The existence problem of compact quotients of pseudo-Riemannian homogeneous spaces.

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Zariski dense subgroups, number theory, and geometric applications.

ICTS (Bengaluru, India), 1 January 2024.

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§ 5 A few words on the proof.

G/H : a homogeneous space, $\Gamma \subset G$: a discrete subgp

If $\Gamma \curvearrowright G/H$ is proper and free,

$\Gamma \backslash G/H$ admits a canonical manifold structure that makes

$$\pi : G/H \longrightarrow \Gamma \backslash G/H \quad \text{a covering map.}$$

Def^n : We call such $\Gamma \backslash G/H$

a Clifford-Klein form of G/H.

(CK form)

Rmk : proper, not free $\rightsquigarrow$ orbifolds

not proper $\rightsquigarrow$ non-Hausdorff, pathological

A G- invariant geometric structure on G/H

$\rightsquigarrow$ the corresponding geometric structure on $\Gamma \backslash G/H$

Conversely, one can sometimes prove that

a manifold equipped with a 'very nice' geometric structure

must be a Clifford-Klein form.

Example : spaces of constant negative curvatures

- **Def^n**: $\mathcal{H}^{p,q} := \left\{ [x_1 : \cdots : x_{p+q}] \in \mathbb{RP}^{p+q} \;\middle|\; \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_{p+j}^2 < 0 \right\}$

  (= negative-definite lines in $\mathbb{R}^{p,q+1}$).

  is called the pseudo-Riemannian hyperbolic space.

* **Rmk** : $\mathcal{H}^{p,0} = \mathcal{H}^p$, $\mathcal{H}^{0,q} = \mathbb{RP}^q$.

$$\mathrm{Isom}(\mathcal{H}^{p,q}) = PO(p,q+1) \left(:= O(p,q+1)/\{\pm 1\} \right),$$

$$PO(p,q+1)/P[O(p,q) \times O(1)] \xrightarrow{\cong} \mathcal{H}^{p,q} \quad \text{diffeo}.$$

Let $\widetilde{\mathcal{H}}^{p,q}$ be the universal cover of $\mathcal{H}^{p,q}$.

$\left(\widetilde{\mathcal{H}}^{p,q} = O(p, q+1) / O(p, q) \quad \text{for } q \geq 2 \right)$.

Fact : (Clifford-Klein forms of $\widetilde{\mathcal{H}}^{p,q}$)

$\updownarrow$ 1 : 1

(Negative space-forms of signature (p, q))

$\parallel$

(Complete pseudo-Riem. mfds with sect. curv. $\equiv -1$)

$\updownarrow$ 1 : 1

(Positive space-forms of signature (q, p))

In this talk, we consider the following problem:

Prob.: Given a homogeneous space G/H, determine if it admits $\underbrace{\text{a compact CK form}}$.

(often simply called a compact quotient)

A special case ($G/H = \widetilde{\mathcal{H}}^{p,q}$):

Determine all $(p,q) \in \mathbb{N}^2$ such that $\exists$ a compact complete pseudo-Riem. mfd of signature (p,q) with sectional curvature $\equiv -1$.

Rmk: If G is linear, by Selberg's lemma,

$\exists$ a compact CK form of G/H

$\Leftrightarrow$ $\exists$ $\Gamma \subset G$: torsion-free discrete subgp s.t.

$\Gamma \curvearrowright G/H$ is proper and cocompact.

The problem is insensitive to

passing to linear finite covers of G,

adding compact factors to H, etc.

We list major approaches for this problem :

- <u>Structure theory of reductive Lie groups</u>

  (properness criterion in terms of the KAK - decomp.)

  Kobayashi '89. '92, <u>Benoist '96</u>. M. '21 +

  (Kobayashi's talk)

  (Systematic search for examples :

  Okuda '13, Bocheński - Tralle '15, ...)

- <u>Homogeneous dynamics</u> (Zimmer's cocycle superrigidity)

  Zimmer '94. Labourie - Mozes - Zimmer '95.

  Labourie - Zimmer '95

- Infinite-dimensional unitary representation theory

  (decay of matrix coefficients)

  Margulis '97, Shalom '00. (Kobayashi's talk)

  (examples: Oh '98, Benoist-Kobayashi)

- de Rham and relative Lie algebra cohomologies

  (Hirzebruch's proportionality principle, Sullivan model)

  Kulkarni '81, Kobayashi-Ono '90,

  Benoist-Labourie '92, M '15, '17, Tholozan '15 +

  (examples: M. '19)

- Geometric group theory

  (Anosov representations, solution to the Sharpness Conj.,)

  Kassel-Tholozan (in preparation).

A nice thing about the existence problem of compact quotients is that different methods often give partially overlapping but different results!

We may see this problem as one of the crossroads in mathematics.

Ex. : Known results for $G/H = \mathbb{H}^{p,q}$ (or $\widetilde{\mathbb{H}}^{p,q}$)

- $\exists$ a compact quotient if :

$$(p,q) = \underbrace{(0,n)}_{\text{(trivial)}},\ \underbrace{(n,0)}_{\text{(hyperbolic manifolds)}},\ \underbrace{(2n,1),\ (4n,3)}_{\text{(Kulkarni '81)}},\ \underbrace{(8,7)}_{\text{(Kobayashi '94)}}.$$

- $\nexists$ a compact quotient if :
  - $q \geq p \geq 1$ (Calabi - Markus '62, Wolf '62, Kobayashi '89)
  - $(p,q) = (2n+1, 2n)$ $(n \geq 1)$ (Benoist '96)
  - p, q : odd (Kulkarni '81)
  - p : odd, $q \geq 1$ (Tholozan '15+, M. '17)

⋮

q \ p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	…
10	∃	∄	∄	∄	∄	∄	∄	∄	∄	∄	∄	∄		∄		∄		
9	∃	∄	∄	∄	∄	∄	∄	∄	∄	∄		∄		∄		∄		
8	∃	∄	∄	∄	∄	∄	∄	∄	∄	∄		∄		∄		∄		
7	∃	∄	∄	∄	∄	∄	∄	∄	∃	∄		∄		∄		∄		
6	∃	∄	∄	∄	∄	∄	∄	∄		∄		∄		∄		∄		
5	∃	∄	∄	∄	∄	∄		∄		∄		∄		∄		∄		
4	∃	∄	∄	∄	∄	∄		∄		∄		∄		∄		∄		
3	∃	∄	∄	∄	∃	∄		∄	∃	∄		∄	∃	∄		∄	∃	
2	∃	∄	∄	∄		∄		∄		∄		∄		∄		∄		
1	∃	∄	∃	∄	∃	∄	∃	∄	∃	∄	∃	∄	∃	∄	∃	∄	∃	
0	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	∃	

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Today, I give a new necessary condition for the existence of compact CK forms, which gives new results for $H^{p,q}$ and many other homogeneous spaces.

It is formulated in terms of algebraic topology of sphere bundles.

Before stating our new necessary condition, let me explain the Geometric Fibration Conjecture.

In the mid-'10s, Tholozan proposed a conjectural picture of compact CK forms, which he calls the Geometric Fibration Conjecture.

(GF Conj.)

(first formulated in '15 +,
discussed in detail in his Habilitation Thesis '21)

Our main theorem is inspired by this conjecture.

Setting : G : a linear reductive Lie group
$\cup$
H : reductive in G.

$\Gamma \curvearrowright G/H$: proper, cocpt. Γ : torsion-free.

Take a max. compact subgp K of G
so that $K \cap H$ is a max. compact subgp of H.

$\rightsquigarrow$ We obtain the following double fibration :

$$\begin{tikzcd}
 & \Gamma \backslash G/K\cap H \arrow[dl, "\pi_\Gamma"', "\substack{\text{fibre } H/K\cap H \\ (\text{contractible})}"] \arrow[dr, "p_\Gamma", "\substack{\text{fibre } K/K\cap H \\ (\text{compact})}"'] & \\
\Gamma \backslash G/H & & \Gamma \backslash G/K
\end{tikzcd}$$

$\Gamma\backslash G/H$: a compact CK form with Γ torsion-free

$\rightsquigarrow$
$$\begin{tikzcd} & \Gamma\backslash G/K\cap H \arrow[dl, "\pi_\Gamma"'] \arrow[dr, "p_\Gamma"] & \\ \Gamma\backslash G/H & & \Gamma\backslash G/K \end{tikzcd}$$

fibre $H/K\cap H$ (contractible)

fibre $K/K\cap H$ (compact)

Geometric Fibration Conjecture (Tholozan, around '15)

In this situation, $\exists\, M_\Gamma \subset \Gamma\backslash G/K$: a submfd

s.t. π_Γ induces a diffeo. $p_\Gamma^{-1}(M_\Gamma) \xrightarrow[\cong]{\pi_\Gamma|_{p_\Gamma^{-1}(M_\Gamma)}} \Gamma\backslash G/H$.

(In other words, $\exists\, \sigma_\Gamma : \Gamma\backslash G/H \longrightarrow \Gamma\backslash G/K\cap H$ a section of π_Γ whose image is a $K/K\cap H$-foliated submanifold of $\Gamma\backslash G/K\cap H$)

Rmk : The Geometric Fibration Conjecture asserts in particular that

$\exists$ a smooth fibre bundle $\Gamma\backslash G/H \xrightarrow{K/K_H} M$

with total space $\Gamma\backslash G/H$ and typical fibre K/K_H.

<u>Ex</u> : $G/H = \mathcal{H}^{n,1} = PO(n,2) / P(O(n,1) \times O(1))$

(... the <u>anti-de Sitter space</u> <u>AdS^{n+1}</u>)

$K = P(O(n) \times O(2))$, $K \cap H = P(O(n) \times O(1) \times O(1))$.

(negative-definite direction)

$\rightsquigarrow$

$$
\begin{array}{ccccc}
 & & \left(\begin{array}{c}\text{based timelike geodesics} \\ \text{in } \Gamma \backslash AdS^{n+1}\end{array}\right) & & \\
 & \swarrow^{\pi_\Gamma} & & \searrow^{p_\Gamma} & \\
\Gamma \backslash AdS^{n+1} & & & & \left(\begin{array}{c}\text{timelike geodesics} \\ \text{in } \Gamma \backslash AdS^{n+1}\end{array}\right)
\end{array}
$$

fibre $\mathcal{H}^n$ fibre $\mathbb{RP}^1$

The GF Conj. says that $\Gamma \backslash AdS^{n+1}$ should admit a foliation by timelike geodesics.

Obviously, one can rephrase GF Conj. as follows:

GF Conj., rephrased:

$\Gamma \backslash G/H$: a compact CK form with Γ torsion-free

$$
\begin{tikzcd}
 & G/K\cap H \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\
G/H & & G/K
\end{tikzcd}
$$

fibre $H/K\cap H$ (contractible)

fibre $K/K\cap H$ (compact)

Then, $\exists\, M \subset G/K$: a Γ-invariant submfd

s.t. π induces a diffeo. $\rho^{-1}(M) \xrightarrow[\cong]{\pi|_{\rho^{-1}(M)}} G/H$.

Supporting evidences :

[(1) : GF Conj. is true for 'standard' compact CK forms]

($\begin{cases} \cdot\ L \subset G : \text{a connected closed subgp.} \\ \qquad\qquad L \curvearrowright G/H : \text{proper, transitive} \\ \cdot\ \Gamma \subset L : \text{a torsion-free discrete subgp.} \end{cases}$

$\rightsquigarrow$ $M := L/K \cap L$ has the required property)

(2) : Γ_0 : an (abstract) torsion-free discrete group,

$$\mathcal{M} := \left\{ \begin{array}{l} j : \Gamma_0 \hookrightarrow G \quad \text{a discrete embedding s.t.} \\ \qquad j(\Gamma_0) \backslash G/H \quad \text{is a compact CK form} \end{array} \right\} \Big/ \text{conjugate}$$

$$\Rightarrow \left\{ [j] \in \mathcal{M} \;\middle|\; \text{GF Conj. is true for } j(\Gamma_0) \backslash G/H \right\}$$

is open in $\mathcal{M}$.

In particular, GF Conj. is true for

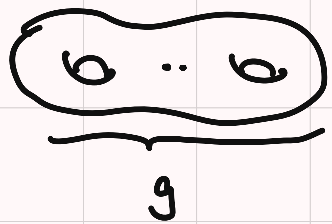
a small deformation of standard compact CK forms.

$\underline{(3)}$: GF Conj. is true for the compact CK forms of

$$G/H = (PO(n.1) \times PO(n.1)) \Big/ \operatorname{diag} PO(n.1)$$

(Guéritaud - Kassel '17)

($n = 2 \rightsquigarrow$ anti - de Sitter 3 - manifolds)

<u>Rmk</u> : Let $n = 2$ and $\Gamma_0 := \pi_1\Big($  $\Big)$.

$$\mathcal{M} := \left\{ \begin{array}{l} j : \Gamma_0 \hookrightarrow G \quad \text{a discrete embedding s.t.} \\ j(\Gamma_0) \backslash G/H \quad \text{is a compact CK form} \end{array} \right\} \Big/ \text{conj}.$$

is <u>not connected</u> (Salein '00).

But (3) says that GF Conj. is true for <u>every</u> point of $\mathcal{M}$.

(4) : GF Conj. is true for the compact CK forms of

$$G/H = O(2n,2)/U(n,1).$$

(follows from Monclair - Schlenker - Tholozan. '23+ and Kassel - Tholozan)

*

Rmk : Monclair - Schlenker - Tholozan '23+ proved that, for $n \geq 2$, $\exists$ a compact Clifford-Klein form $\Gamma \backslash G/H$ s.t. Γ is not isomorphic to any cocompact lattice of any reductive Lie group.

(Lee - Marquis '19 : $n = 2, 3, 4$)

But (4) says that GF Conj. is true even for such Γ.

*

$\underline{N}$: the normal bundle of $K/K\cap H$ in G/H.

Prop. : Assume the GF Conj. is true.

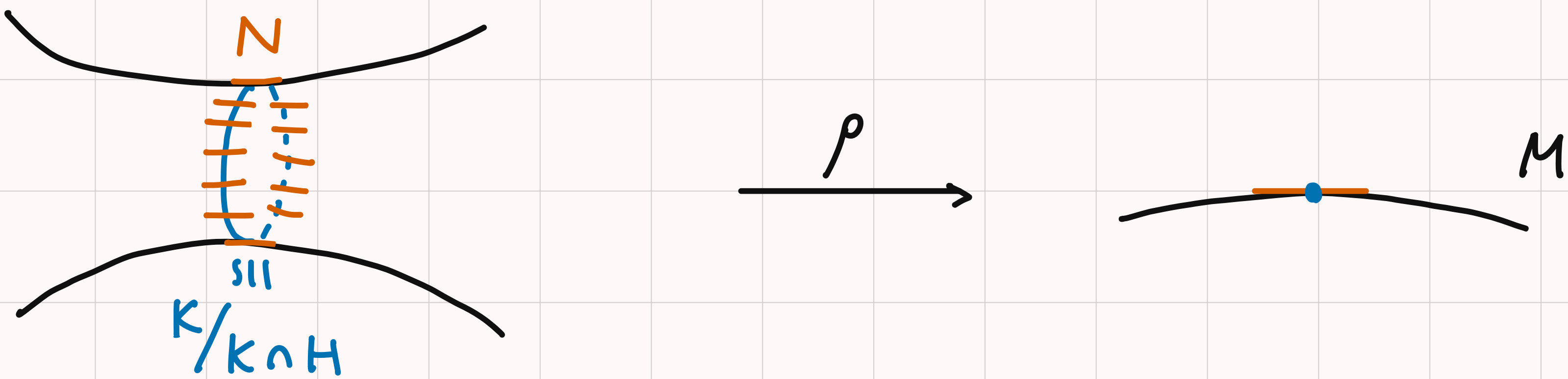
Then, $\boxed{G/H \text{ admits a compact CK form } \underline{\text{only when}} \ N \text{ is a trivial vector bundle.}}$

(Rmk : conjectured by Kobayashi – Yoshino in mid-'00s)

Sketch of proof: If GF Conj. were true, we have

$$\begin{tikzcd}
& G/K\cap H \arrow[dl, "\pi"'] \arrow[dr, "\rho"] \arrow[d, phantom, "\cup"] & \\
G/H \arrow[d, equal] & \rho^{-1}(M) \arrow[dl, "\cong"', "\text{diffeo}"] \arrow[dr, "K/K\cap H\text{ - fibre bundle}"'] & G/K \arrow[d, phantom, "\cup"] \\
G/H & & \exists M
\end{tikzcd}$$

$G/H = \rho^{-1}(M)$



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We proved a somewhat weaker result without assuming the QF Conj. :

Main Theorem (Kassel – M – Tholozan, in preparation):

G, H, K, N : as before,

$S(N)$: the unit sphere bundle for N.

Then, G/H admits a compact CK form only when $S(N)$ has the same fibrewise-homotopy type as the trivial sphere bundle on $K/K \cap H$

(next slide)

B : a topological space

$\pi : X \longrightarrow B$, $\pi' : X' \longrightarrow B$ cont. maps

Def^n : (1) $f_0, f_1 : X \rightrightarrows X'$ cont. maps over B

(compatible with π, π')

$f_0 \sim_B f_1 \overset{\text{def.}}{\Longleftrightarrow} {}^\exists F : X \times [0, 1] \longrightarrow X'$ cont. over B

(fibrewise homotopic)

s.t. $F|_{X \times \{0\}} = f_0$, $F|_{X \times \{1\}} = f_1$.

(2) X has the same fibrewise-homotopy type as X'

$\overset{\text{def.}}{\Longleftrightarrow} {}^\exists f : X \to X'$, ${}^\exists f' : X' \longrightarrow X$ cont. over B

s.t. $f' \circ f \sim_B \mathrm{id}_X$, $f' \circ f \sim_B \mathrm{id}_{X'}$.

Rmk : $\boxed{N : \text{trivial as a vector bundle}}$

$$\Updownarrow$$

$\boxed{S(N) : C^1\text{-trivial as a fibre bundle}}$

$$\Downarrow \quad \leftarrow \left(\text{I do not know if } \Uparrow \text{ holds} \right)$$

$\boxed{S(N) : C^0\text{-trivial as a fibre bundle}}$

$$\Downarrow \quad \not\Uparrow \quad \leftarrow \left(\underline{\text{Counter example}} : PU(24.2) / P(U(24.1) \times U(1)) \right)$$

$\boxed{S(N) \text{ has the same fibrewise-homotopy type as the trivial sphere bundle}}$

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① Constant sectional curvature : (cf. Kobayashi-Yoshino '05 on the tangential version.)

$$G/H = \mathcal{H}^{p,q} \quad \left(:= PO(p,q+1) / P(O(p,q) \times O(1)) \right)$$

$$\rightsquigarrow \quad K/K \cap H = \mathbb{R}P^q, \quad N = (\text{tautological line bundle})^{\oplus p}$$

Fact (Adams '62, 'Vector fields on spheres'):

In this situation, $S(N)$ has the same fibrewise-homotopy type as the trivial sphere bundle

if and only if p is divisible by $2^{h(q)}$.

$$h(q) := \begin{cases} \lfloor \frac{q}{2} \rfloor & (q \equiv 0, 6, 7 \mod 8) \\ \lfloor \frac{q}{2} \rfloor + 1 & (q \equiv 1, 2, 3, 4, 5 \mod 8) \end{cases}$$

Thus, we obtain :

[**Th** (Kassel - M - Tholozan) :

$H^{p,q}$ admits a compact CK form only when

p is divisible by $2^{h(q)}$.]

$\updownarrow$

p	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	$8\mathbb{N}$	$16\mathbb{N}$	$32\mathbb{N}$	$64\mathbb{N}$	$128\mathbb{N}$	$\cdots$
q	0	1	$2, 3$	$4, 5, 6, 7$	8	9	$10, 11$	$12, 13, 14, 15$	$\cdots$

i.e. a compact complete pseudo-Riem. manifold

of sectional curvature $\equiv -1$ can exist only when

the signature (p, q) belongs to the above table.

$q \backslash p$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
$\vdots$																		
10	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		
9	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		
8	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		
7	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		
6	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		
5	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		
4	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		
3	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$		$\nexists$	$\exists$	$\nexists$		$\nexists$	$\exists$	$\nexists$		$\nexists$	$\exists$	
2	$\exists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		$\nexists$		
1	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	
0	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	

⋮

q \ p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
10	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
9	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
8	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	
7	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	
6	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	
5	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	
4	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	
3	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\exists$	$\nexists$	$\nexists$	$\nexists$	$\exists$
2	$\exists$	$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$		$\nexists$	$\nexists$	$\nexists$	
1	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\exists$
0	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$	$\exists$

...

② $SL(n, \mathbb{K}) / SL(m, \mathbb{K})$:

[**Thm** (Kassel - M - Tholozan and Kassel - Tholozan) :

$SL(n, \mathbb{K}) / SL(m, \mathbb{K})$ $(n > m \geq 2, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

does not admit a compact CK form.]

*

Rmk : Today's main theorem : all cases but

$SL(4, \mathbb{R}) / SL(3, \mathbb{R})$ and $SL(8, \mathbb{R}) / SL(7, \mathbb{R})$.

Kassel - Tholozan : $n = m + 1$

*

③ Simple symmetric spaces :

Let G/H be a simple symmetric space, i.e.

G : a simple Lie group

$\cup$

H : the fixed-point subgp by an involution on G

(**Assume** : H : noncompact. $G \neq H$)

Fact (Kulkarni '81. Kobayashi '90s):

G/H admits a compact CK form when

it is locally isomorphic to one of the following:

- $SO(2n,2)/S(O(2n,1)\times O(1))$ $H^{2n,1}$
- $SO(2n,2)/U(n,1)$.
- $SO(4n,4)/S(O(4n,3)\times O(1))$ $H^{4n,3}$
- $SU(2n,2)/Sp(n,1)$
- $SU(2n,2)/S(U(2n,1)\times U(1))$
- $SO(8,8)/S(O(8,7)\times O(1))$ $H^{8,7}$
- $SO(4,3)/S(O(4,1)\times O(2))$
- $SO(4,4)/S(O(4,1)\times O(3))$
- $SO(8,\mathbb{C})/S(O(7,\mathbb{C})\times O(1,\mathbb{C}))$
- $SO(8,\mathbb{C})/SO(7,1)$

$\underline{\text{Th}^m}$ (Kassel - M - Tholozan and Kobayashi '92, Benoist '96, Tholozan '15+, M. '17, ...) :

G/H : a simple sym. space, not belonging to the previous list.

Then, G/H does not admit a compact CK form

except possibly for the following four cases :

(i) $SO(p, q+q') / S(O(p, q) \times O(q'))$ (p is divisible by $\underline{j_{\mathbb{R}}}(q, q')$)

(ii) $SU(p, q+q') / S(U(p, q) \times U(q'))$ (〃 $\underline{j_{\mathbb{C}}}(q, q')$)

(iii) $Sp(p, q+q') / (Sp(p, q) \times Sp(q'))$ (〃 $\underline{j_{\mathbb{H}}}(q, q')$)

(iv) $SU(n, n) / SO^*(2n)$ (n : odd)

(definitions omitted.

$j_{\mathbb{R}}(q, 1) = 2^{h(q)}$)

Proved by (hard!) case-by-case algebraic topology computations.

(KO-theory, the Adams conjecture on the J-group, etc.)

*

<u>Rmk</u> : If we could prove the C^0-triviality of N, the cases (ii), (iii), and [(i) with ~~$q' \geq 2$~~ $q, q' \geq 2$] would be excluded.

If, furthermore, we could prove the GF Conj., the case (iv) would be excluded.

[(i) with $\underline{q = 1 \text{ or } q' = 1}$] seems to be the hardest case.

(i.e. $G/H = H^{p,q}$, p is divisible by $2^{h(q)}$)

*

* **Rmk** : Although our new necessary condition is quite powerful in many cases, it does not cover some 'easy' cases.

For instance, our new method does not cover the case of $G/H = GL(4, \mathbb{R}) / (GL(3, \mathbb{R}) \times GL(1, \mathbb{R}))$,

whereas many previous methods cover it.

(e.g. Kobayashi '89. Kobayashi-Ono '90. Benoist-Labourie '92)
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Setting :

- G : a linear reductive Lie group
  $\cup$
  H : reductive in G.
- $K \subset G$: max. compact, $K \cap H$ is max compact in H.
- N : the normal bundle of $K/K\cap H$ in G/H.
- $S(N)$: the unit sphere bundle for N.

Main Theorem (Kassel - M - Tholozan) :
G/H admits a compact CK form only when $S(N)$ has the same fibrewise-homotopy type as the trivial sphere bundle on $K/K \cap H$.

Sketch of proof of main theorem :

① : $N = K \times_{K\cap H} (\mathfrak{p} \cap \mathfrak{q})$. $\quad \left(\begin{array}{l} \mathfrak{p} := \mathfrak{k}^{\perp} \\ \mathfrak{q} := \mathfrak{h}^{\perp} \end{array} \text{ in } \mathfrak{g}. \right)$

$$\begin{array}{ccc} K \times_{K\cap H} (\mathfrak{p} \cap \mathfrak{q}) & \xrightarrow{\cong} & G \quad \text{diffeo.} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [k, X] & \longmapsto & k \cdot \exp(X) \end{array}$$

(Kobayashi '89).

② :

$$\begin{tikzcd} & G/K\cap H \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ G/H \arrow[ur, bend right, "\sigma"'] & & G/K \end{tikzcd}$$

$\exists \sigma$: a Γ-equivariant section

($\because$ the fibre $H/K\cap H$ is contractible)

③ :

$$\begin{tikzcd}
 & & G \arrow[d] \\
\mathfrak{p} \cap \mathfrak{q} \arrow[urr, "\exists\, \tilde{\sigma}"] \arrow[r, hook] & G/H \arrow[r, hook, "\sigma"] & G/(K \cap H) \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$$X \longmapsto \exp(X) \cdot H$$

(☺ $\mathfrak{p} \cap \mathfrak{q}$ is contractible)

④ : Put $\Phi : K/(K \cap H) \times (\mathfrak{p} \cap \mathfrak{q}) \longrightarrow G/H \cong N$.

$$(x, X) \longmapsto \tilde{\sigma}(X) \cdot x$$

(⚠ Φ is incompatible with the projections onto $K/(K \cap H)$)

⑤ : Construct a fibrewise-homotopy equivalence from Φ.

(use $\Gamma \curvearrowright G/H$: cpt. σ : Γ-equivariant) $\blacksquare$

Thank you !!

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§ 6 Additional remarks

Rmk : G : semisimple. $P \subset G$: parabolic

⇝ the concept of P-Anosov subgroups of G.

(Labourie '06, Guichard - Wienhard '12, ...)

Fact : $\Gamma \subset PO(p, q+1)$: a discrete subgp. $p \geq q+2$.

Then, ($\Gamma \backslash \mathbb{H}^{p,q}$ is a compact CK form)

$$\Updownarrow$$

(Γ is P_q-Anosov and $\mathrm{cd}_{\mathbb{R}}(\Gamma) = p$).

($\Uparrow$: Guéritaud - Guichard - Kassel - Wienhard '17

$\Downarrow$: Kassel - Tholozan.)

Rmk : The simplest unsolved case $\cdots$ $\mathcal{H}^{4.2}$ (or $\tilde{\mathcal{H}}^{4.2}$)

In this case there is a special 'subgeometry' :

$$\underbrace{G_{2(2)}/SU(2.1)}\xrightarrow[\text{diffeo.}]{\cong} O(4.3)/O(4.2) = \tilde{\mathcal{H}}^{4.2}$$

($\exists$ almost complex str, similarly to $S^6 = G_{2.\text{cpt}}/SU(3)$)

We even do not know if $G_{2(2)}/SU(2.1)$ admits a compact CK form.

(Very naive hope : $\mathcal{H}^{4.2}$ (or even $G_{2(2)}/SU(2.1)$) actually admits a compact CK form ???)

Thank you !!