Quantization in covariant field theory with Poisson algebra bundles

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Joint work with O. Kravchenko and L. Ryvkin, arXiv:2407.15287

and many projects with the network team Dijon-Lyon-Metz

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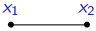
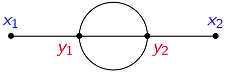
Correlation function for $\lambda\phi^4$ on $M = \mathbb{R}^{1,3}$:

$$\langle\Omega| T\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{\langle 0| T\phi(x_1)\phi(x_2)\ \exp\{-\frac{i\lambda}{\hbar}\int d^4y\ \phi(y)^4\}|0\rangle}{\langle 0| T\exp\{-\frac{i\lambda}{\hbar}\int d^4y\ \phi(y)^4\}|0\rangle}$$

$$= \Delta_F(x_1, x_2) + \frac{\lambda^2\hbar^2}{2}\int_{M\times M} dy_1\, dy_2 \Big[48\,\Delta_F(x_1, y_1)\Delta_F(y_1, y_2)^3\Delta_F(y_2, x_2)$$

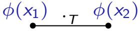
$$+ 48\,\Delta_F(x_1, y_2)\Delta_F(y_2, y_1)^3\Delta_F(y_1, x_2)\Big] + \mathcal{O}(\lambda^3)$$

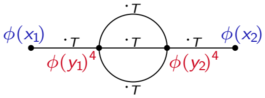
= 

$+\ \lambda^2\hbar^2$ 

$+\ \mathcal{O}(\lambda^3)$

= amplitude of a classical deformation:

$= \Big\langle 0\Big|$ 

$+\ \lambda^2\hbar^2$ 

$+\ \mathcal{O}(\lambda^3)\Big|0\Big\rangle$

Which deformation?

QFT as Hopf algebra deformation of classical observables

- **Twist deformation [Sweedler 1968, Rota-Stein 1987, Drinfeld 1990]:**
  Hopf algebra: H (graded) commutative
  Laplace pairing/2-cocycle: $(\ |\ ) : H \otimes H \to \mathbb{K}$ such that $\begin{cases} (1|a) = (a|1) = \varepsilon(a) \\ (ab|c) = \sum (a|c_{(1)})(b|c_{(2)}) \end{cases}$

  Twisted algebra: H with associative circle product $\boxed{a \circ b = \sum (a_{(1)}|b_{(1)})\, a_{(2)} b_{(2)}}$

- **QFT as twist deformation [Brouder-Fauser-F-Oeckl 2004] [Brouder 2009]:**
  Free fields: Hopf (super)algebra $H = "S(\phi^n(x) \,|\, n \geqslant 0, x \in M)"$

  $$\boxed{(\phi(x)|\phi(y)) = \begin{cases} \hbar\Delta_W(x,y) \text{ Wightmann} & \Rightarrow \quad \circ = \text{operator product} \\ \hbar\Delta_F(x,y) \text{ Feynman} & \Rightarrow \quad \circ = \cdot_T = \text{time-ordered product} \end{cases}}$$

  $$\boxed{\text{Expectation value} = \text{counit} \quad \begin{cases} \langle \phi(x_1)\cdots\phi(x_k)\rangle = \varepsilon(\phi(x_1)\circ\cdots\circ\phi(x_k)) \\ \langle T\phi(x_1)\cdots\phi(x_k)\rangle = \varepsilon(\phi(x_1)\cdot_T\cdots\cdot_T\phi(x_k)) \end{cases}}$$

  Interacting fields: $\mathcal{L}_{int}(\phi) = \phi^p \quad \Rightarrow \quad \exp_\circ(-\frac{i}{\hbar}\lambda\phi^p) = \sum_{n\geqslant 0} \frac{(-i\lambda)^n}{\hbar^n n!}(\phi^p)^{\circ n}$
  insert in counit, explicit Feynman graphs with symmetry factors!

  Renormalization: problem with $\phi^p(x)$ in time-ordered product
  [Brouder-Schmitt 2007]: second Hopf algebra on $\mathrm{Span}\{\phi^n(x)\} = "S(\phi(x))"$
  [Herscovich 2019]: two tensor products $\otimes$ and $\boxtimes$ on LCVS of sections

- **Problems:** 1) Formally consistent, but what is $H = "S(S(\phi(x)))"$?
  2) Which relation with renormalization Hopf algebras [Connes-Kreimer 2000, 2001]?

Classical observables as sections of a bundle

- **Data:** vector bundle $E \to M$ over spacetime + Lagrangian $\mathcal{L} : JE \to \mathrm{Dens}_M$

  Fields: $\mathcal{E}(M,E) = \{\text{smooth sections } \varphi : M \to E\}$ cf. [Rejzner 2016]

  $\mathcal{E}_{\mathcal{L}}(M,E) = \{\varphi \mid \text{Euler-Lagrange eq. } EL(\varphi) = 0\}$

  Observables: on-shell $C^\infty(\mathcal{E}_{\mathcal{L}}(M,E)) = \{F : \mathcal{E}_{\mathcal{L}}(M,E) \to \mathbb{R} \text{ smooth}\} \cong \frac{C^\infty(\mathcal{E}(M,E))}{V(\mathcal{E}_{\mathcal{L}}(M,E))}$

  start with off-shell $C^\infty(\mathcal{E}(M,E)) = \{F : \mathcal{E}(M,E) \to \mathbb{R} \text{ smooth}\}$

  both algebras with multiplication $(F_1 \cdot F_2)(\varphi) = F_1(\varphi) F_2(\varphi)$

  microcausal restrictions for Poisson bracket $\{F_1, F_2\}_{\mathcal{L}}(\varphi) = F_1'(\varphi)\Delta_C F_2'(\varphi)$

- **Bundle description of observables?**

  Local observables: $F_f(\varphi) = \int_M \langle f(x), j\varphi^n(x)\rangle \, dx$ represented by $f : M \to S^n(JE^*)$

  Multilocal observables: $(F_{f_1} \cdot F_{f_2})(\varphi) = \int_{M\times M} \langle f_1(x) \otimes f_2(y), j\varphi^{n_1}(x) \otimes j\varphi^{n_2}(y)\rangle \, dx \, dy$

  rep. by $f_1 \boxtimes f_2 : M \times M \to S^{n_1}(JE^*) \boxtimes S^{n_2}(JE^*)$ external tensor product of VB

  Problem: $F_{f_1} \cdot F_{f_2}$ is commutative, **but not** $f_1 \boxtimes f_2$ because $(x,y) \neq (y,x)$ in $M \times M$.

  Trick: **change category of vector bundles so that $\boxtimes$ becomes symmetric!**

  Need space of unordered configurations M^k/S_k.

  This is an orbifold, start with configurations of distinct points $(M^k \backslash \Delta^{(k)})/S_k$.

Vector bundles over configuration spaces

- **Configuration space of a manifold** M**:** $\mathrm{Conf}(M) = \bigsqcup_{k \geqslant 0} \mathrm{Conf}_k(M)$

  $\mathrm{Conf}_0(M) = \{\text{vacuum } \varnothing\}$

  $\mathrm{Conf}_k(M) = (M^k \backslash \Delta^{(k)})/S_k = \{\underline{x} = \{x_1, \dots, x_k\} \mid \text{distinct points } x_1, \dots, x_k \text{ in } M\}$

- **Category of** $\mathbb{K}$**-vector bundles over** $\mathrm{Conf}(M)$**:** $VB(\mathrm{Conf}(M))$

  vector bundle: $\mathbf{V} = \bigsqcup_{\underline{x} \in \mathrm{Conf}(M)} \mathbf{V}_{\underline{x}} = \bigsqcup_{k \geqslant 0} \mathbf{V}_k$ with $\mathbf{V}_k \to \mathrm{Conf}_k(M)$ usual VB

  bundle map: $\mathbf{V} \to \mathbf{W}$ = collection of usual bundle maps $\mathbf{V}_k \to \mathbf{W}_k$

- **Theorem:** $VB(\mathrm{Conf}(M))$ is a symmetric 2-monoidal category [Aguiar-Mahajan 2010]

  Hadamard $\quad (\mathbf{V} \otimes \mathbf{W})_{\underline{x}} = \mathbf{V}_{\underline{x}} \otimes \mathbf{W}_{\underline{x}} \qquad (\mathbf{I}_{\otimes})_{\underline{x}} = \mathbb{K}$

  Cauchy $\quad (\mathbf{V} \boxtimes \mathbf{W})_{\underline{x}} = \bigoplus_{\underline{x} = \underline{x}' \sqcup \underline{x}''} \mathbf{V}_{\underline{x}'} \otimes \mathbf{W}_{\underline{x}''} \qquad (\mathbf{I}_{\boxtimes})_n = \begin{cases} \mathbb{K}, & \underline{x} = \varnothing \\ 0, & \text{else} \end{cases}$

$\boxtimes$ is a symmetrized version of the external tensor product of bundles on M

Example: $(\mathbf{V} \boxtimes \mathbf{W})_{\{x,y\}} = \mathbf{V}_{\{x,y\}} \otimes \mathbf{W}_{\varnothing} \oplus \mathbf{V}_x \otimes \mathbf{W}_y \oplus \mathbf{V}_y \otimes \mathbf{W}_x \oplus \mathbf{V}_{\varnothing} \otimes \mathbf{W}_{\{x,y\}}$

- 2-**algebra bundle:** $\mathbf{A} \to \mathrm{Conf}(M)$ with

  $m_\otimes : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ $\qquad u_\otimes : \mathbf{I}_\otimes \to \mathbf{A}$

  $m_\boxtimes : \mathbf{A} \boxtimes \mathbf{A} \to \mathbf{A}$ $\qquad u_\boxtimes : \mathbf{I}_\boxtimes \to A$

  plus usual compatibility conditions.

  2-**coalgebra bundle:** $\mathbf{C} \to \mathrm{Conf}(M)$

  with dual maps $\Delta_\otimes$, $\varepsilon_\otimes$, $\Delta_\boxtimes$, $\varepsilon_\boxtimes$

  and dual relations

  $\Longrightarrow$ $\mathbf{C}^* = \mathrm{Hom}(\mathbf{C}, \mathbf{I}_\otimes)$ is a 2-algebra.

- $\otimes$ **and** $\boxtimes$**-tensors (co)algebras** from $E \to M$ and $E^* \to M$:

$$\begin{tikzcd}
\mathbf{T}^{a,\boxtimes}(E^*) = \bigoplus_n (E^*)^{\boxtimes n},\ \boxtimes \arrow[d, two heads, "\text{algebra}"'] & & \mathbf{T}^{c,\boxtimes}(E) = \bigoplus_n E^{\boxtimes n},\ \Delta_\boxtimes \arrow[ll, "gr*"'] \\
\mathbf{S}^\boxtimes(E^*) = \bigoplus_n (E^*)^{\boxtimes n}/S_n,\ \boxdot \arrow[rr, "\cong", "Sym"'] & & \mathbf{\Sigma}^\boxtimes(E) = \bigoplus_n (E^{\boxtimes n})^{S_n},\ \Delta_\boxdot \arrow[u, hook, "\text{coalgebra}"']
\end{tikzcd}$$

(same for $\otimes$)

- **Tensor** 2-**(co)algebra bundles:** combine $\otimes$ and $\boxtimes$-tensors

$$\boxed{\mathbf{S}^\boxtimes S^\otimes(E^*) \cong \left(\mathbf{\Sigma}^\boxtimes \Sigma^\otimes(E)\right)^*}$$

Poisson 2- and $\boxtimes$-algebra bundles

- **Poisson** 2**-algebra bundle:** $(\mathbf{P}, \bullet_\otimes, \bullet_\boxtimes)$ commutative with $\boxed{\{\ ,\ \} : \mathbf{P} \boxtimes \mathbf{P} \to \mathbf{P}}$

  antisymmetry: $\{a_{\underline{x}}, b_{\underline{y}}\} = -\{b_{\underline{y}}, a_{\underline{x}}\}$ over $\underline{x} \sqcup \underline{y}$

  Jacobi identity: $\{\{a_{\underline{x}}, b_{\underline{y}}\}, c_{\underline{z}}\} + \{\{b_{\underline{y}}, c_{\underline{z}}\}, a_{\underline{x}}\} + \{\{c_{\underline{z}}, a_{\underline{x}}\}, b_{\underline{y}}\} = 0$

  $\bullet_\boxtimes$-Leibniz rule: $\{a_{\underline{x}}, b_{\underline{y}} \bullet_\boxtimes c_{\underline{z}}\} = \{a_{\underline{x}}, b_{\underline{y}}\} \bullet_\boxtimes c_{\underline{z}} + b_{\underline{y}} \bullet_\boxtimes \{a_{\underline{x}}, c_{\underline{z}}\}$

  $\bullet_\otimes$-Leibniz rule: $\{a_{\underline{x}}, b_{\underline{y}} \bullet_\otimes c_{\underline{y}}\} = \{a_{\underline{x}}, b_{\underline{y}}\} \bullet_\otimes (1_{\underline{x}} \bullet_\boxtimes c_{\underline{y}}) + (1_{\underline{x}} \bullet_\boxtimes b_{\underline{y}}) \bullet_\otimes \{a_{\underline{x}}, c_{\underline{y}}\}$

- **Theorem:** In $VB(\mathrm{Conf}(M))$ we have:
  1. A vector bundle $E \to M$ determines a 2-algebra bundle $\mathbf{S}^\boxtimes S^\otimes (JE)^*$ with $\odot$, $\boxdot$.
  2. Any antisymmetric bundle map $k : (JE)^* \boxtimes (JE)^* \to I_\otimes \boxdot I_\otimes \cong \mathbb{K}$ over $\mathrm{Conf}_2(M)$ determines a Poisson bracket on $\mathbf{S}^\boxtimes S^\otimes (JE)^*$ with $\boxdot$.
  3. Densities $\mathrm{Dens}_{\mathrm{Conf}(M)} \cong \mathbf{S}^\boxtimes(\mathrm{Dens}_M)$ form a commutative algebra bundle with $\boxdot$.
  4. Hence k gives a **Poisson $\boxtimes$-algebra bundle on** $\mathrm{Conf}(M)$ with $\boxdot$

$$\boxed{\mathbf{P}(E) = \mathbf{S}^\boxtimes S^\otimes (JE)^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)} \cong \left(\mathbf{\Sigma}^\boxtimes \Sigma^\otimes (JE)\right)^\vee}$$

where $\mathbf{V}^\vee = \mathbf{V}^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}$ is the functional dual.

Poisson algebras of sections and distributions

- **Sections:** $\phi : \mathrm{Conf}(M) \to \mathbf{V}$ = collection of usual sections $\phi_k : \mathrm{Conf}_k(M) \to \mathbf{V}_k$
- **Distributions:**

$$
\left\{
\begin{array}{lccc}
 & \text{compact support} & & \text{smooth} \\
\text{regular} & \mathcal{D}(\mathrm{Conf}(M), \mathbf{V}^\vee) & \hookrightarrow & \mathcal{E}(\mathrm{Conf}(M), \mathbf{V}^\vee) \\
 & \downarrow & & \downarrow \\
\text{all} & \mathcal{E}'(\mathrm{Conf}(M), \mathbf{V}^\vee) & \hookrightarrow & \mathcal{D}'(\mathrm{Conf}(M), \mathbf{V}^\vee) \\
 & = \mathcal{E}(\mathrm{Conf}(M), \mathbf{V})' & & = \mathcal{D}(\mathrm{Conf}(M), \mathbf{V})'
\end{array}
\right.
$$

- **Theorem:** If $\mathbf{P}$ is a Poisson $\boxtimes$-algebra bundle on $\mathrm{Conf}(M)$, then

$$\boxed{\mathcal{E}(\mathrm{Conf}(M), \mathbf{P}) \quad \text{and} \quad \mathcal{D}'(\mathrm{Conf}(M), \mathbf{P}) \quad \text{are Poisson algebras}}$$

(While $\mathcal{D}(\mathrm{Conf}(M), \mathbf{P}) \subset \mathcal{E}(\mathrm{Conf}(M), \mathbf{P})$ is not a subalgebra!)

- **Corollary:** Apply to $\mathbf{P}(E) = \mathbf{S}^{\boxtimes} S^{\otimes}(JE)^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}$ with Poisson bracket from an antisymmetric bundle map $k : (JE)^* \boxtimes (JE)^* \to \mathbb{K}$ over $\mathrm{Conf}_2(M)$, same as a smooth section $k \in \mathcal{E}(\mathrm{Conf}_2(M), \Lambda^{\boxtimes 2}(JE))$, i.e. a regular distribution $k \in \mathcal{D}'(\mathrm{Conf}_2(M), \Lambda^{\boxtimes 2}(JE))$.

- **Claim:** 1) The causal propagator is an antisymmetric distributional kernel k yielding a Poisson bracket on sections of $\mathbf{P}(E) = \mathbf{S}^{\boxtimes} S^{\otimes}(JE)^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}$.
  2) There is a Poisson map (non injective!)

$$\boxed{F : \mathcal{D}'_{\mu c}(\mathrm{Conf}(M), \mathbf{P}(E)) \longrightarrow C^\infty(\mathcal{E}(M,E))}$$

  sending a distribution T to the function $F_T : \mathcal{E}(M,E) \to \mathbb{R}$ defined on a field φ by

$$\boxed{F_T(\varphi) = \langle T, e(j\varphi)\rangle = \int_{\mathrm{Conf}(M)} \langle T(\underline{x}), e(j\varphi)(\underline{x})\rangle}$$

  where $e(j\varphi) = \sum \frac{1}{n_1! \cdots n_k!} (j\varphi)^{\otimes n_1} \boxtimes \cdots \boxtimes (j\varphi)^{\otimes n_1}$ is a section of $\mathbf{\Sigma}^{\boxtimes}\hat{\Sigma}^{\otimes}(JE)$.

- **Idea:** Bornology + microcausality for $\mathcal{D}'_{\mu c}$ [Brunetti-Fredenhagen-Ribero 2012]

$$\begin{tikzcd}
\mathcal{D}(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[r, hook] \arrow[d, hook, "\text{not algebra}"'] & \mathcal{E}_{\mu c}(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[r, hook] \arrow[d, hook, "\text{algebra}"'] & \mathcal{E}(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[d, hook, "\text{algebra}"'] \\
\mathcal{E}'(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[r, hook] \arrow[dr, "F"'] & \mathcal{D}'_{\mu c}(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[r, hook] \arrow[d, "F?"'] & \mathcal{D}'(\mathrm{Conf}(M), \mathbf{P}(E)) \arrow[dl, squiggly, "\nexists F"] \\
 & C^\infty(\mathcal{E}(M,E), \mathbb{R}) &
\end{tikzcd}$$

- Quantize $\mathcal{D}'_{\mu c}(\mathrm{Conf}(M), \mathbf{P}(E))$ using the Feynman propagator as a Laplace pairing: works on bundles (formally already done), to do on distributions, wait for orbifolds! [with P. Bonneau and G. Dito, cf. E. Herscovich]
- Include diagonals with multi-configuration orbifolds M^k/S_k: renormalization should show up classically as blow up on singular diagonals [PhD H.C. Nguyên, and with L. Ryvkin, I. Khavkine, A. Kotov]
- Double Poisson bracket and r-matrix [with M. Fairon, O. Kravchenko, cf. Appendix]
- Multilocal observables in Multisymplectic Geometry and Lagrangian Field Theory: higher Poisson algebras to resolve the kernel of F [with A. Miti, L. Ryvkin and C. Laurent-Gengoux, T. Wurzbacher]
- Factorization algebras as sections of Poisson bundles over $\mathrm{Conf}(M)$ [with O. Kravchenko, A. Thomas], links with pAQFT à la Fredenhagen, Rejzner et al.
- Higher groupoid symmetries for gauge fields [PhD S. Amiel and with M. D'Agostino, A. Miti] and for spinors [someone interested?]
- Which structure to resolve on-shell observables? [ideas? someone interested?]

Thank you for the attention

- **Double Poisson algebra [Van den Bergh 2008]:** unital associative algebra (A, m) with double bracket $\{\!\{\ ,\ \}\!\} : A \otimes A \to A \otimes A$ satisfying suitable skew-symmetry, Jacobi and Leibniz identities. Then $A/[A, A]$ is a Poisson algebra with $\{\ ,\ \} = m \circ \{\!\{\ ,\ \}\!\}$.

- **Theorem [Fairon-F-Kravchenko 2025]:**
  Any skew-symmetric bundle map $\tilde{k} : V \boxtimes V \to I_\otimes \boxtimes I_\otimes$ over $\mathrm{Conf}_2(M)$ gives a quadratic double Poisson bracket $\{\!\{\ ,\ \}\!\}$ on $\mathbf{S}^{\boxtimes} S^{\otimes}(V)$ over $\mathrm{Conf}(M)$ such that $\{\ ,\ \} = \boxdot \circ \{\!\{\ ,\ \}\!\}$ is the Poisson bracket induced by $k = \boxdot \circ \tilde{k}$.

- **Corollary after [Odesskii-Roubtsov-Sokolov 2013]:**
  On $\mathbf{S}^{\boxtimes} S^{\otimes}(JE)^*$ there is a classical antisymmetric (associative) r-matrix: if $\{e_N(x) = e_1^{n_1}(x) \cdots e_\ell^{n_\ell}(x) \mid N = (n_1, \dots, n_\ell) \in \mathbb{N}^\ell\}$ is a basis of $S^{\otimes}(JE)^*_x$, then:
  on 2-generators $(JE)^*$: $\{\!\{e_i(x), e_j(y)\}\!\} = \tilde{k}(e_i(x), e_j(y)) \in (I_\otimes \boxtimes I_\otimes)_{\{x,y\}}$
  on $\boxtimes$-generators $S^{\otimes}(JE)^*$:
  $$\begin{aligned}\{\!\{e_N(x), e_M(y)\}\!\} &= \sum_{i,j} n_i m_j \tilde{k}(e_i(x), e_j(y))\, e_{n_1,\dots,n_i-1,\dots,n_\ell}(x) \boxtimes e_{m_1,\dots,m_j-1,\dots,m_\ell}(y) \\ &= \sum_{JK} r_{NM}^{JK}(x, y)\, e_J(x) \boxtimes e_K(y).\end{aligned}$$

- **Ongoing:** what can we say on distributions of $\mathbf{P}(E) = \mathbf{S}^{\boxtimes} S^{\otimes}(JE)^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}$?