Conformal limits of parabolic Higgs bundles

Richard A. Wentworth (with B. Collier and L. Fredrickson)

UNIVERSITY OF MARYLAND

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MAX PLANCK INSTITUTE FOR MATHEMATICS

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- C = compact Riemann surface of genus $g \geq 2$.
- $\mathcal{N}$ = moduli of stable holomorphic $\mathsf{SL}(n)$ bundles on C.
- $\mathcal{M}_{Dol}$ = moduli of stable holomorphic $\mathsf{SL}(n)$ Higgs bundles.
- $\mathcal{M}_{dR}$ = moduli of irreducible flat $\mathsf{SL}(n)$ connections.

$$
(T^*\mathcal{N} =) \qquad \overset{\circ}{\mathcal{M}}_{Dol} \longrightarrow \mathcal{N} \longleftarrow \overset{\circ}{\mathcal{M}}_{dR} \qquad (T^*\mathcal{N} \text{ torsor})
$$

Nonabelian Hodge correspondence:

$$\begin{tikzcd}
\mathcal{M}_{Dol} \arrow[rr, "\simeq", "NAH"'] & & \mathcal{M}_{dR} \\
\cup & & \cup \\
\mathring{\mathcal{M}}_{Dol} \arrow[dr] & & \mathring{\mathcal{M}}_{dR} \arrow[dl] \\
& \mathcal{N} &
\end{tikzcd}$$

This diagram *does not commute*.

There *is* a commuting diagram:

$$\begin{tikzcd}
\mathring{\mathcal{M}}_{Dol} \arrow[rr, "\simeq", "CL"'] \arrow[dr] & & \mathring{\mathcal{M}}_{dR} \arrow[dl] \\
& \mathcal{N} \arrow[ur, dashed, bend right, "NS"'] &
\end{tikzcd}$$

The *conformal limit* extends this map (almost) everywhere:

$$\mathcal{M}_{Dol} \overset{}{\underset{CL}{\dashrightarrow}} \mathcal{M}_{dR}$$

It is *not continuous*, but interchanges two natural stratifications.

- The closed strata in $\mathcal{M}_{Dol}$ are the *Hitchin components*.

- The closed strata in $\mathcal{M}_{dR}$ are the *Opers*.

Conjecture (Gaiotto '14) – **Theorem** (Dumitrescu, Fredrickson, Kydonakis, Mazzeo, Mulase, Neitzke '16).

On a Hitchin component, the conformal limit exists and gives a biholomorphism between Hitchin components and components of the Opers.

Theorem (Collier, W. '18)

The conformal limit exists for almost all Higgs bundles and gives a biholomorphism between fibers of the BB stratifications in $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ (Simpson) .

- *Higgs bundle*: $(\bar{\partial}_E, \Phi),\ \Phi \in \Omega^{1,0}(\mathfrak{sl}_E),\ \bar{\partial}_E \Phi = 0.$
- *Flat connection*: $D = \bar{\partial}_V + \nabla,\ [\bar{\partial}_V, \nabla] = 0.$

$$\nabla : V \longrightarrow V \otimes K_C$$

- *NAH*: (assume stability) there is a *harmonic* metric h such that:

$$\bar{\partial}_V = \bar{\partial}_E + \Phi^{*_h} \quad , \quad \nabla = \partial_E^h + \Phi$$

- In this case: $D_\xi = \bar{\partial}_E + \partial_E^h + \xi^{-1}\Phi + \xi \Phi^{*_h}$ is **flat** for all $\xi \in \mathbb{C}^*$.
- *Twistor space*:

$$\begin{tikzcd} \mathcal{M}_{Hod} \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & \mathcal{M}_{dR} \times \mathbb{C} \arrow[dl, "pr_2"] \\ & \mathbb{C} & \end{tikzcd}$$

- There is a $\mathbb{C}^*$ action on $\mathcal{M}_{Dol}$:

$$\lambda \cdot [(\bar{\partial}_E, \Phi)] = [(\bar{\partial}_E, \lambda\Phi)]$$

- Limits $\lim_{\lambda \to 0} \lambda \cdot [(\bar{\partial}_E, \Phi)]$ always exist in $\mathcal{M}_{Dol}$.[†]

- The fixed points are called *Hodge bundles*.

- The $\mathbb{C}^*$-action extends to one on $\mathcal{M}_{Hod}$ covering the usual action on $\mathbb{C}$.

[†] or rather, in the completion by semistable Higgs bundles

- A Hodge bundle is a split Higgs bundle $E = E_1 \oplus \cdots \oplus E_\ell$:

$$\bar{\partial}_E = \bar{\partial}_{E_1} \oplus \cdots \oplus \bar{\partial}_{E_\ell} \quad , \quad \Phi : E_i \to E_{i+1} \otimes K_C$$

- A filtration $V_\bullet : 0 \subset V_1 \subset \cdots \subset V_\ell = V$ on $(\bar{\partial}_V, \nabla)$ is called *Griffiths transverse* if:

$$\nabla : V_i \to V_{i+1} \otimes K_C$$

- Given a Griffiths transverse filtration we get a Hodge bundle:

$$E = \mathrm{Gr}(V_\bullet) \quad , \quad \Phi = \nabla : V_i / V_{i-1} \to (V_{i+1}/V_i) \otimes K_C$$

Theorem (Simpson). For any $(\bar{\partial}_V, \nabla)$ there is a Griffiths transverse filtration such that the associated Hodge bundle is semistable, and it is unique if the Hodge bundle is stable.

One can show that in $\mathcal{M}_{Hod}$,

$$\lim_{\lambda \to 0} [(\bar{\partial}_V, \nabla)] = [(\bar{\partial}_E, \Phi)]$$

where $\bar{\partial}_E$ corresponds to $\mathrm{Gr}(V_\bullet)$ and Φ to ∇.

Example.

- Hodge bundle:

$$E = K_C^{-(n-1)/2} \oplus \cdots \oplus K_C^{(n-1)/2}$$
$$\Phi : K_C^j \to K_C \otimes K_C^{j-1} \text{ is the identity}$$

- Uniformizing Oper: $V = J^{n-1}(K_C^{-(n-1)/2})$. The connection ∇ comes from the Fuchsian projective structure on C.
- Then: (E, Φ) is obtained from the associated graded of (V, ∇).

Stratifications: Fix a Hodge bundle $(\bar{\partial}_0, \Phi_0)$.

- $W^0(\bar{\partial}_0, \Phi_0) := \{[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_{Dol} \mid \lim_{\lambda\to 0}[(\bar{\partial}_E, \Phi)] = [(\bar{\partial}_0, \Phi_0)]\}$
- $W^1(\bar{\partial}_0, \Phi_0) := \{[(\bar{\partial}_V, \nabla)] \in \mathcal{M}_{dR} \mid \lim_{\lambda\to 0}[(\bar{\partial}_V, \nabla)] = [(\bar{\partial}_0, \Phi_0)]\}$
- **Fact**. NAH does *not* take $W^0(\bar{\partial}_0, \Phi_0)$ to $W^1(\bar{\partial}_0, \Phi_0)$.

Gaiotto: Combine the $\mathbb{C}^*$ action with the twistor line:

$$\bar{\partial}_E + \partial_E^{h_R} + \xi^{-1} R\Phi + \xi R \Phi^{*_{h_R}}$$

and keep the ratio $\xi R^{-1} = \hbar$ *fixed*.

$$D_{R,\hbar} = \bar{\partial}_E + \partial_E^{h_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*_{h_R}}$$

The limit $\lim_{R\to 0} D_{R,\hbar}$ (if it exists) is called the *$\hbar$-conformal limit.*[*]

Theorem (CW '18). If $\lim_{\lambda\to 0} \lambda \cdot [(\bar{\partial}_E, \Phi)] = (\bar{\partial}_0, \Phi_0)$ is a stable Hodge bundle, then the conformal limit exists and lies in $W^1(\bar{\partial}_0, \Phi_0)$. Moreover, this gives a biholomorphism $W^0(\bar{\partial}_0, \Phi_0) \xrightarrow{\sim} W^1(\bar{\partial}_0, \Phi_0)$.

[*]from now on $\hbar = 1$

- The conformal limit is easy to describe – it comes from a certain *gauge fixing*.
- Given a Higgs bundle $(\bar{\partial}_E, \Phi)$ with harmonic metric h and associated flat connection $D = D'' + D'$ from NAH,

$$D'' = \bar{\partial}_E + \Phi \quad , \quad D' = \partial_E^h + \Phi^{*_h}$$

- Consider the pair of equations:

$$(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}_E) \oplus \Omega^{1,0}(\mathfrak{sl}_E) \quad , \quad \begin{cases} D''(\beta, \varphi) + [\beta, \varphi] = 0 \\ D'(\beta, \varphi) = 0 \end{cases}$$

  The first says $(\bar{\partial}_E + \beta, \Phi + \varphi)$ is another Higgs bundle, and the second is a natural gauge fixing condition.
- If $(\bar{\partial}_E, \Phi)$ is stable, these equations define a *slice*: $\mathcal{S}(\bar{\partial}_E, \Phi)$.
- Notice that $D + \beta + \varphi$ is again flat!

- Now suppose $E = E_1 \oplus \cdots \oplus E_\ell$, $\Phi : E_i \to E_{i+1} \otimes K_C$, is a Hodge bundle.
- Taking (β, φ) to be upper triangular with respect to this splitting defines a *subslice* $\mathcal{S}^+(\bar\partial_E, \Phi)$.
- One can show that $\mathcal{S}^+(\bar\partial_E, \Phi)$ is biholomorphic to the BB-fiber over $(\bar\partial_E, \Phi)$.
- Moreover, for $(\beta, \varphi) \in \mathcal{S}^+(\bar\partial_E, \Phi)$, the conformal limit of the point $(\bar\partial_E + \beta, \Phi + \varphi)$ is $D + \beta + \varphi$.
- To *prove* this is the conformal limit, one needs a singular perturbation argument to relate h_R to the harmonic metric of the Hodge bundle (see [DFKMMN]).

- Gaiotto's original conjecture (part of it) was in the context of Higgs bundles with *parabolic structure*: $D = p_1 + \cdots + p_d$, consisting of:
  - a flag in E_p, $p \in D$;
  - weights $\alpha(p)$, $0 \leq \alpha_1 < \cdots < \alpha_{\ell(p)} < 1$;
  - Higgs field $\Phi \in H^0(C, \mathrm{End}(E) \otimes K_C(D))$;
  - $\mathrm{res}_p \Phi$ preserves the flag for each $p \in D$.
- Simpson '90 proves the NAH in this case as well, where the correspondence is with logarithmic connections:

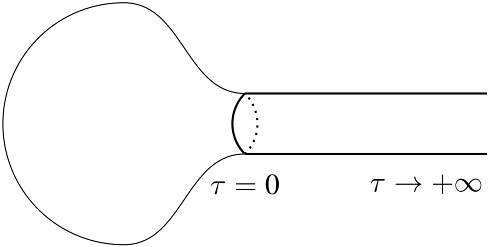
$$\nabla : V \longrightarrow V \otimes K_C(D)$$

- V has a parabolic structure (filtered) at D, and $\mathrm{res}(\nabla)$ preserves the flags.

- $(\bar{\partial}_E, \Phi)$ is *strongly parabolic* if $\operatorname{res}\Phi$ is strictly upper triangular with respect to the flag structure.
- **Assumption**. *Either* strongly parabolic *or* full flags.
- *Extended* moduli space of parabolic Higgs bundles (see Yokogawa, Logares-Martens):

$$\mathrm{eigres} : \mathcal{M}^{par}_{Dol}(\alpha) \longrightarrow \prod_{p \in D} \mathfrak{l}_p$$

  Here, $\mathfrak{l}_p \simeq$ traceless diagonal matrices.
- $\mathrm{eigres}^{-1}(0)$ is the strongly parabolic locus.
- We give a gauge theoretic construction of the moduli space (as a complex manifold) using weighted Sobolev spaces and Lockhart-McOwen analysis (Taubes, Mrowka, Matić, Biquard, Daskalopoulos-W., Konno, Nakajima, Biquard-Boalch, ...).



- Model connection: $d_{A_0} = d + i\alpha d\theta$, $\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_\ell)$.
- Sobolev spaces $L^2_{k,\delta}$ with weight $e^{\tau\delta}$.
- $0 < \delta$ is chosen small depending on weights α.
- Connections are of the form $d_A = d_{A_0} + a$, $a \in L^2_{1,\delta}$.

- Admissible Higgs fields: $\Phi \in L^2_{1,-\delta}$ with $\bar{\partial}_{A_0}\Phi \in L^2_\delta$.
- Then $(\bar{\partial}_A, \Phi)$ defines a parabolic Higgs bundle with $\mathrm{eigres}(\Phi)$ possibly nonzero.
- **Technical point**. Need to control $[\Phi_1, \Phi_2^*]$. This is *not* a priori in L^2_δ, but it is *under* the assumption of full flags.
- The same method gives a construction of $\mathcal{M}^{par}_{dR}(\alpha)$, consisting of logarithmic connections with residues preserving flags.

Theorem (Collier-Fredrickson-W.) Assume either *strongly parabolic or full flags*. Then:

- the conformal limit exists under the same assumptions as in the closed surface case;
- it gives biholomorphisms between BB-strata in $\mathcal{M}^{par}_{Dol}(\alpha)$ and $\mathcal{M}^{par}_{dR}(\alpha)$.

Simpson's table:

NAH	$(\bar{\partial}_E, \Phi)$	$(\bar{\partial}_V, \nabla)$	D
jump	α	$\alpha - 2b$	$-2b$
eigenvalue	$b + ic$	$\alpha + 2ic$	$\exp(-2\pi i\alpha + 4\pi c)$

Conformal limit table:

CL	$(\bar{\partial}_E, \Phi)$	$(\bar{\partial}_V, \nabla)$	D
jump	α	α	$-b$
eigenvalue	$b + ic$	$\alpha + b + ic$	$\exp(-2\pi i(\alpha + b + ic))$

Thanks Sushmita & Nuno!!!