^0(\mathbb{R}, \mathbb{Z}_l(2)) \to H^1_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to H^1(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to$$
$$H^1(\mathbb{Q}_l, \mathbb{Z}_l(2)) \to H^2_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to H^2(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to$$
$$H^2(\mathbb{Q}_l, \mathbb{Z}_l(2)) \oplus H^2(\mathbb{R}, \mathbb{Z}_l(2)) \to H^3_c(\mathbb{Z}[\frac{1}{l}], \mathbb{Z}_l(2)) \to 0$$

$H^2(\mathbb{R}, \mathbb{Z}_l(2)) \cong \mathbb{Z}/2\mathbb{Z}$ for $l = 2$

Insertion: Determinants of perfect complexes

What is a perfect complex?
R ring. A complex $C^\bullet$ of R-Modules is called **perfect** if $C^\bullet$ is quasi-isomorphic to a complex

$$\cdots \to 0 \to P^n \to P^{n+1} \to \cdots \to P^{n+k} \to 0 \to \cdots$$

where each P^i is a finitely generated projective R-module.

Example: R regular Noetherian, e.g. $R = \mathbb{Z}$,

$$C^\bullet \text{ perfect} \Leftrightarrow H^i(C^\bullet) \text{ finitely generated and zero for almost all } i$$

R commutative, $C^\bullet$ perfect. Define rank

$$\mathrm{rank}_R\, C^\bullet := \sum_{i\in\mathbb{Z}} (-1)^i\, \mathrm{rank}_{\mathbb{Z}}\, P^i \in H^0(\mathrm{Spec}(R), \mathbb{Z})$$

and the determinant

$$\det{}_R\, C^\bullet = \bigotimes_{i\in\mathbb{Z}} \left(\bigwedge^{\mathrm{rank}_R P^i} P^i \right)^{(-1)^i}$$

an invertible R-module.

Determinants of perfect complexes, ctd

Main properties:

A) A short exact sequence of complexes

$$0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$$

induces canonical isomorphism

$$\det_R C_2^\bullet \cong \det_R C_1^\bullet \otimes_R \det_R C_3^\bullet$$

B) For acyclic $C^\bullet$ there is a canonical isomorphism

$$\det_R C^\bullet \cong R$$

C) $C_1^\bullet \xrightarrow{\sim} C_2^\bullet$ quasi-isomorphism induces

$$\det_R C_1^\bullet \xrightarrow{\sim} \det_R C_2^\bullet$$

D) For R regular Noetherian there are canonical isomorphisms

$$\det_R C^\bullet \cong \bigotimes_{i\in\mathbb{Z}} \det_R^{(-1)^i} H^i(C^\bullet)$$

Determinants of perfect complexes, ctd

The isomorphism in **A)** depends on the ordering. Take free, rank one R-module V. The diagram

$$
\begin{tikzcd}
\det_R(V \oplus V) \arrow[r] \arrow[d, "\text{flip}"] & \det_R V \otimes \det_R V \arrow[r, equal] \arrow[d, "\text{flip}"] & V \otimes V \arrow[d, "\text{flip}"] \\
\det_R(V \oplus V) \arrow[r] & \det_R V \otimes \det_R V \arrow[r, equal] & V \otimes V
\end{tikzcd}
$$

does not commute since $v_1 \wedge v_2 = -v_2 \wedge v_1$ but $v_1 \otimes v_2 = v_2 \otimes v_1$. Hence define **graded line bundles** (L, α) where $\alpha \in H^0(\mathrm{Spec}(R), \mathbb{Z})$ with

$$(L, \alpha) \otimes (M, \beta) := (L \otimes M, \alpha + \beta)$$

and commutativity constraint

$$l \otimes m = (-1)^{\alpha\beta} m \otimes l.$$

$$\mathrm{Det}_R : \text{Perfect complexes} \to \text{Graded line bundles}$$

$$\mathrm{Det}_R\, C^\bullet = (\det_R C^\bullet, \mathrm{rank}_R\, C^\bullet)$$

Determinants of perfect complexes, ctd

If $R = \mathbb{Z}_l$ and M is a finite R-module then

$$\det_{\mathbb{Q}_l}(M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \det_{\mathbb{Q}_l}(0) = \mathbb{Q}_l$$

and

$$\det_{\mathbb{Z}_l}(M) = \mathbb{Z}_l \cdot |M|^{-1} \subset \mathbb{Q}_l$$

- R a finite free $\mathbb{Z}_l$-algebra
- T a finitely generated projective R-module with continuous R-linear action of $\pi_{1,et} \operatorname{Spec}(\mathbb{Z}\left[\frac{1}{S}\right])$
- $R\Gamma_c(\mathbb{Z}\left[\frac{1}{S}\right], T)$, $R\Gamma(\mathbb{Q}_l, T)$ are perfect complexes of R-modules, $R\Gamma(\mathbb{Z}\left[\frac{1}{S}\right], T)$ is not in general.

The Class number formula from the point of view of TNC

$$\operatorname*{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} \cdot h \cdot R}{w \cdot \sqrt{|D|}}$$

Motive: $M = \mathbb{Q}(1)_F$
Betti realization: $M_B = \bigoplus_{\tau \in \Sigma} (2\pi i)\mathbb{Q}\tau$, $\Sigma = \mathrm{Hom}(F, \mathbb{C})$
Fundamental Line: $\Xi(M) :=$

$$\mathrm{Det}_{\mathbb{Q}}^{-1}(H^1_f(M)) \otimes \mathrm{Det}_{\mathbb{Q}}^{-1}(H^0_f(M^*(1))^*) \otimes \mathrm{Det}_{\mathbb{Q}}(M_{dR}/M^0_{dR}) \otimes \mathrm{Det}_{\mathbb{Q}}^{-1}(M_B^+)$$

$$\cong \mathrm{Det}_{\mathbb{Q}}^{-1}((\mathcal{O}_F^\times)_{\mathbb{Q}}) \otimes \mathrm{Det}_{\mathbb{Q}}^{-1}(\mathbb{Q}) \otimes \mathrm{Det}_{\mathbb{Q}} F \otimes \mathrm{Det}_{\mathbb{Q}}^{-1}\Big(\bigoplus_{\substack{v|\infty \\ v \text{ complex}}} (2\pi i)(\tau_v - \bar{\tau}_v)\mathbb{Q}\Big)$$

$$0 \to M^+_{B,\mathbb{R}} \xrightarrow{\alpha_M} \bigoplus_{v|\infty} F_v \cong F_{\mathbb{R}} \to \mathrm{coker}(\alpha_M) \cong \bigoplus_{v|\infty} \mathbb{R} \to 0$$

$$0 \to (\mathcal{O}_F^\times)_{\mathbb{R}} \xrightarrow{\log|\ |_v} \mathrm{coker}(\alpha_M) \to \mathbb{R} \to 0$$

induces

$$\vartheta_\infty : \mathbb{R} \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

$$\vartheta_\infty\left(\frac{(2\pi)^{r_2} R}{\sqrt{|D|}}\right) = (\eta_1 \wedge \cdots \wedge \eta_{r_1+r_2-1})^{-1} \otimes (1^*)^{-1} \otimes (\omega_1 \wedge \cdots \wedge \omega_d) \otimes \bigwedge_v (\tau_v - \bar{\tau}_v)^{-1}$$

The Class number formula from the point of view of TNC, ctd

$$\eta_1, \cdots, \eta_{r_1+r_2-1} \quad \mathbb{Z}\text{-basis of } \mathcal{O}_F^\times/\mathrm{tor}$$
$$\omega_1, \cdots, \omega_d \quad \mathbb{Z}\text{-basis of } \mathcal{O}_F$$
$$1 \quad \mathbb{Z}\text{-basis of } H^0(\mathrm{Spec}\, F, \mathbb{Z})$$

$$R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(1)) \to R\Gamma_f(F, \mathbb{Z}_l(1)) \to R\Gamma_f(F_{\mathbb{Q}_l}, \mathbb{Z}_l(1)) \oplus R\Gamma_f(F_{\mathbb{R}}, \mathbb{Z}_l(1))$$

$$0 \to M_{B,\mathbb{Z}}^+ \otimes \mathbb{Z}_l \cong \bigoplus_{v|\infty} H^0(F_v, \mathbb{Z}_l(1)) \to H_c^1(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(1)) \to \mathcal{O}_F^\times \otimes \mathbb{Z}_l$$
$$\to \bigoplus_{v|l} \hat{\mathcal{O}}_{F_v}^\times \oplus \bigoplus_{v|\infty} H^1(F_v, \mathbb{Z}_l(1)) \to H_c^2(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(1)) \to \mathrm{Pic}(\mathcal{O}_F) \otimes \mathbb{Z}_l \to 0$$
$$H_c^2(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(1)) \cong \mathbb{Z}_l$$

$$H_f^1(F_v, \mathbb{Z}_l(1)) := \hat{\mathcal{O}}_{F_v}^\times \xrightarrow{\subset} F_v^\times \otimes \mathbb{Z}_l \cong H^1(F_v, \mathbb{Z}_l(1))$$

The Class number formula from the point of view of TNC, ctd

The exponential series $exp(x) = 1 + x + \frac{x^2}{2} + \cdots$ gives an isomorphism

$$\mathcal{O}_{F_v} \supset (l) = \mathfrak{m}_v^e \cong 1 + \mathfrak{m}_v^e \subset \mathcal{O}_{F_v}^\times$$

hence

$$\det_{\mathbb{Z}_l} \mathcal{O}_{F_v} = \frac{Nv - 1}{Nv} \det_{\mathbb{Z}_l} \hat{\mathcal{O}}_{F_v}^\times$$

inside

$$\det_{\mathbb{Q}_l} \hat{\mathcal{O}}_{F_v}^\times \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = \det_{\mathbb{Q}_l} H_f^1(F_v, \mathbb{Q}_l(1)).$$

But

$$\vartheta_l : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong \det_{\mathbb{Q}_l} R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l(1))$$

also involves a factor $1 - Nv^{-1}$ by definition. Hence

$$\mathbb{Z}_l \cdot \vartheta_l \vartheta_\infty^{-1}(\zeta_F(1)^*) = \mathrm{Det}_{\mathbb{Q}_l} R\Gamma_c(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(1))$$

Advantages of the Fontaine/Perrin-Riou formulation

- Generalizes to a description of the leading Taylor-coefficients $\zeta_F^*(s)$ at $s = 0, -1, -2, \ldots$
- Generalizes to Hasse-Weil L-functions $L(M, s)$ of motives M over number fields at any $s \in \mathbb{Z}$
- Generalizes to motives with coefficients in a semisimple $\mathbb{Q}$-algebra A.

Time line

Year	Mathematician	Theorem or Conjecture
1673	Leibniz	formula for $\frac{\pi}{4}$
1734	Euler	formula for $\zeta(2k)$
1838	Dirichlet	Analytic Class number formula for quadratic fields
1850	Kummer	Analytic Class number formula for cyclotomic fields
1859	Riemann	Analytic continuation and functional equation of $\zeta(s)$, Riemann hypothesis
1896	Dedekind	Analytic Class number formula for number fields
1917	Hecke	Analytic continuation and functional equation of $\zeta_F(s)$
1921	Artin	Zeta function for global fields of finite characteristic
1934	Hasse	Riemann hypothesis in char. p and $g = 1$
1949	Weil	Weil-Conjectures
1960	Dwork	Rationality of Zeta-Functions in char. p
1963	Ono	Tamagawa numbers of Tori
1965	Birch, Sw.-Dyer	Birch and Swinnerton-Dyer (BSD) conjecture for elliptic curves

Year	Mathematician	Theorem or Conjecture
1963	Grothendieck et al	Cohomological formula for Zeta-Functions in char. p
1966	Tate	BSD-Conjecture for abelian varieties, Tate Conjecture
1969	Serre	Definition of motivic L-Functions
1972	Quillen	Definition of algebraic K-groups
1974	Borel	Relation between $\zeta_F(m)$ and $K_{2m-1}(\mathcal{O}_F)$
1974	Deligne	Riemann hypothesis in char. p
1976	Coates, Wiles	rank-part of BSD for CM elliptic curves of rank 0
1978	Bloch	$K_2(E) \sim L(E,2)$ for certain elliptic curves
1979	Deligne	rationality conjecture for $L(M,0)$ for critical motives M
1983	Gross, Zagier	Gross-Zagier formula for $L'(E,1)$
1984	Mazur, Wiles	Iwasawa main conjecture for cyclotomic fields
1985	Beilinson	rationality conjecture for $L(M,s)$
1986	Rubin	first examples of finite Tate-Shafarevich groups
1988	Bloch, Kato	integrality conjecture for $L(M,s)$ for $w(M)-2s<0$
1988	Kolyvagin	Euler systems, rank-part of BSD for $\mathrm{ord}_{s=1}L(E,s) \leq 1$

Year	Mathematician	Theorem or Conjecture
1991	Fontaine, Perrin-Riou	integrality conjecture for $L(M, s)$ for all s, determinants
1999	Burns, Flach	equivariant conjecture with non-commutative coefficients, Galois module theory
2000	Kings	TNC for $L(E, s)$, E CM, $s = 2, 3, 4...$
2003	Huber, Kings	TNC for Dirichlet L-functions
2003	Burns, Greither	ETNC for Dirichlet L-functions
2005	Fukaya, Kato	General Iwasawa main conjecture
2014	Skinner, Urban, Kato	Iwasawa main conjecture for elliptic curves over $\mathbb{Q}$

Partly based on: Tejaswi Navilarekallu, On the Equivariant Tamagawa Number Conjecture, Thesis, Caltech 2006.