Inflation II

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Slow roll

$$\cancel{\ddot{\phi}} + 3H\dot{\phi} + V'(\phi) = 0$$

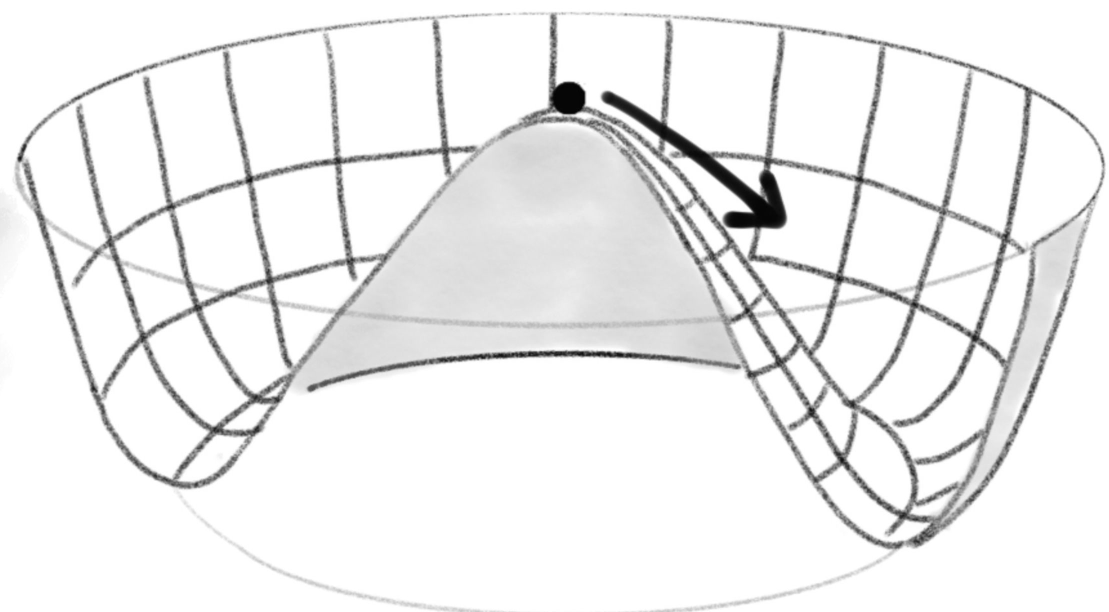
$$H^2 = \frac{1}{3M_P^2}\left[\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi)\right]$$

$$\left.\begin{aligned} \rho &= \cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi) \\ p &= \cancel{\frac{1}{2}\dot{\phi}^2} - V(\phi) \end{aligned}\right\} \quad p \simeq -\rho$$

Attractor solution:

$$\dot{\phi} \simeq -\frac{V'(\phi)}{3H}$$

$$H^2 \simeq \frac{1}{3M_P^2} V(\phi)$$



Hamilton-Jacobi Formalism

Exercise:

Show that for $\phi(t)$ monotonic:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \qquad H^2 = \frac{1}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

can be re-written as the equivalent system of equations:

$$H^2(\phi)\left[1 - \frac{2M_P^2}{3}\left(\frac{H'(\phi)}{H(\phi)}\right)^2\right] = \frac{1}{3M_P^2}V(\phi) \qquad H'(\phi) = -\frac{1}{2M_P^2}\dot{\phi}$$

$H(\phi) \equiv H\left[\dot{\phi}(t), \phi(t)\right]$

$\dot{\phi}(t) = \dot{\phi}[\phi(t)]$

Slow roll (formal approximation)

Slow roll parameter:

$$\epsilon\left(\phi\right) \equiv 2M_P^2\left(\frac{H'\left(\phi\right)}{H\left(\phi\right)}\right)^2 \simeq \frac{M_P^2}{2}\left(\frac{V'\left(\phi\right)}{V\left(\phi\right)}\right)^2$$

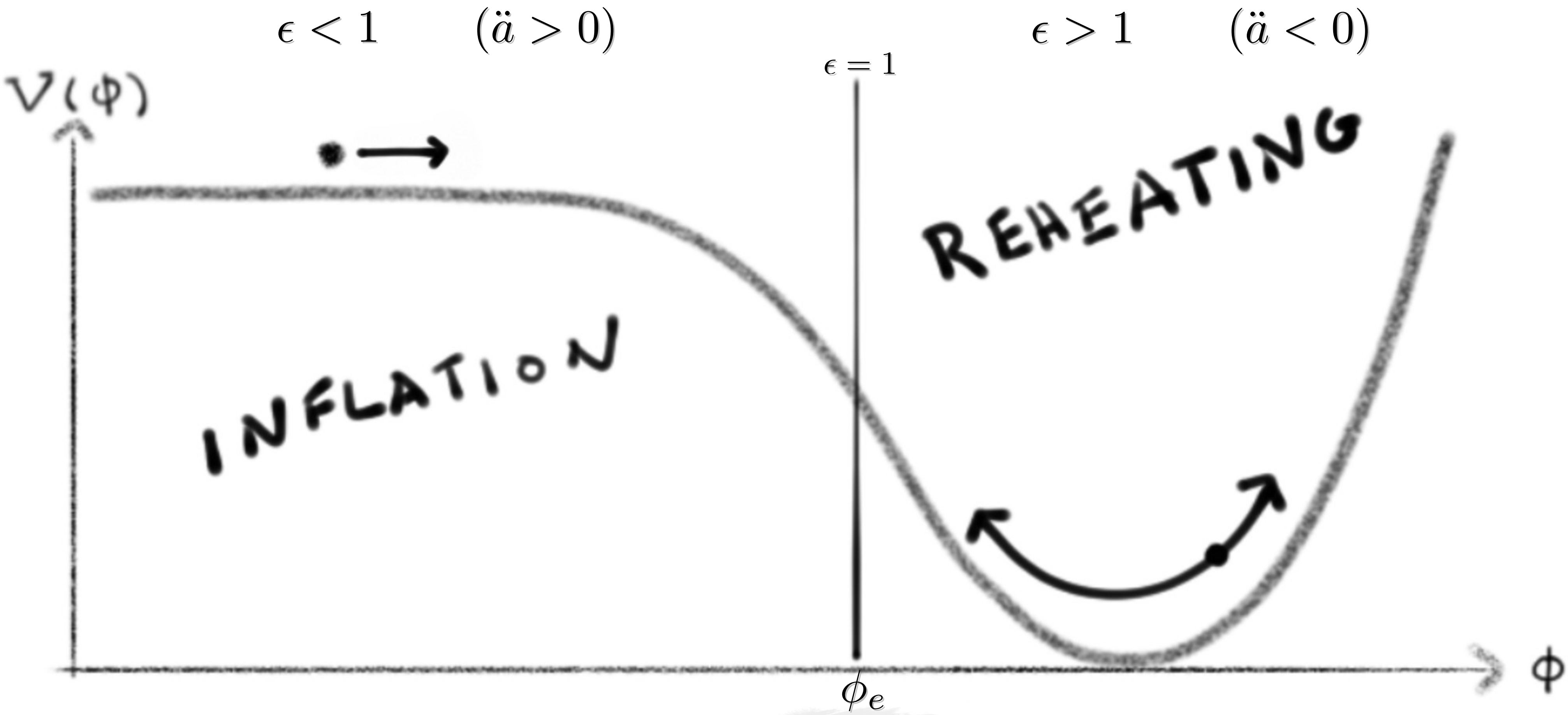
$$p = \rho\left[\cancelto{\epsilon \ll 1}{\frac{2}{3}\epsilon\left(\phi\right)} - 1\right] \simeq -\rho$$

Equations of motion:

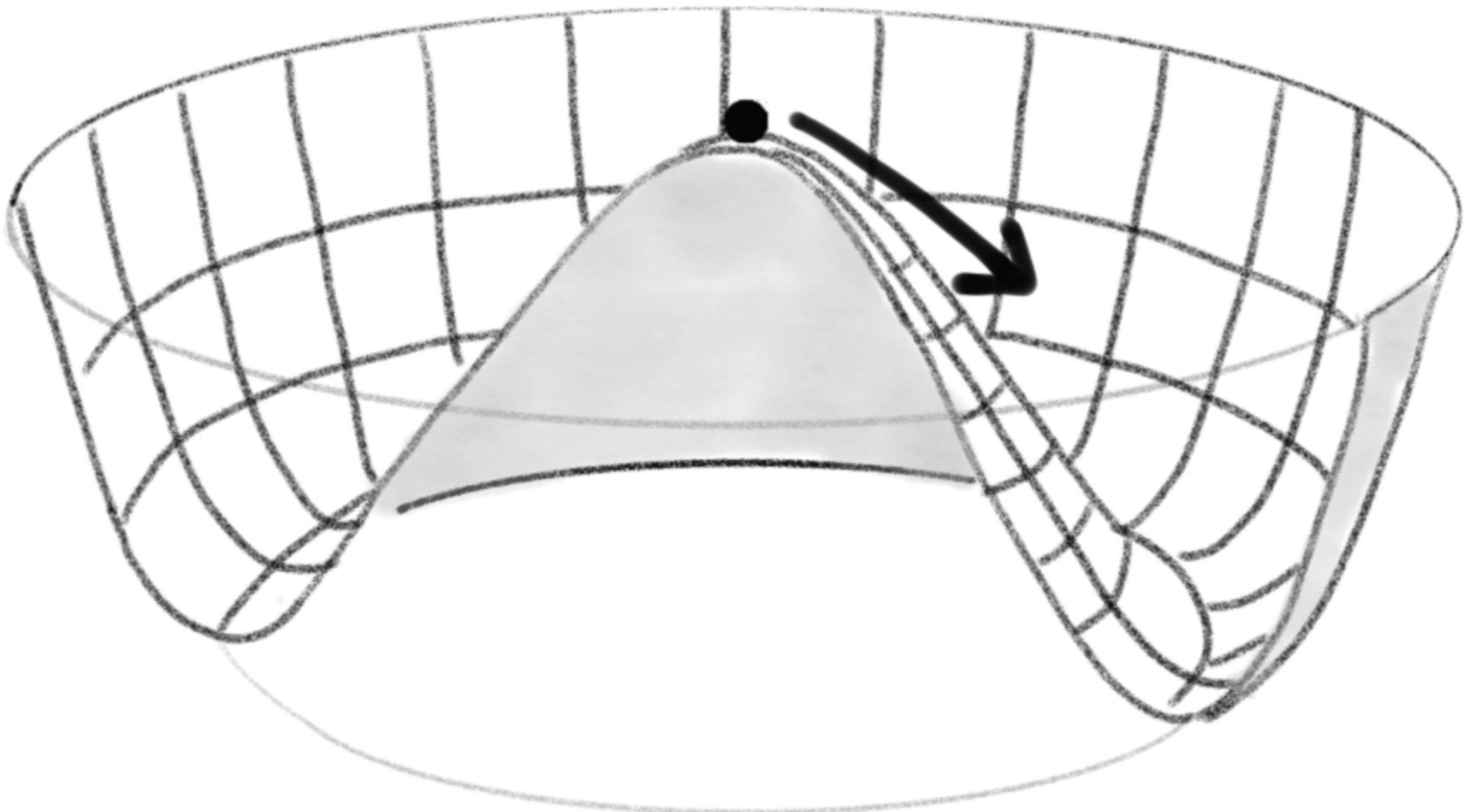
$$H^2\left(\phi\right)\left[1 - \cancelto{\epsilon \ll 1}{\frac{1}{3}\epsilon\left(\phi\right)}\right] = \frac{1}{3M_P^2}V\left(\phi\right) \qquad H^2\left(\phi\right) \simeq \frac{1}{3M_P^2}V\left(\phi\right)$$

$$\dot{\phi} = -2M_P^2 H'\left(\phi\right) \qquad \dot{\phi} \simeq \frac{V'\left(\phi\right)}{3H\left(\phi\right)}$$

Slow roll inflation



$$\frac{\ddot{a}}{a} = -\frac{2}{3M_P^2}(\rho + 3p) = H^2(\phi)\left[1 - \epsilon(\phi)\right]$$ **(EX)**



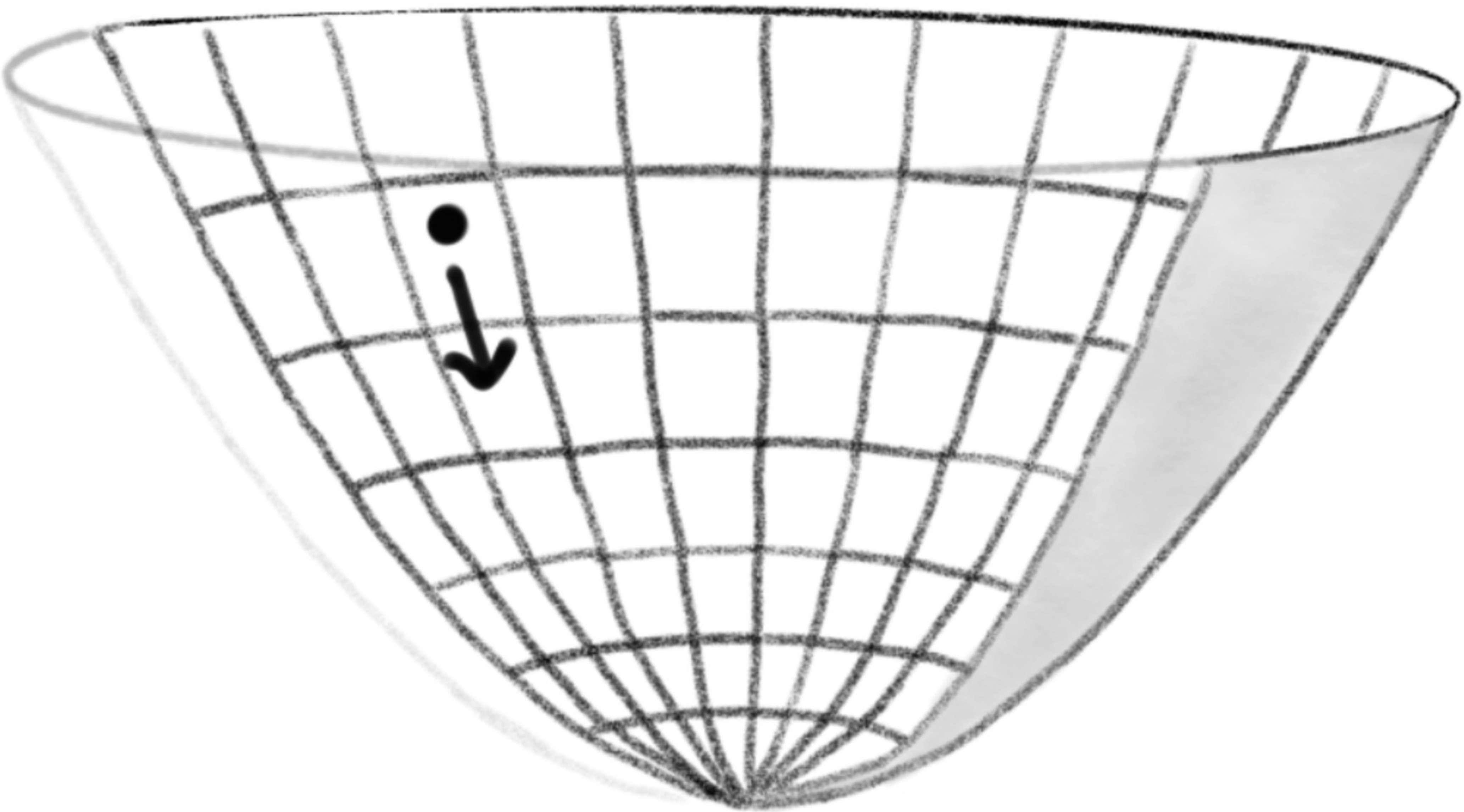
Worked example: hilltop inflation

$$V(\phi) = V_0 - \frac{1}{2}m^2\phi^2$$

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{M_P^2}{2}\left(\frac{m^2\phi}{V_0 - \frac{1}{2}m^2\phi^2}\right)^2 \simeq \frac{M_P^2 m^4 \phi^2}{2V_0^2}$$

End of inflation:

$$\epsilon(\phi_e) = 1 \Rightarrow \phi_e = \frac{\sqrt{2}V_0}{M_P m^2}$$



Worked example: chaotic inflation

$$V\left(\phi\right) = \frac{1}{2}m^2\phi^2$$

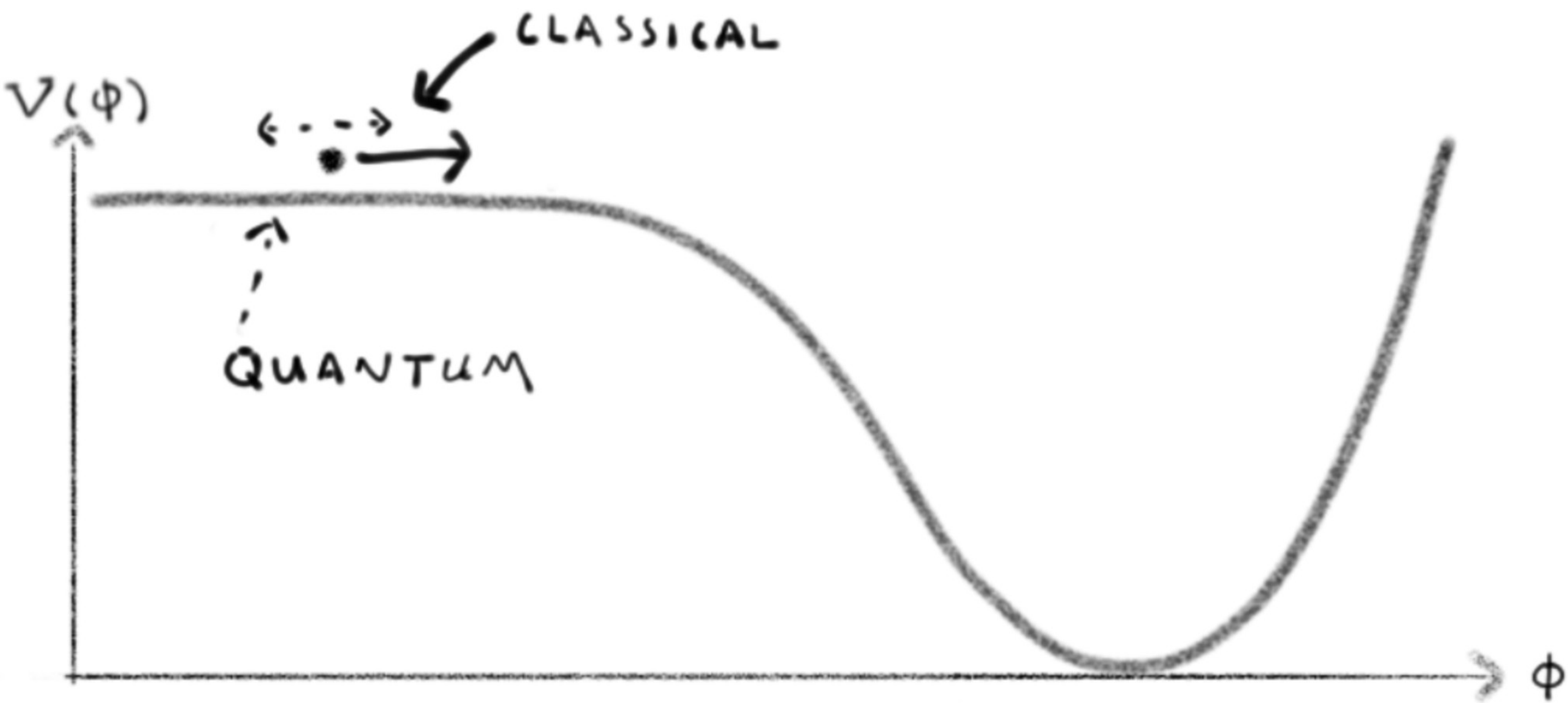
$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'\left(\phi\right)}{V\left(\phi\right)}\right)^2 = \frac{M_P^2}{2}\left(\frac{m^2\phi}{\frac{1}{2}m^2\phi^2}\right)^2 \simeq \frac{2M_P^2}{\phi^2}$$

End of inflation:

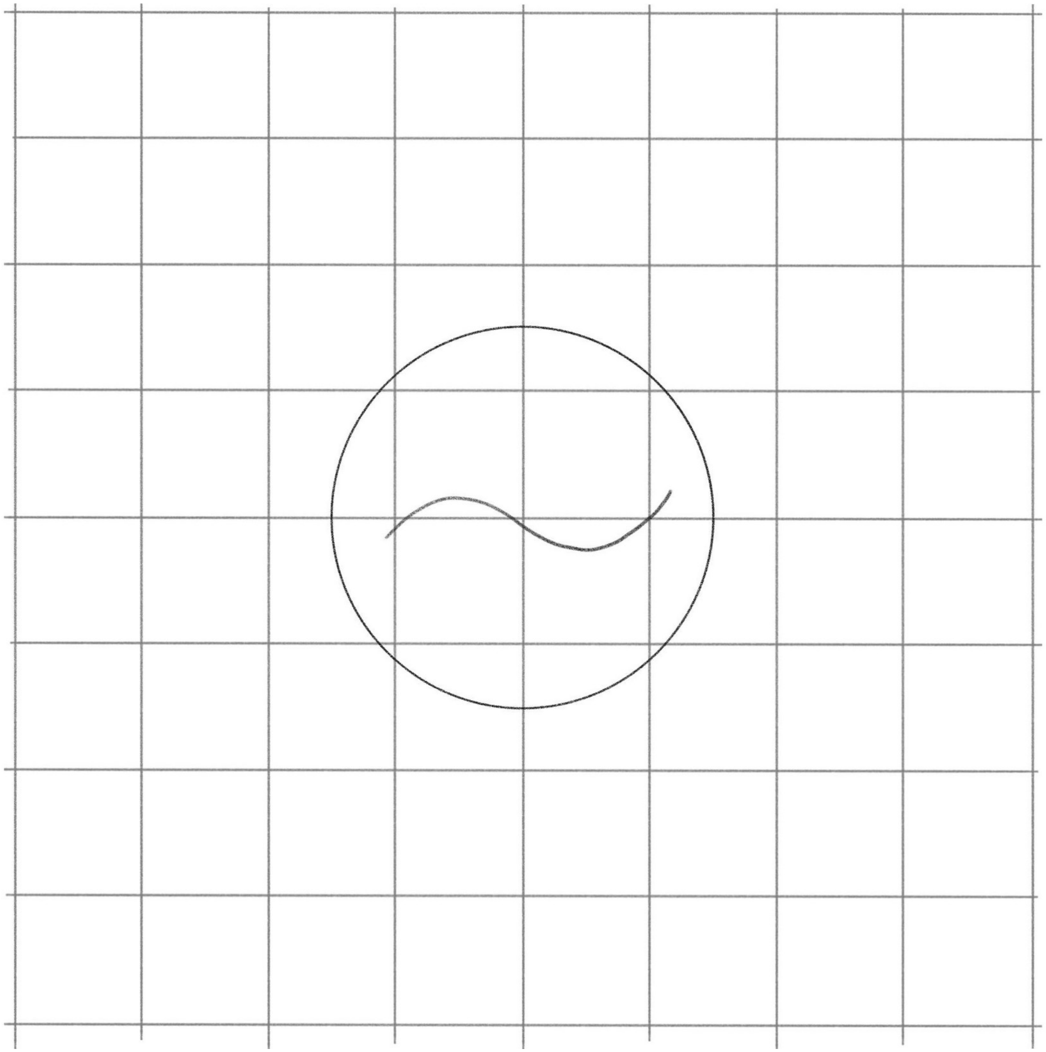
$$\epsilon\left(\phi_e\right) = 1 \Rightarrow \phi_e = \sqrt{2}M_P$$

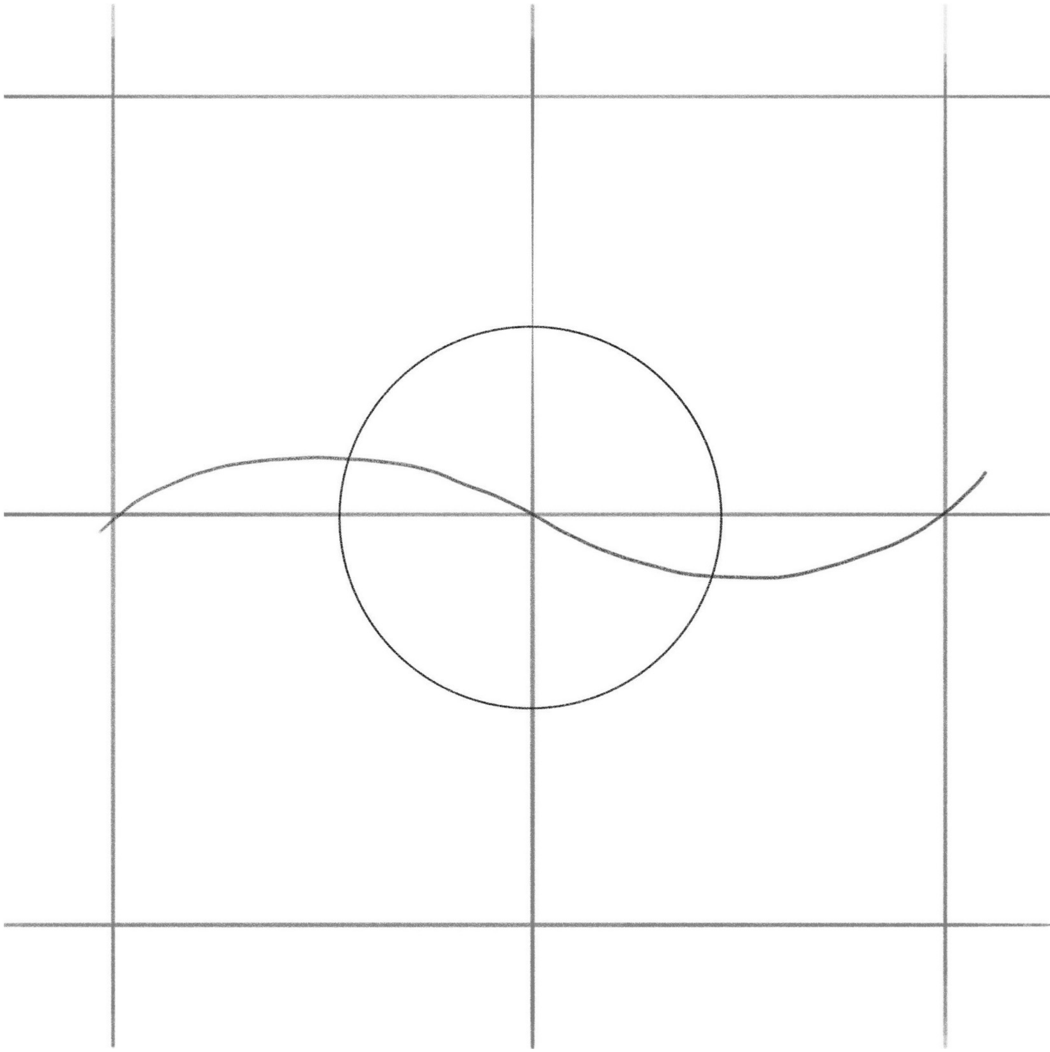
Perturbations

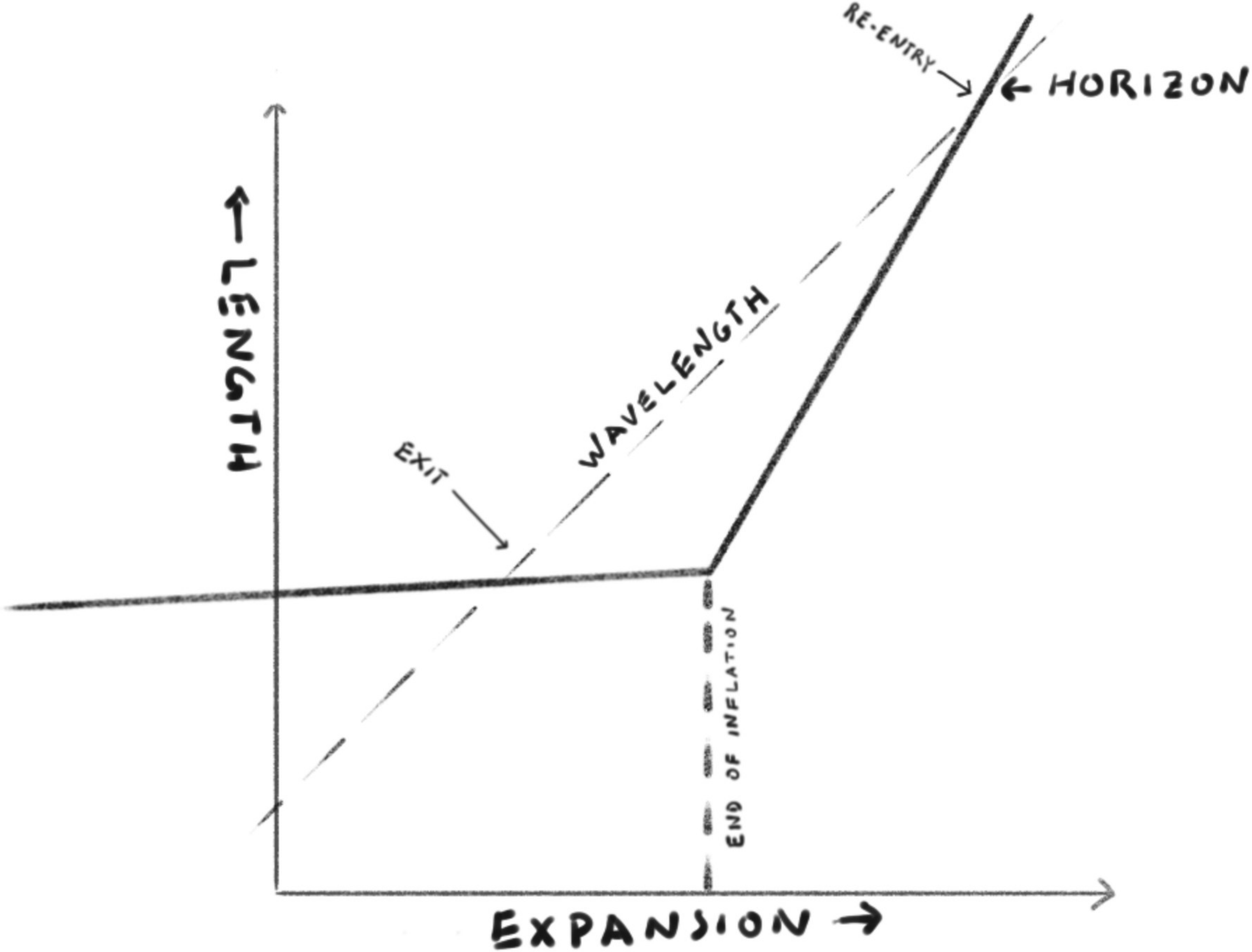
Quantum fluctuations

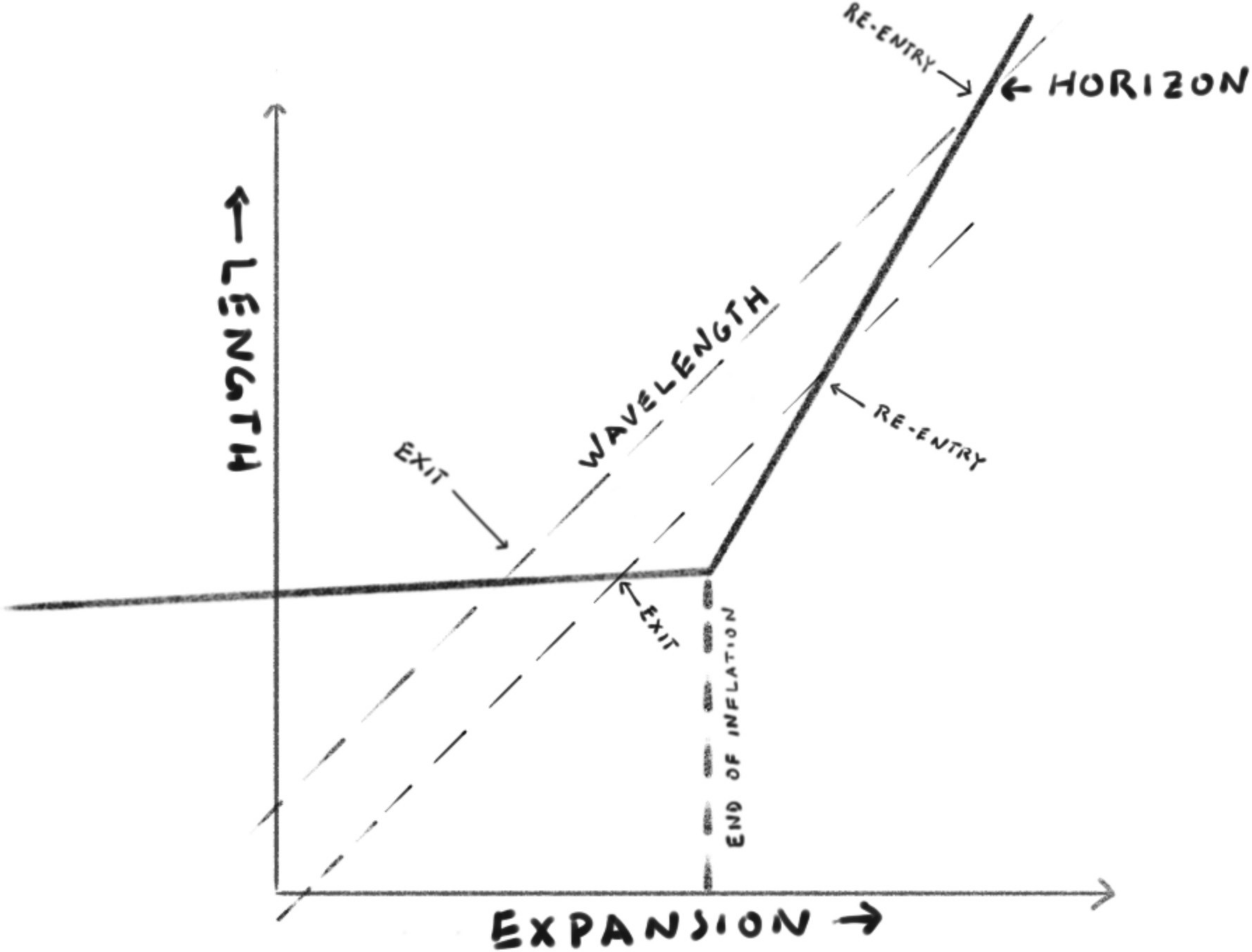


Quantum fluctuations





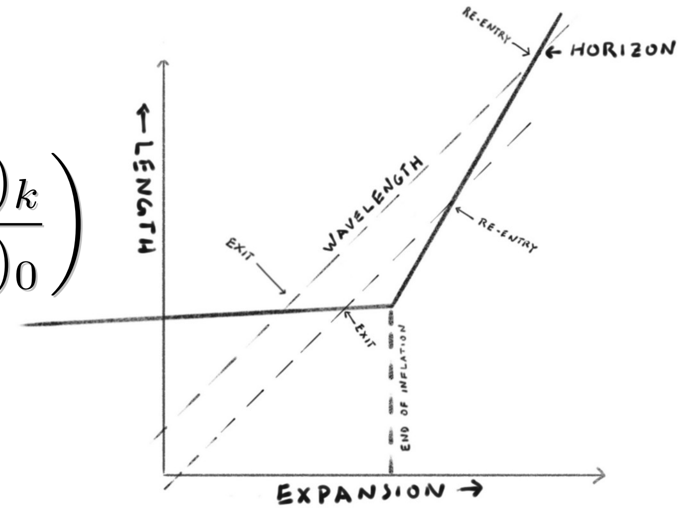

RE-ENTRY
HORIZON
LENGTH
WAVELENGTH
EXIT
END OF INFLATION
EXPANSION


RE-ENTRY
HORIZON
LENGTH
WAVELENGTH
EXIT
RE-ENTRY
EXIT
END OF INFLATION
EXPANSION

How much inflation?

Wavenumber in units of current horizon size

$$\ln\left(\frac{k}{a_0H_0}\right) = \ln\left(\cancelto{1}{\frac{k}{(aH)_k}}\,\frac{(aH)_k}{a_0H_0}\right) = \ln\left(\frac{(aH)_k}{(aH)_0}\right)$$



value at horizon exit $k = aH$

$$\ln\left(\frac{k}{a_0H_0}\right) = \ln\left(\frac{(aH)_k}{(aH)_{\mathrm{end}}}\right) + \ln\left(\frac{(aH)_{\mathrm{end}}}{(aH)_{\mathrm{eq}}}\right) + \ln\left(\frac{(aH)_{\mathrm{eq}}}{(aH)_0}\right)$$

end of inflation

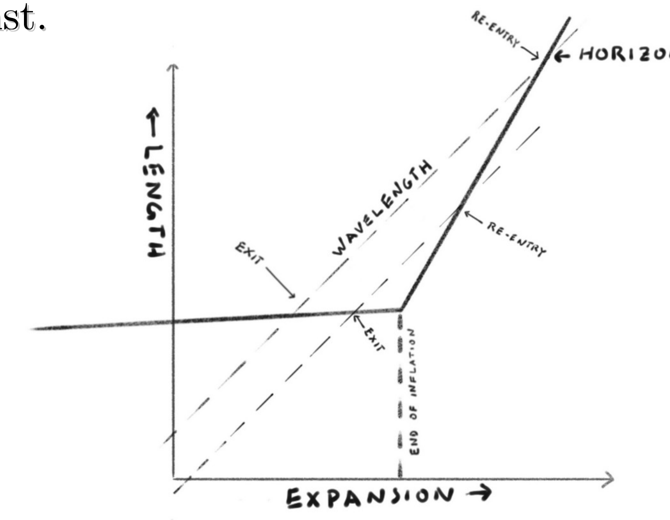
instantaneous
reheating

How much inflation?

$H \simeq \text{const.}$

$$\ln\left(\frac{(aH)_k}{(aH)_{\text{end}}}\right) = N_k - \cancelto{0}{N_{\text{end}}} + \ln\left(\cancel{\frac{H_k}{H_{\text{end}}}}\right)$$

$a \propto e^{-N}$



$aH \propto a^{-(1+3w)/2} = a^{-1}$

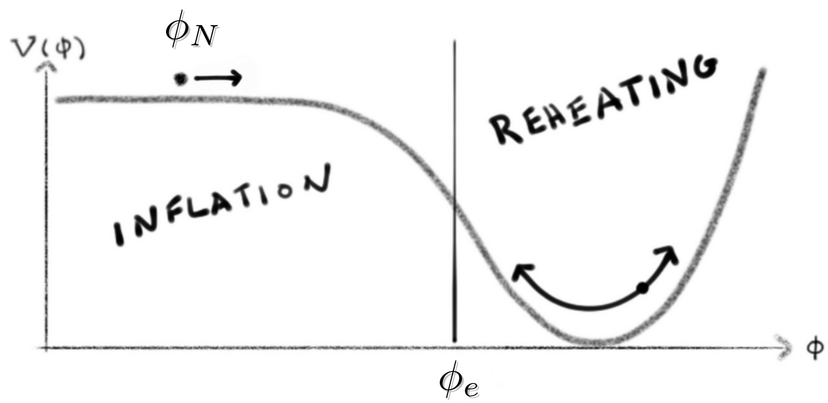
$$\ln\left(\frac{(aH)_{\text{end}}}{(aH)_{\text{eq}}}\right) = \ln\left(\frac{a_{\text{eq}}}{a_{\text{end}}}\right)$$

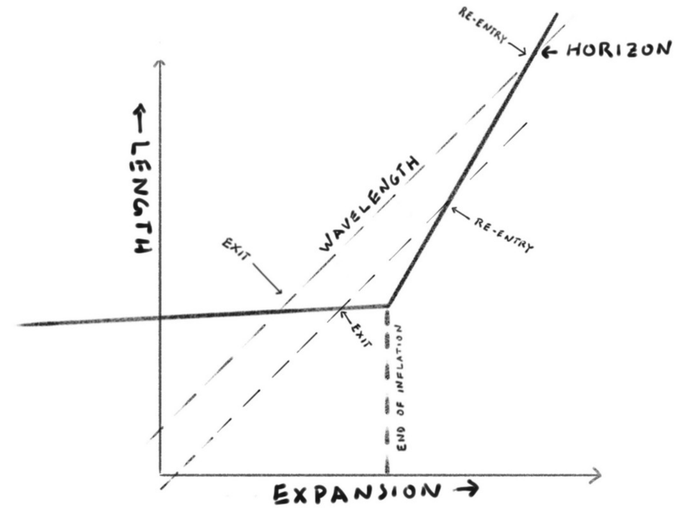
$$= \ln\left(\frac{T_R}{T_{\text{eq}}}\right) + \frac{1}{3}\ln\left(\frac{g_{*S}[T_R]}{g_{*S}[T_{\text{eq}}]}\right)$$

\# of relativistic degrees of freedom

$$\ln\left(\frac{(aH)_{\text{eq}}}{(aH)_0}\right) = 3.839$$

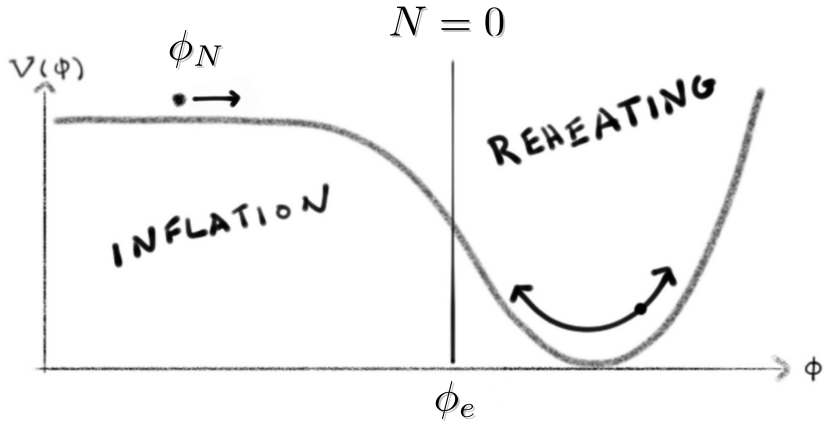
How much inflation?





$$N_k = -\ln\left(\frac{k}{a_0 H_0}\right) + \ln\left(\frac{T_R}{10^{25}\ \mathrm{eV}}\right) + \frac{1}{3}\ln\left(\frac{g_{*S}[T_R]}{g_{*S}[T_{\mathrm{eq}}]}\right) + 61.6$$

How much inflation?



$$a(t) \propto e^{-N}$$

$$N \equiv - \int H dt$$

note sign convention!

$$N(\phi_N) = \int_{t_e}^{t} H dt = \int_{\phi_e}^{\phi_N} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_e}^{\phi_N} \frac{H(\phi)}{\left[-2M_P^2 H'(\phi)\right]} d\phi$$

$$N(\phi_N) = \frac{1}{\sqrt{2}M_P} \int_{\phi_N}^{\phi_e} \frac{d\phi}{\sqrt{\epsilon(\phi)}} d\phi$$

Worked example: hilltop inflation

$$V\left(\phi\right) = V_0 - \frac{1}{2}m^2\phi^2 \qquad \epsilon\left(\phi\right) \simeq \frac{M_P^2 m^4 \phi^2}{2V_0^2} \qquad \phi_e = \frac{\sqrt{2}V_0}{M_P m^2}$$

$$N\left(\phi_N\right) = \frac{1}{\sqrt{2}M_P}\int_{\phi_N}^{\phi_e}\frac{d\phi}{\sqrt{\epsilon\left(\phi\right)}}$$

$$= \frac{V_0}{m^2 M_P^2}\int_{\phi_N}^{\phi_e}\frac{d\phi}{\phi} = \frac{V_0}{m^2 M_P^2}\ln\left(\frac{\phi_e}{\phi_N}\right)$$

$$\phi_N = \phi_e \exp\left(-\frac{m^2 M_P^2}{V_0}N\right)$$

Metric Perturbations

$$g_{\mu\nu} = a^2(\tau)\left(\eta_{\mu\nu} + \delta g_{\mu\nu}\right) \quad \left\{ \begin{aligned} \delta g_{00} &= -2A \\ \delta g_{0i} &= \cancel{\partial_i B} \\ &\qquad\qquad\qquad\quad k^2 \to 0 \\ \delta g_{ij} &= -2\left(\zeta\delta_{ij} + \cancel{\partial_i\partial_j H_T}\right) \end{aligned} \right.$$

$$\Theta \equiv u^{\mu}{}_{;\mu} = -3H\left[1 + A + \frac{1}{aH}\frac{\partial\zeta}{\partial\tau}\right]$$

<u>Unperturbed:</u>

$$dN \equiv -H\,dt$$

$$= -aH\,d\tau$$

<u>Perturbed (comoving):</u>

$$d\mathcal{N} \equiv \frac{1}{3}\Theta\,ds$$

$$= \frac{1}{3}\Theta\left[a\left(1 - A\right)d\tau\right]$$

Worked example: chaotic inflation

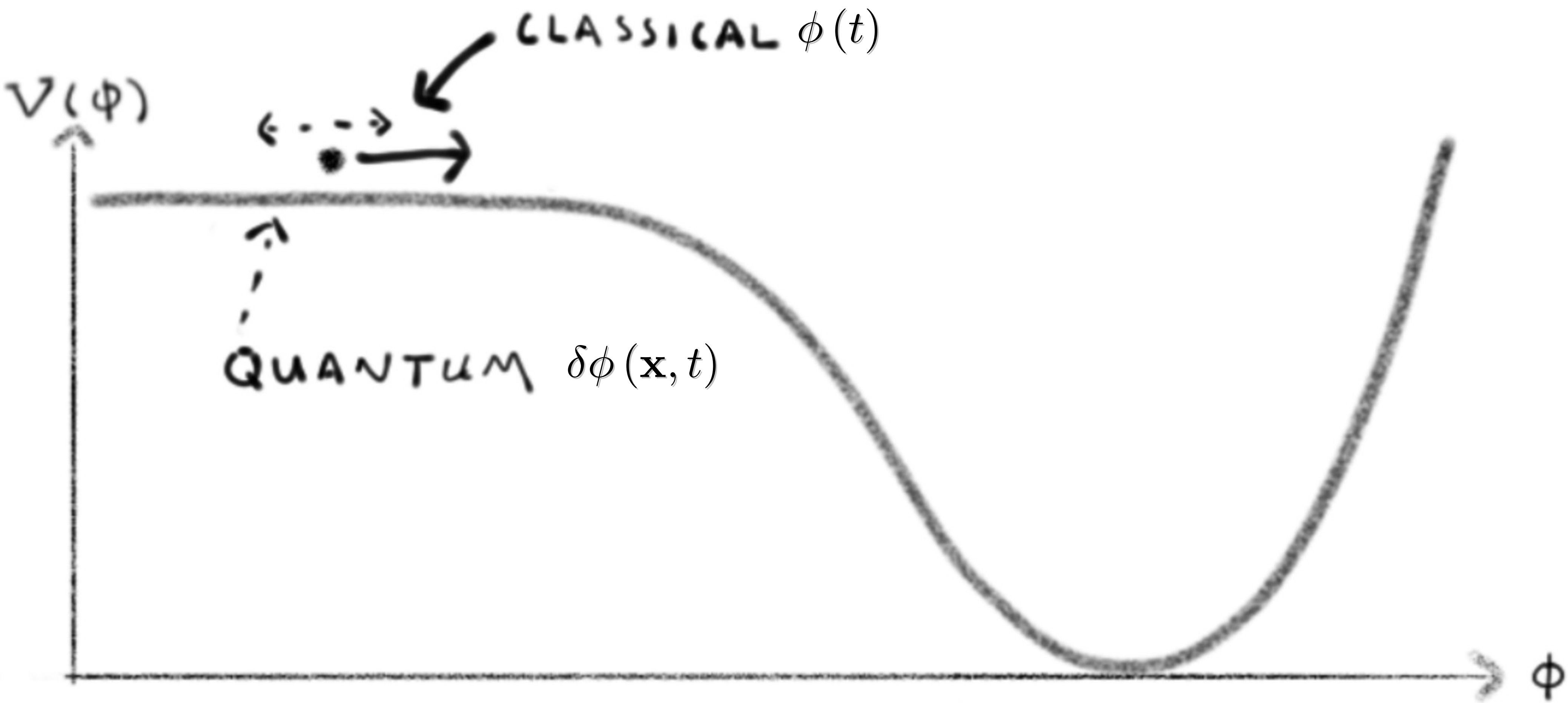
$$V\left(\phi\right) = \frac{1}{2}m^2\phi^2 \qquad \epsilon\left(\phi_e\right) = \frac{2M_P^2}{\phi^2} = 1 \Rightarrow \phi_e = \sqrt{2}M_P$$

$$N\left(\phi_N\right) = \frac{1}{\sqrt{2}M_P}\int_{\phi_N}^{\phi_e}\frac{d\phi}{\sqrt{\epsilon\left(\phi\right)}}$$

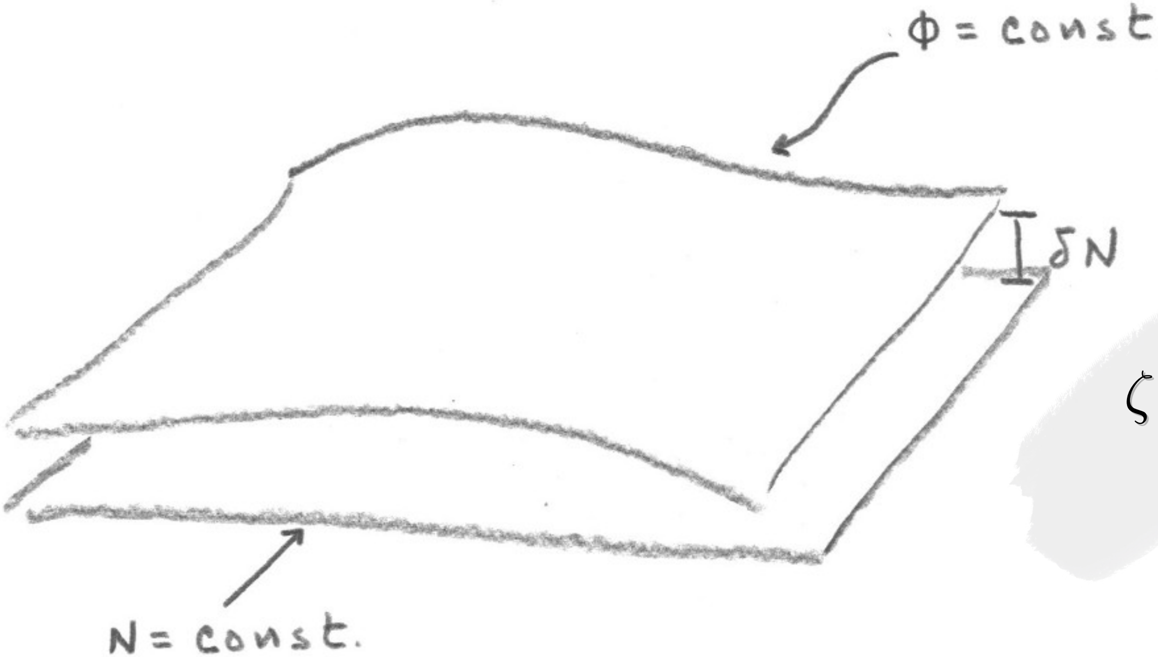
$$= \frac{1}{2M_P^2}\int_{\phi_e}^{\phi_N}\phi d\phi = \frac{\phi_N^2}{4M_P^2} - \frac{1}{2}$$

$$\phi_N = M_P\sqrt{4N+2}$$

Quantum Fluctuations



The comoving curvature perturbation



$$\zeta = N - \mathcal{N} = \frac{\delta N}{\delta \phi}\delta\phi$$

$$\mathcal{N} = \frac{1}{3}\int \Theta ds = -\zeta - \int H dt + \mathcal{O}(A^2)$$

Scalar field perturbations

$$\phi(\mathbf{x},\tau) = \phi(\tau) + \delta\phi(\mathbf{x},\tau)$$

Equation of motion:

$$\delta\phi_k'' + 2aH\delta\phi_k' + k^2\delta\phi_k = 0 \qquad \delta\phi' \equiv \frac{d(\delta\phi)}{d\tau}$$

(EX)

Mode function:

$$v_k \equiv a(\tau)\,\delta\phi_k$$

$$v_k'' + \left(k^2 - \underbrace{\frac{a''}{a}}_{m^2(\tau)}\right) v_k = 0$$

Scalar field perturbations

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0$$

$\underline{\mathbf{k^2 \gg a''/a}}$

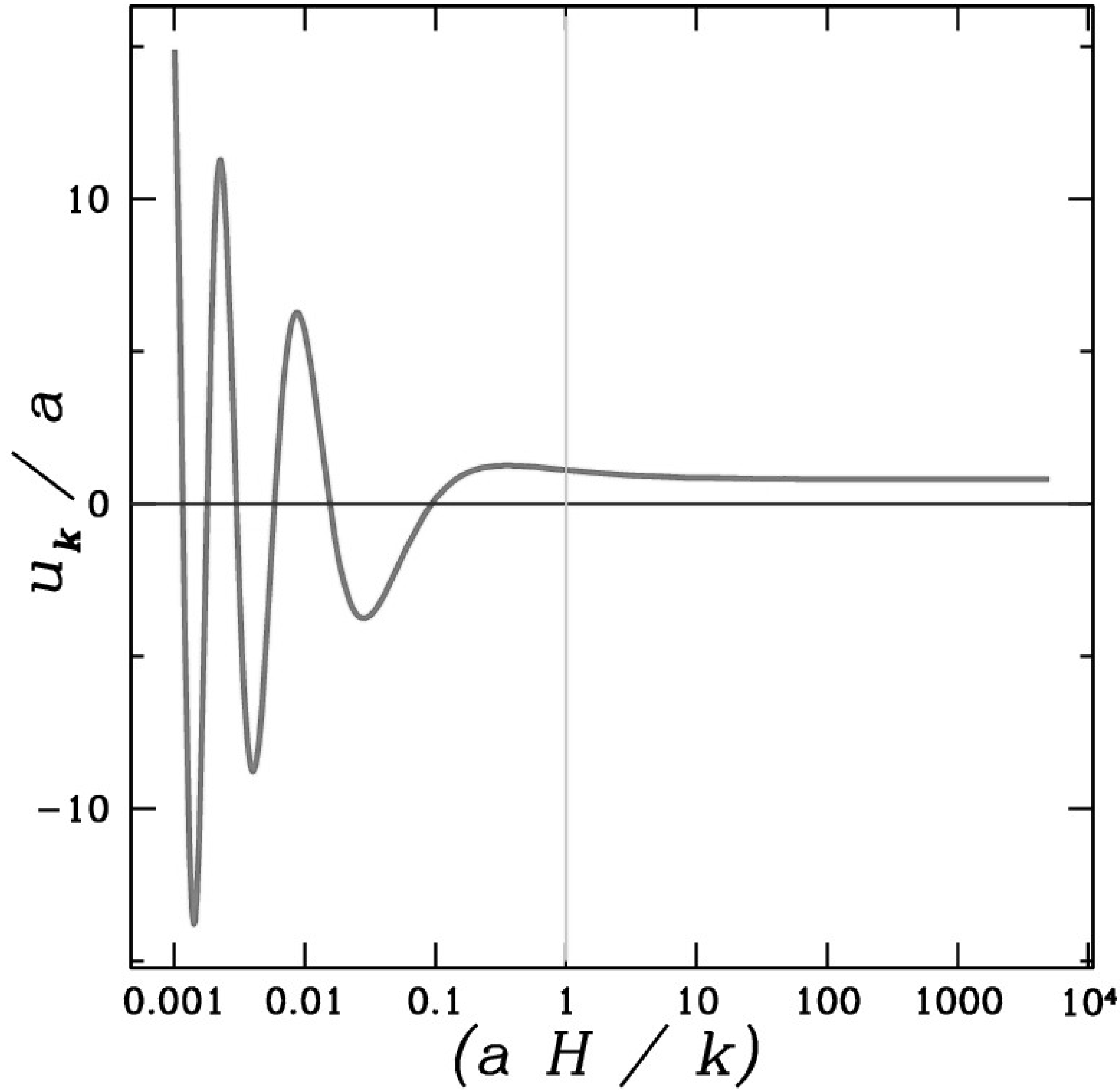
$$v_k'' + k^2 v_k = 0 \Rightarrow \quad v_k \propto e^{\pm ik\tau}$$

$\underline{\mathbf{k^2 \ll a''/a}}$

$$a'' v_k = a v_k'' \Rightarrow \quad v_k \propto a$$

mode freezing $\delta\phi_k = \dfrac{v_k}{a} \to \text{const.}$

Quantum fluctuations



Scalar field perturbations: exact solutions

Let: $\epsilon = \text{const.} \quad \Rightarrow \quad a(\tau) \propto \tau^{1/\epsilon}$

Mode equation: $v_k'' + \left[k^2 - (aH)^2 (2-\epsilon)\right] = 0$ **(EX)**

Solution:

$$v_k(\tau) \propto \sqrt{-k\tau}\left[J_\nu(-k\tau) \pm i Y_\nu(-k\tau)\right]$$
$$\propto \sqrt{-k\tau} H_{\pm\nu}(-k\tau)$$

$$\nu = \frac{3-\epsilon}{2(1-\epsilon)}$$

Quantization

To quantize the field fluctuations, replace Fourier amplitudes with creation/annihilation operators:

$$\delta\phi\left(\mathbf{x},\tau\right) = \int \frac{d^3k}{\left(2\pi\right)^{3/2}} \left[\hat{a}_{\mathbf{k}}\ \delta\phi_k\left(\tau\right) e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{H.C.}\right]$$

$$= a^{-1}\left(\tau\right) \int \frac{d^3k}{\left(2\pi\right)^{3/2}} \left[\hat{a}_{\mathbf{k}} v_k\left(\tau\right) e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{H.C.}\right]$$

scale factor

operator!

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = \delta^3\left(\mathbf{k} - \mathbf{k}'\right)$$

Quantization

Canonical momentum

$$\Pi(\mathbf{x},\tau) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} = a^2(\tau)\frac{\partial \phi}{\partial \tau}$$

$$[\phi, \Pi]_{\tau=\tau'} = i\delta^3(\mathbf{x}-\mathbf{x}') \Rightarrow$$

$$v_k \frac{\partial v_k^*}{\partial \tau} - v_k^* \frac{\partial v_k}{\partial \tau} = i$$

(EX)

Mode normalization

Short wavelength limit $-k\tau \to \infty$

$$v_k \to \frac{1}{\sqrt{2k}} \left[A_k e^{-ik\tau} + B_k e^{+ik\tau} \right]$$

Quantization

$$v_k \frac{\partial v_k^*}{\partial \tau} - v_k^* \frac{\partial v_k}{\partial \tau} = i \qquad \Rightarrow |A_k|^2 - |B_k|^2 = 1$$

(EX)

Bunch-Davies vacuum

$$A_k = 1, \quad B_k = 0 \qquad v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

Example: de Sitter space

$$v_k(\tau) \propto \sqrt{-k\tau}\left[J_\nu(-k\tau) \pm i Y_\nu(-k\tau)\right] \qquad \nu = \frac{3-\epsilon}{2(1-\epsilon)} = \frac{3}{2}$$

$$\propto \left(\frac{k\tau - i}{k\tau}\right) e^{-ik\tau}$$

Bunch-Davies vacuum

Normalized solution (from quantization):

$$v_k = \frac{1}{\sqrt{2k}}\left(\frac{k\tau - i}{k\tau}\right) e^{-ik\tau}$$

Example: de Sitter space

$$v_k = \frac{1}{\sqrt{2k}} \left(\frac{k\tau - i}{k\tau} \right) e^{-ik\tau}$$

Long wavelength limit $-k\tau \to 0$

$$v_k \to \frac{1}{2k} \left(\frac{i}{-k\tau} \right) = \frac{i}{2k} \left(\frac{aH}{k} \right)$$ **(EX)**

$$|\delta\phi_k| = \left| \frac{v_k}{a} \right| \to \frac{H}{\sqrt{2}k^{3/2}} = \text{const.}$$

mode freezing

The power spectrum

Exercise: show that the vacuum two-point correlation is

$$\langle 0 \left| \delta\phi\left(\tau, \mathbf{x}\right) \; \delta\phi\left(\tau', \mathbf{x}'\right) \right| 0 \rangle_{\tau'=\tau} = \int \frac{d^3k}{\left(2\pi\right)^3} \left| \frac{v_k}{a} \right|^2 e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right)}$$

then

$$\left\langle \delta\phi^2 \right\rangle_{\mathbf{x}'=\mathbf{x}} = \int \frac{d^3k}{\left(2\pi\right)^3} \left| \frac{v_k}{a} \right|^2 \equiv \int \frac{dk}{k} P\left(k\right)$$

power spectrum

$$P\left(k\right) = \left(\frac{k^3}{2\pi^2}\right) \left| \frac{v_k}{a} \right|^2 \longrightarrow \left(\frac{H}{2\pi}\right)^2 \quad -k\tau \to 0$$

Exercise

Show that for $\epsilon = \text{const.} \quad \nu = \frac{3-\epsilon}{2(1-\epsilon)}$

(a) The Bunch-Davies boundary condition corresponds to the positive mode:

$$v_k \propto J_\nu(-k\tau) + iY_\nu((-k\tau)) = H_{+\nu}(-k\tau)$$

(b) Quantization fixes the normalization

$$v_k = \frac{1}{2}\sqrt{\frac{\pi}{k}}\sqrt{-k\tau}H_\nu(-k\tau)$$

(c) The power spectrum in the limit $-k\tau \to 0$ is a power-law

$$P(k)^{1/2} = 2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{H}{2\pi}\right)\left(\frac{k}{aH(1-\epsilon)}\right)^{3/2-\nu}$$

The curvature perturbation

We can then take

$$\delta\phi \equiv \sqrt{\langle\delta\phi^2\rangle} = \frac{H}{2\pi} \qquad \frac{\delta N}{\delta\phi} = \frac{dN}{dt}\frac{dt}{d\phi} = \frac{H}{\dot{\phi}}$$

to write the comoving curvature perturbation

$$P_\zeta^{1/2} = \frac{\delta N}{\delta\phi}\delta\phi = \frac{\langle\delta\phi\rangle_{\mathrm{Q}}}{\langle\delta\phi\rangle_{\mathrm{Cl}}}$$

$$P_\zeta^{1/2} = \frac{H^2}{2\pi\dot{\phi}} = \frac{H}{2\pi M_P\sqrt{2\epsilon}}$$

this should bother you!

Tensor modes

Gravitational wave (tensor) perturbations are described by free scalar fields, so for tensors, we're done!

$$P_T = \frac{8\left\langle \delta\phi^2 \right\rangle}{M_P^2} = \frac{2H^2}{\pi^2 M_P^2}$$

We can then define the *tensor fraction*, or *tensor/scalar ratio:*

$$r \equiv \frac{P_T}{P_\zeta} = 16\epsilon$$

Consistency Relation

$$P_T = \frac{8\left\langle \delta\phi^2 \right\rangle}{M_P^2} = \frac{2H^2}{\pi^2 M_P^2} \propto k^{n_T} \qquad r \equiv \frac{P_T}{P_\zeta} = 16\epsilon$$

$$r \equiv \frac{P_T}{P_\zeta} = 16\epsilon \qquad n_T \simeq -2\epsilon = -\frac{r}{8}$$

single-field consistency relation