Automorphic forms and the Bloch–Kato conjecture

Mini Courses

Mini Course I: Anticyclotomic CM main conjecture

Part I

Speaker: Jacques Tilouine (University of Paris-Nord, France)

Title: A divisibility towards the Anticyclotomic Main Conjecture for CM fields.

Abstract: In a joint work with H. Hida, we proved in the 90's that the anticyclotomic Katz p-adic L function associated to a p-adic CM type divides the characteristic power series of the Iwasawa module associated to this p-adic CM type. The goal of these two talks is to sketch this proof. Note that we couldn't treat the divisibility at the prime p of the Iwasawa algebra. This has been treated in subsequent works by H. Hida.

Part II

Speaker: Haruzo Hida (University of California Los Angeles, USA)

Title: Lectures on the Anticyclotomic main conjecture.

Abstract : We first prove, for a prime $p > 3$ unramified in a CM quadratic extension of a totally real field F, $h(M/F)L(\chi) \mid H(\psi) \mid h(M/F)F(\chi)$ $(h(M/F) = h(M)/h(F))$ in Λ for the congruence power serie $H(\psi)$ of ψ lifting a fixed anti cyclotomic character χ and anticyclotomic Katz p-adic L-function $L(\chi)$ of branch character χ, built on the lectures by Tilouine proving this over $\Lambda[1/p]$. Here Λ is the many variable Iwasawa algebra of M. In the second lecture, we give a sketch of the proof of the reverse divisibility: $H(\psi) \mid h(M/F)L(\chi)$ resulting in the main conjecture, as $H(\psi) = h(M/F)F(\chi)$ for the anticyclotomic Iwasawa power series $F(\chi)$ by the "$R = T$"-theorem.

Mini Course II: Lectures on the local epsilon conjecture

Speaker: Kentaro Nakamura (Kyushu University, Japan)

Abstract: The local epsilon conjecture is one of a series of Kato's conjectures on a generalization of the Iwasawa main conjecture to general families of p-adic Galois representations. It gives a precise description of a p-adic variation of the p-adic Hodge theoretic invariants, like local (L-, and epsilon) factors, Bloch-Kato's cohomologies, and Hodge-Tate weights which are only defined for de Rham representations, in p-adic families of local p-adic Galois representations. In my lectures, I will explain the formulation of this conjecture, the proof of the conjecture for the rank two case using the p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$, and it's application to a generalization of Rubin's local sign decomposition conjecture.

Mini Course III: Special cycles on moduli spaces of unitary shtukas, and higher derivatives of L-functions

Speaker: Benjamin Howard (Boston College, USA)

Abstract: The arithmetic Siegel-Weil formula, conjectured by Kudla-Rapoport and proved by Li-Zhang, expresses the degrees of certain 0-cycles on integral models of unitary Shimura varieties in terms of the

nondegenerate Fourier coefficients of the central derivative of an Eisenstein series.

Feng-Yun-Zhang proved a higher derivative version of this arithmetic Siegel-Weil formula in the function field setting, now expressing degrees of 0-cycles on moduli spaces of unitary shtukas to the nondegenerate Fourier coefficients of higher central derivatives of an Eisenstein series.

The goal of my lecture series is (1) to explain all of this background, (2) extend the results of Feng-Yun-Zhang to include some degenerate coefficients, and (3) deduce from this extension an arithmetic application: the nonvanishing of higher central derivatives of certain Langlands L-functions implies the nonvanishing of classes in the Chow groups of moduli spaces of shtukas.

All of the new results are joint work with Tony Feng and Mikayel Mkrtchyan.

Mini Course IV: An introduction to the Gan–Gross–Prasad conjectures

Speaker: Dipendra Prasad (IIT Bombay, India)

Abstract: The speaker will try to give an introduction to the GGP conjectures, keeping in mind that he will be speaking to a very mixed audience some of whom may be seeing representation theory of groups over local fields for the first time. I will try not to presume much beyond a basic introduction to representation theory of finite groups over complex numbers, and familiarity with p-adic fields, and p-adic groups. There will be four lectures whose outline I give below.

Lecture 1: Branching laws illustrated with some finite dimensional examples, emphasizing the need of a parametrization, Gelfand pairs, strong Gelfand pairs. Automorphic representations and period integrals, Local-global principle, L-functions.

Lecture 2: Review of Classical groups in general, and their classification over local and global fields; their parabolics and Levi subgroups, Whittaker models, degenerate Whittaker models, Bessel and Fourier-Jacobi models, the last will need a bit of the Weil representations.

Lecture 3: A bit of representation theory of groups over local fields, parabolic induction, cuspidal representations. Review of the Local Langlands correspondence, L-functions and epsilon factors. L-packets, the Jacquet-Langlands correspondence, The GGP conjectures: both local and global conjectures.

Lecture 4: Spill-over from the last lecture, and finish with some low dimensional examples, including the fundamental work of Waldspurger; illustrative examples from finite fields.

References:

- D. W. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, 55, Cambridge Univ. Press, Cambridge, 1997; MR1431508
- C. J. Bushnell and G. M. Henniart, *The local Langlands conjecture for GL(2)*, Grundlehren der mathematischen Wissenschaften, 335, Springer, Berlin, 2006; MR2234120
- *Automorphic forms, representations and L-functions. Part 1*, Proceedings of Symposia in Pure Mathematics, XXXIII, American Mathematical Society, Providence, RI, 1979; MR0546586
- *Automorphic forms, representations, and L-functions. Part 2*, Proceedings of Symposia in Pure Mathematics, XXXIII, American Mathematical Society, Providence, RI, 1979; MR0546606
- W. T. Gan, B. H. Gross and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups, Astérisque No. 346 (2012), 1–109; MR3202556
- W. T. Gan, B. H. Gross and D. Prasad, Restrictions of representations of classical groups: examples, Astérisque No. 346 (2012), 111–170; MR3202557

Mini Course V: Degenerate automorphic forms and Euler systems

Speakers: Chris Skinner (Princeton University, USA) & Marco Sangiovanni Vincentelli (Columbia University, USA)

Abstract: Euler Systems have proven to be versatile tools for understanding Selmer groups and their connections to special values of L-functions. However, despite the key role they have played in making progress toward foundational conjectures in number theory like the Birch–Swinnerton-Dyer and Bloch–Kato Conjectures, only a handful of provably non-trivial Euler systems have been constructed to date. A significant obstacle to constructing Euler Systems lies in producing candidate Galois cohomology classes. This lecture series presents a method to overcome this obstacle that does not rely on rare (known) motivic classes. We will focus on building étale cohomology classes originating from automorphic data: Eisenstein series and Theta series. This framework not only retrieves most classical Euler systems but can also be applied to construct an Euler system for the adjoint of an elliptic modular form.

References:

- C. Skinner, L-values and nonsplit extensions: a simple case, `https://msp.org/ent/2024/3-1/p03.xhtml`
- H. Darmon etal, p-adic L-functions and Euler systems: a tale in two trilogies.