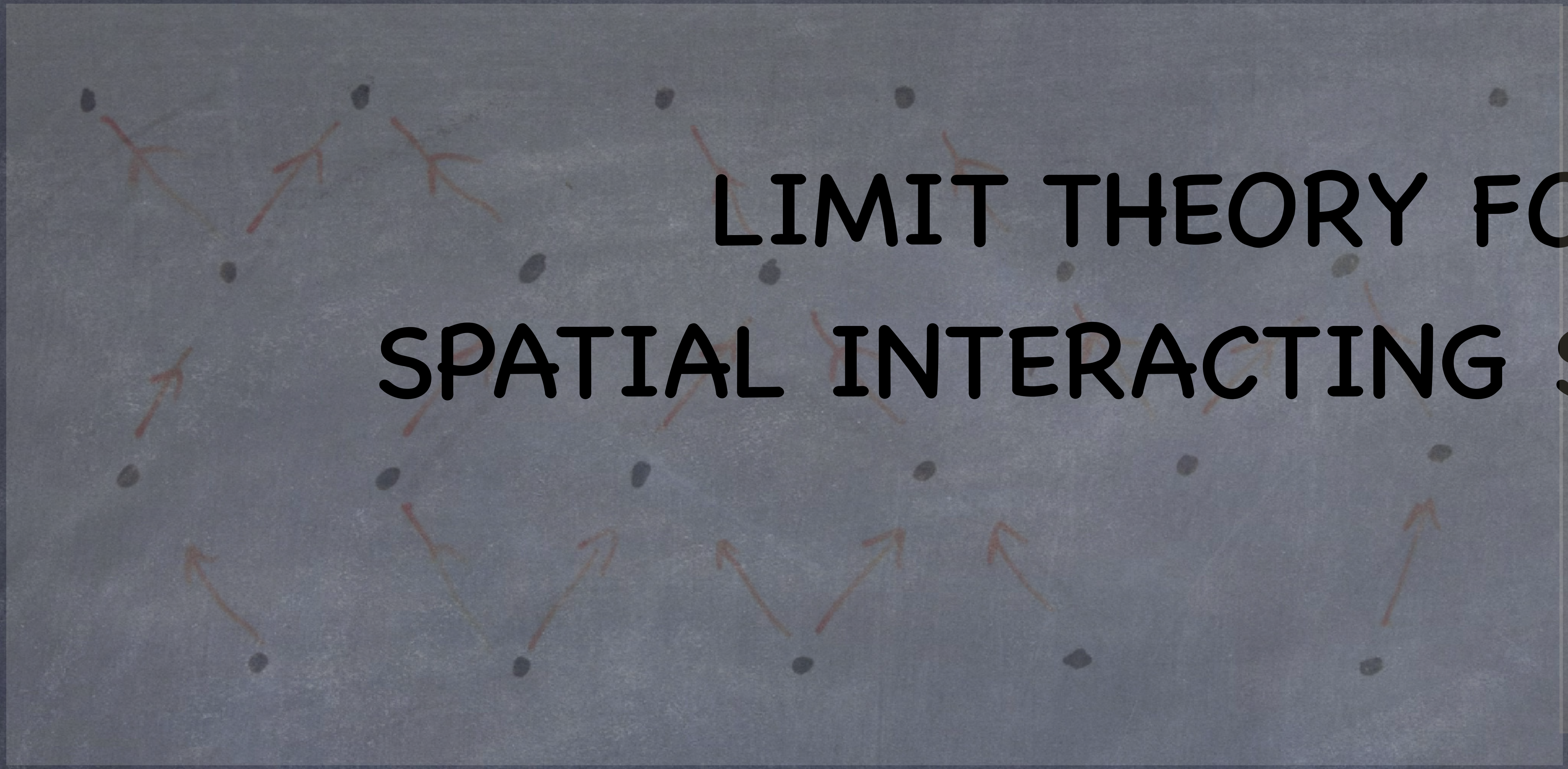

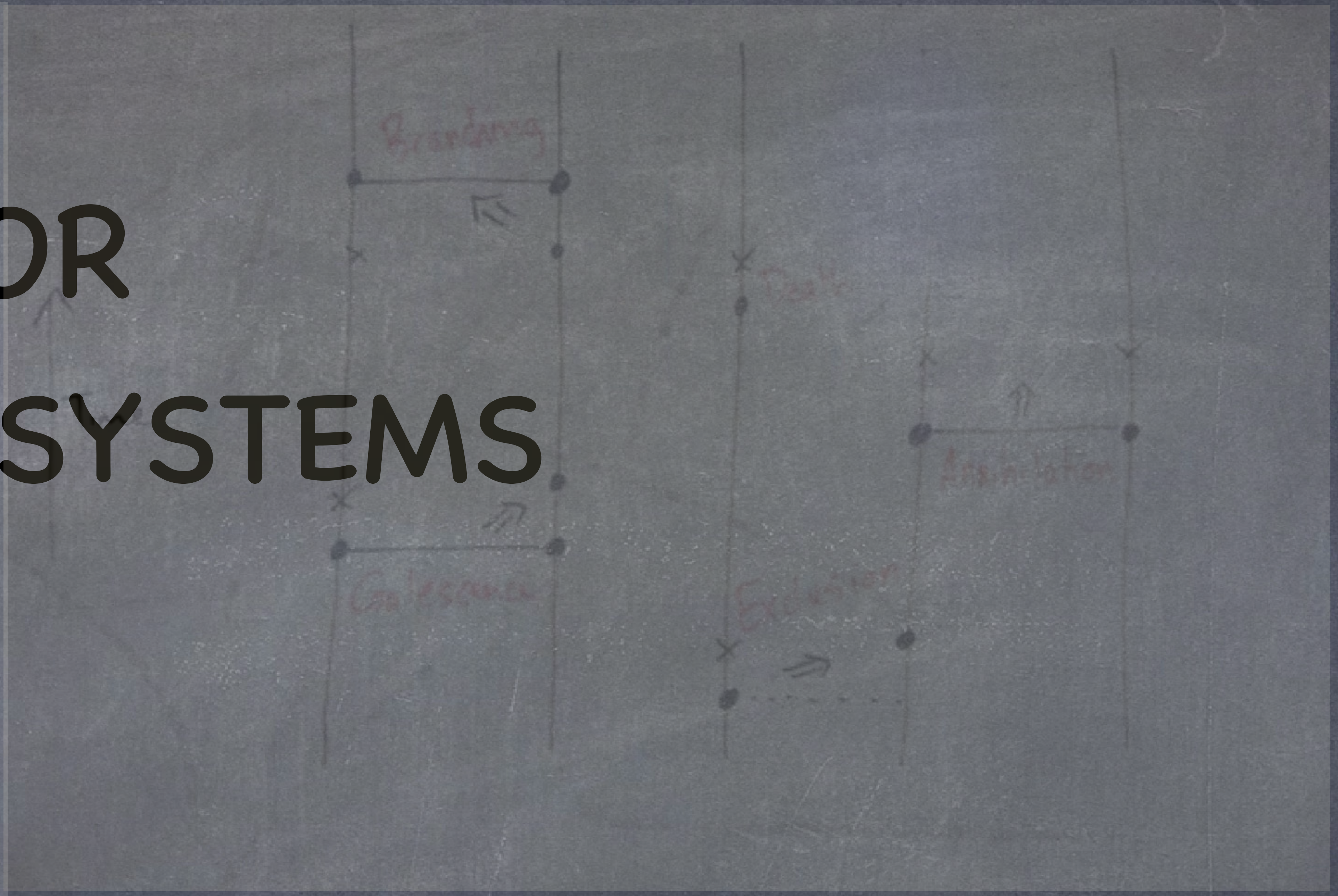

LIMIT THEORY FOR SPATIAL INTERACTING SYSTEMS

D. Yogeshwaran, ISI Bangalore.

ICTS, Bangalore. Jan 2023.

Slides at https://www.isibang.ac.in/~d.yogesh/IPSJan23.pdf

on-going work with

Bartek Błaszczyszyn

Ecole Normale Superieure-INRIA, Paris.

Joe Yukich.

Lehigh University, Bethlehem.

OUTLINE

1. Framework and Question.
2. Examples.
   - Co-operative Sequential Adsorption (CSA)
   - Exclusion process.
   - Other spatial random models.
3. Overview of General limit theorems with some proof ideas.
4. Bounded Lipschitz stabilization / localization.

FRAMEWORK AND QUESTION

Some basic point process notions

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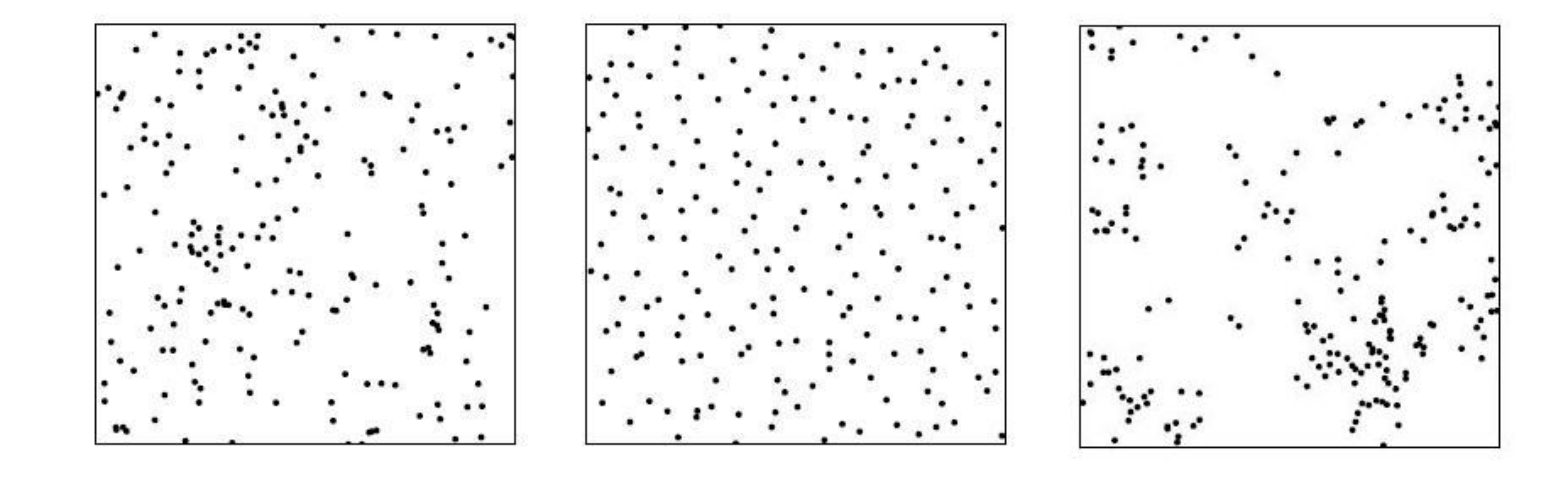

Poisson process — Ginibre process — Cox process

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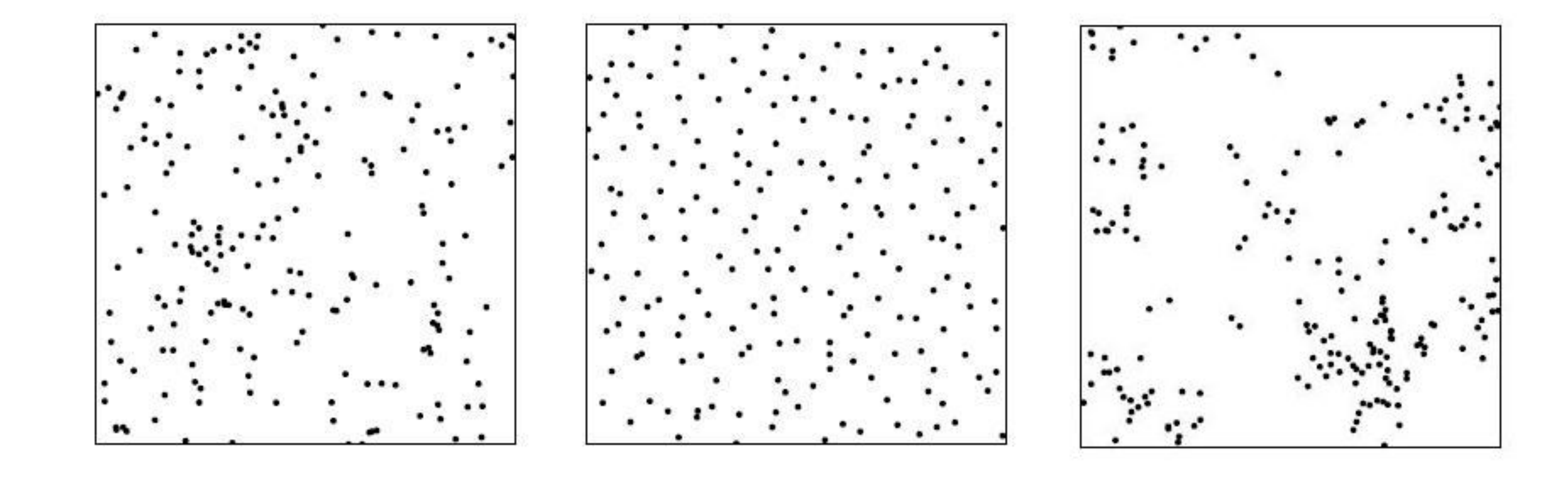

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- **Simple marked point process** - $\tilde{\mathscr{P}} = \{(X_i, M_i)\} \subset \mathbb{R}^d \times \mathbb{M}$ such that $\mathscr{P} = \{X_i\}$ is a point process,

  Equivalently, $\tilde{\mathscr{P}} = \sum_{x \in \mathscr{P}} \delta_{(x, M(x))}$ i.e., each point of $\mathscr{P}$ is marked / labelled with $M(x) \in \mathbb{M}$, Polish space.

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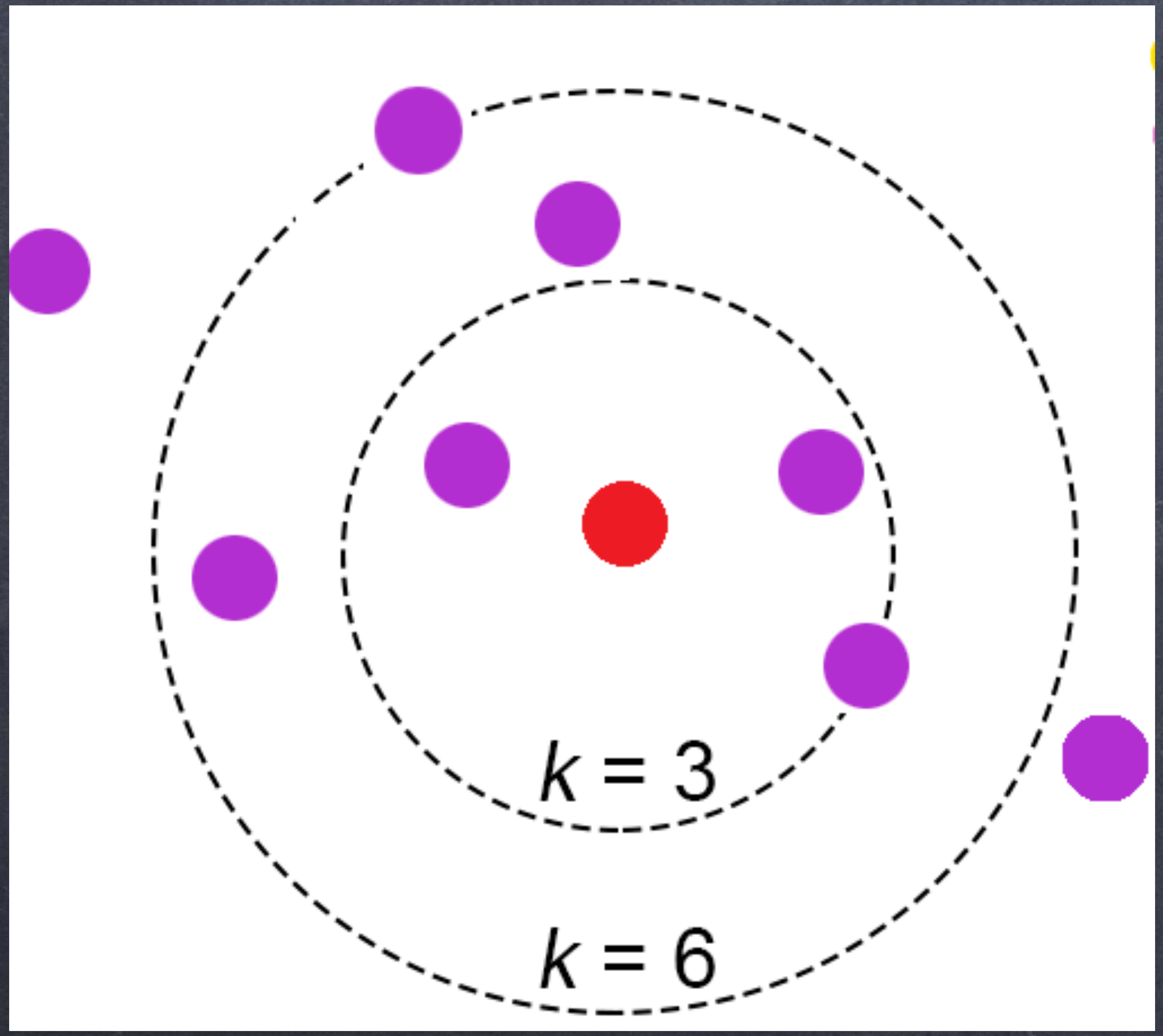

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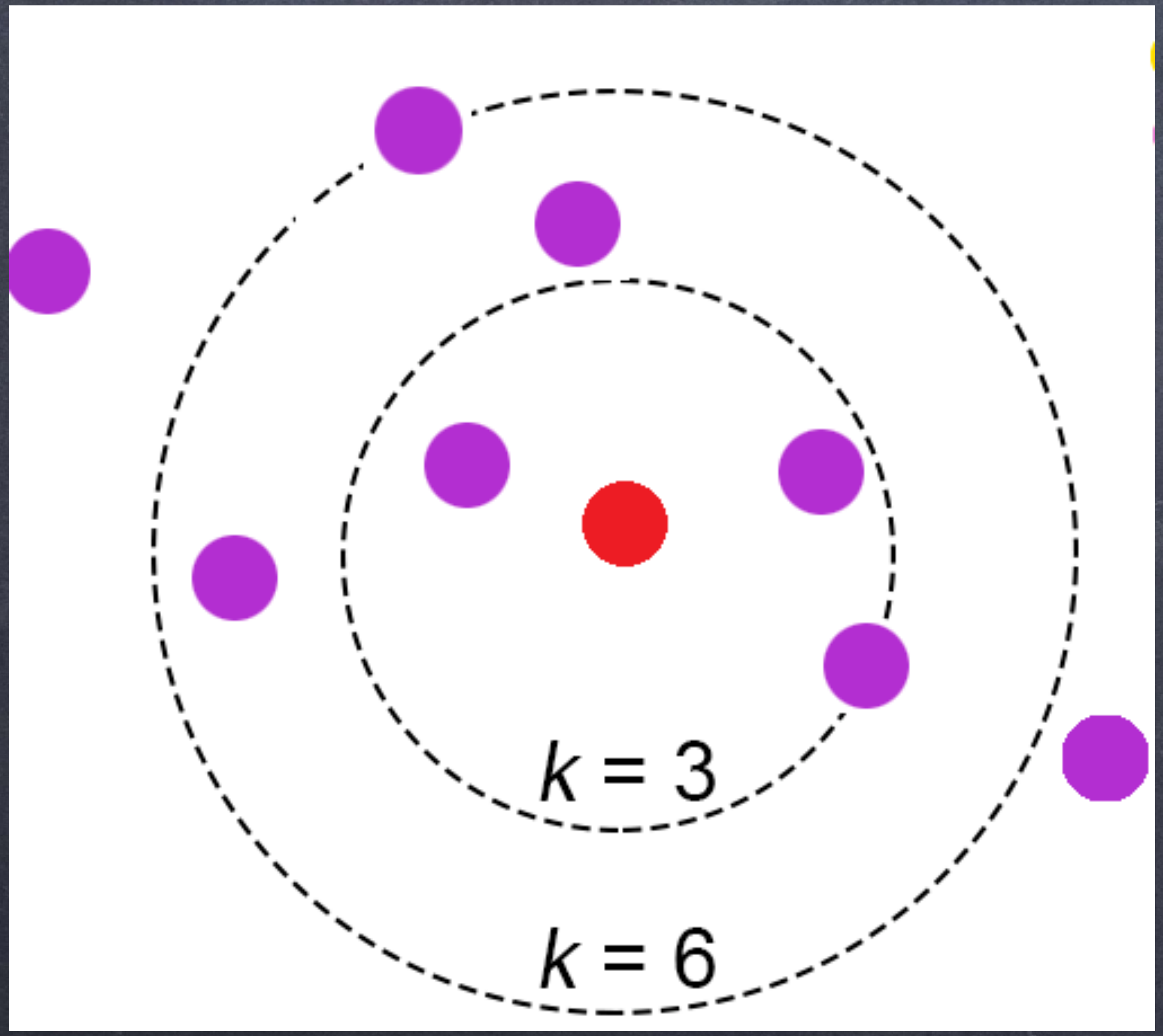

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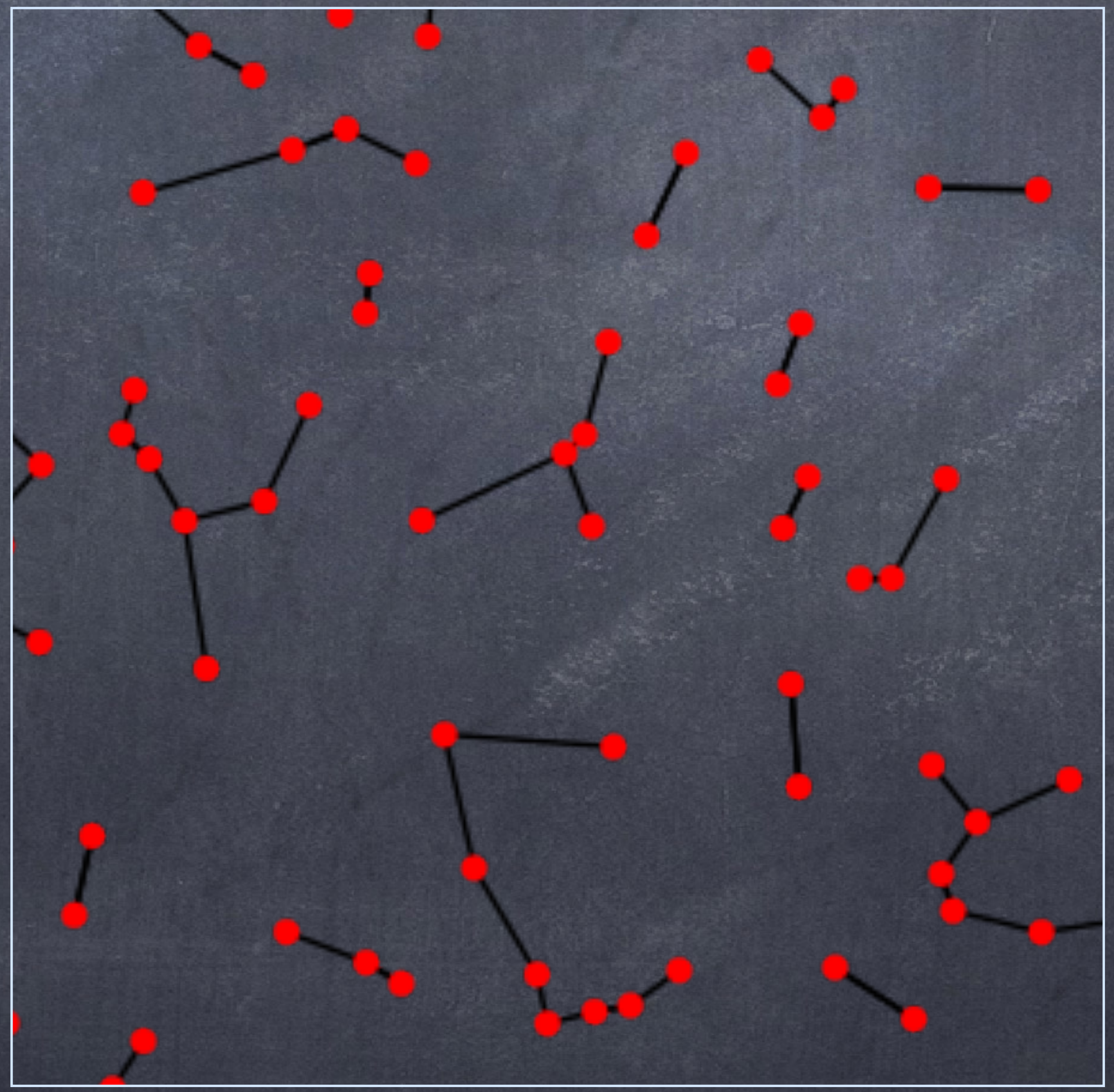

1-NNG on 100 points ; Fig. by D. Eppstein

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Lacker, Ramanan & Wu (2019) - Local weak convergence approach, expectation asymptotics.

THE GENERAL FRAMEWORK

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  Equivalently, $\tilde{\mathscr{P}} = \sum_{x \in \mathscr{P}} \delta_{(x, M(x))}$ i.e., each point of $\mathscr{P}$ is marked\labelled with $M(x) \in \mathbb{M}$, Polish space.

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Proved via an extension of cumulant method CLT and for linear statistics of $\mu_n := \sum_{x \in \mathscr{P}_n} M(x)\, \delta_{n^{-1/d}x}$

Pre-cursors for point processes and special cases - Malyshev (1975), Martin-Yalcin (1980), Nazarov-Sodin (2012), Błaszczyszyn, Y. and Yukich (2019), Fenzl (2019).

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- Other Applications - Empirical random fields, Geostatistical models.

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