Semiclassical kinetic theory of integrable systems

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Collaborators

Main focus will be recent work (arXiv 2107.06157) on Calogero models with

- Xiangyu Cao (ENS)
- Manas Kulkarni (ICTS)
- Joel E. Moore (Berkeley)

Other collaborators on one-dimensional hydrodynamics:

- Christoph Karrasch (TU Braunschweig)
- Romain Vasseur (UMass)
- Herbert Spohn (TUM)

Motivations

- Understand how external trapping potentials "weakly" break integrability
- Apply recent advances in hydrodynamics of quantum integrable systems to model *classical* integrable systems
- Resulting notion of **classical quasiparticle** sheds light on dynamics and transport at/near integrability

The original non-thermalizing system

- In 1955, Fermi-Pasta-Ulam-Tsingou looked at dynamics in some anharmonic chains, e.g. the FPU β-model

$$\ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \beta[(x_{n+1} - x_n)^3 - (x_n - x_{n-1})^3].$$

- Expected ergodicity for $\beta \neq 0$. But not observed - perfect revivals at long times instead.
- Informally: continuum limit is KdV (integrable), close to harmonic chain (integrable), there exist integrable anharmonic chains (e.g. Toda), there is the KAM theorem.
- **But hard to make any of this precise.**

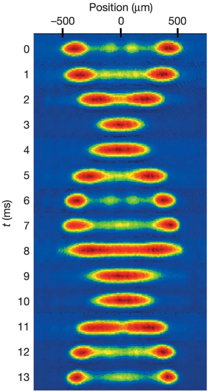
The quantum Newton's cradle

- Quantum Newton's Cradle (*Kinoshita, Wenger, Weiss, '08*) realized

$$H = \sum_{i=1}^{N} \frac{1}{2}\left(-\partial_i^2 + \omega^2 x_i^2\right) + c \sum_{i<j} \delta(x_i - x_j)$$

  with quasi-1D clouds of $^{87}\mathrm{Rb}$.

- **No thermalization on accessible timescale** (around 30τ).
- Like FPU, integrable limits at $\omega = 0$, $c = 0$, $c = \infty$ so always "near integrable". Again difficult to quantify.



How does classical integrability break?

- Classical integrability breaking understood since 50s-60s (Kolmogorov-Arnold-Moser) but the result is subtle to use.
- Rough idea: let H_0 be integrable, V an integrability-breaking perturbation. Consider $H = H_0 + \epsilon V$.
- A torus $(\vec{I}, \vec{\omega})$ of H_0 survives if $\vec{\omega}$ satisfies

$$|\sum_{i=1}^{N} n_i \omega_i| > K/\|\vec{n}\|_1^{\tau}, \tag{1}$$

  with $\tau > N - 1$ and $K \gg \epsilon^{1/2}$.
- Cantor-like set ("nowhere dense") - integrability preserved in a fraction $1 - \mathcal{O}(\epsilon^{1/2})$ of phase space (in measure)

Why is KAM not "useful" for physicists?

- Only yields sufficient condition for tori to survive.
- Set of "good" frequencies Δ highly fractal - impossible to tell with finite precision if $\vec{\omega} \in \Delta$.
- Underappreciated non-degeneracy condition $|\partial\vec{\omega}/\partial\vec{I}| \neq 0$ i.e. **no direct application to perturbed harmonic systems (like FPU and QNC...)**
- Proof does not appear to generalize to quantum systems.

How does quantum integrability break?

- Poorly understood in general. But nice physical picture for $N = 3$ (*Lamacraft, '12*, after Sutherland)
- In reduced 3-body problem, Yang-Baxter relations imply "geometrical optics" for scattering:

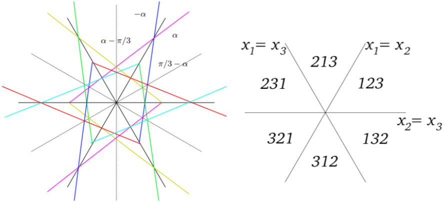
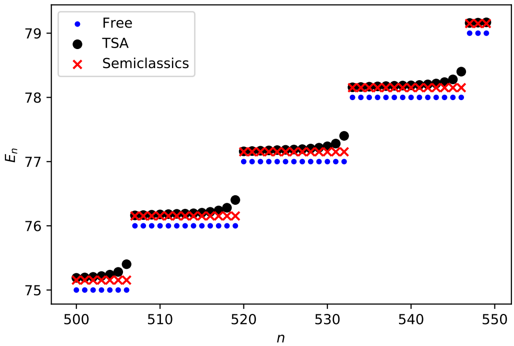


Figure: visualization of Yang-Baxter relations for reduced three-body problem, from *Lamacraft, '12*

- **Amplitude of diffracted wave proportional to violation of Yang-Baxter**

The three-body quantum Newton's cradle

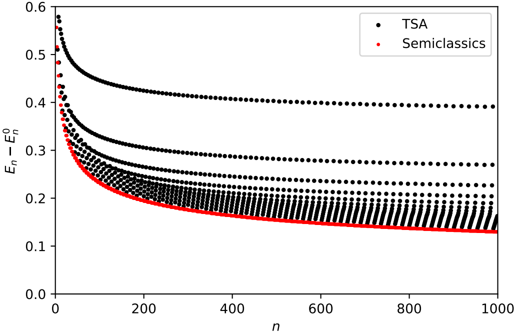
- Let's tackle $N = 3$ quantum Newton's cradle directly (*VBB, PhD thesis, '20*)



- Nowhere near Poisson/GOE - closer to oscillator.
- Nice "diffraction" intuition difficult to apply (wavefunctions decay at infinity)

Semiclassical features of three-body quantum Newton's cradle

- The differences between energy levels of QNC and harmonic oscillator have **remarkable regular features**.



- Semiclassical red line seems to capture infimum of quantum spectrum. Where does this come from?

Bethe's ansatz as Einstein-Brillouin-Keller quantization

- One viewpoint on Bethe ansatz equations is that they represent "exact" EBK quantization e.g. Lieb-Liniger on a ring:

$$p_a L = 2\pi n_a + \sum_{a \neq b} \varphi(p_a - p_b), \quad a = 1, 2, \ldots, N$$

  with $\varphi(p) = -2 \arctan p/c$ the Lieb-Liniger phase shift.
- Write this as

$$\frac{1}{2\pi} \oint p_a dq_a = n'_a + \frac{1}{2} \sum_{a \neq b} b(p_a - p_b),$$

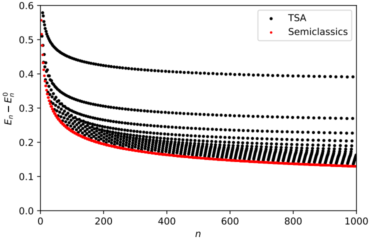
  where $n'_a = n_a - N + 1 \in \mathbb{Z}$, and to each collision plane $\{x_a = x_b\}$, we assign a fractional Maslov index

$$b(p) = 1 + \frac{1}{\pi} \varphi(p),$$

  depending on the momentum incident on that plane.

Semiclassical quantization of quantum Newton's cradle

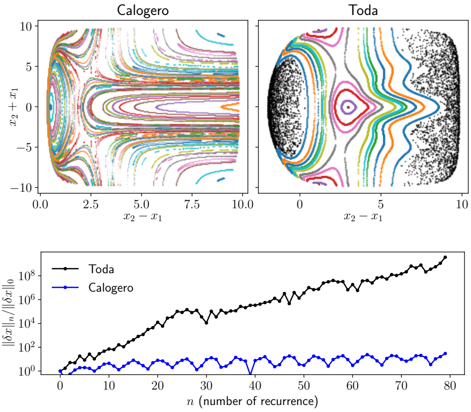
- Let's now add harmonic trap. Substituting interactions for collision planes in semiclassics no longer exact.
- For $0 < c < \infty$, classical orbits are no longer obvious (cf. Gutzwiller). One tractable case is circular orbits; treating phase-shift like Maslov index yields non-linear quantization condition $E = 2n + 3l + 1 + 3b(\sqrt{E}), \quad l, n = 0, 1, \ldots$.



- **Current kinetic theories of quantum Newton's cradle in similar regime of approximation.**

Related classical physics

- It appears that classical integrable systems in integrability-breaking traps can also behave non-ergodically (*Cao, VBB, Moore, '18, Lebowitz, Scaramazza, '18, Di Cintio, Iubini, Lepri, Livi, '18, Dhar, Kundu, Lebowitz, Scaramazza, '19, VBB, Kulkarni, Moore, Cao, '21*)



...back to hydrodynamics

- "Weak integrability breaking" by trap leads to rich physics whose particulars continue to pose a theoretical challenge.
- Advent of generalized hydrodynamics yields testable predictions for large-scale dynamics in trap (...see *Bastianello, De Luca, Vasseur, '21* for up-to-date review.)
- Concurrent resurgence of interest in "quantum Newton's cradle" experiments (see *Bouchoule, Dubail, '21* for up-to-date review.)
- **To probe theoretical questions of chaos and equilibration, classical systems easier for now.**

Introduction: integrability breaking

Kinetic theory of Calogero particles

- Background
- Derivation

The Bethe-Boltzmann equation

- Basic ingredient in applying "generalized hydrodynamics" formalism is kinetic equation for the single-(quasi)particle distribution function

$$\partial_t \rho_k + \partial_x (v_k[\rho]\rho_k) = 0.$$

  with velocity dressed by semiclassical time delays (*Wigner, '53*):

$$v_k[\rho] = v_k^0 + \int dk' \, \rho_{k'} \left(v_k[\rho] - v_{k'}[\rho]\right) \underbrace{\left(\hbar \frac{d\varphi_{k,k'}}{dE_k} v_k^0\right)}_{\Delta x_{k,k'}}$$

- As noted above, Bethe ansatz can be viewed as "exact semiclassical quantization"; one way to understand validity of this expression (rigorous justification increasingly available, e.g. *Spohn,Yoshimura,'20, Pozsgay,'20*)

Quasiparticles in external trapping potentials

- One puts Boltzmann in a trap by adding a force term $-V'(x)\partial_k \rho_k$. Same reasoning here yields

$$\partial_t \rho_k + \partial_x (v_k[\rho]\rho_k) - V'(x)\partial_k \rho_k = 0.$$

- We saw above that traps interfere with simple underlying kinematic model even for $N = 3$. So this is a conjecture to be tested.
- Seems to work well vs experiments/numerics at short times. Deeper questions around time to chaos remain.

Traps that preserve integrability

- The Calogero model has the remarkable property of **remaining integrable in certain traps**.
- e.g. the Hamiltonian (quantum or classical)

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i<j} \frac{g}{(x_i - x_j)^2} + \sum_i \frac{1}{2} \omega^2 x_i^2$$

  is **integrable with perfect revivals** (isochronous).
- Non-trivial test case for naïve ansatz {quasiparticle kinetics + Boltzmann force term}
- First we need quantum and classical hydrodynamics.

Introduction: integrability breaking

Kinetic theory of Calogero particles
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What goes into quasiparticle kinetic theory?

- Ingredients:
  - **Differential phase-shift**:

$$K_{k,k'} = \frac{1}{2\pi}\varphi'(k-k')$$

  - **Quasiparticle energies** (Yang-Yang equation):

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} - \mu + \frac{1}{\beta}\int_{-\infty}^{\infty} dk'\, K_{k,k'} \log\left(1 + e^{-\beta\epsilon_{k'}}\right)$$

  which defines **Fermi factor**

$$\theta_k = (1 + e^{\beta\epsilon_k})^{-1}$$

  - **Kinetic equation** for density of states:

$$\partial_t \rho_k + \partial_x(\rho_k v_k[\rho]) = 0, \quad v_k[\rho] = \frac{\epsilon'_k}{{p^{\mathrm{dr}}}'_k}$$

Kinetic theory of quantum Calogero I

- Write quantum Hamiltonian as ($m = 1$, keep $\hbar$)

$$H = \sum_{i=1}^{N} -\frac{1}{2}(\hbar \partial_i)^2 + \sum_{i<j} \frac{\hbar^2 \alpha(\alpha - 1)}{(x_i - x_j)^2}.$$

- Differential phase shift just a delta function

$$K_{k,k'} = (\alpha - 1)\delta(k - k')$$

so **Bethe integral equations become algebraic equations**.

- In particular, dressing of energy/momentum is just a prefactor...

$$(1 + (\alpha - 1)\theta_k)\epsilon'_k = \hbar^2 k\,,$$
$$(1 + (\alpha - 1)\theta_k){p^{\mathrm{dr}}}'_k = \hbar\,.$$

- ...and so "quasiparticle group velocity" equals the bare velocity:

$$v_k = \epsilon'_k / {p^{\mathrm{dr}}}'_k = \hbar k.$$

Kinetic theory of quantum Calogero II

- Thus kinetic equation of quantum Calogero plus naïve force term

$$\partial_t \rho_k + (\hbar k)\partial_x \rho_k - (V'(x)/\hbar)\partial_k \rho_k = 0$$

is the "freely streaming Boltzmann equation".

- In harmonic trap, has **non-trivial property of microscopic quantum dynamics** - perfect revivals!
- Extension of earlier "superfluid" hydrodynamics (*Abanov, Wiegmann, '05, Abanov, Bettelheim, Wiegmann, '08, Abanov, Gromov, Kulkarni, '11*) to non-zero temperatures, minus dispersion.

Kinetic theory of classical Calogero

- For microscopic tests, we need classical equation. Follow the prescription we developed for Toda (*VBB, Cao, Moore, '19, after Theodorakopoulus, '84, Gruner-Bauer, Mertens, '88*).
- Set $p = \hbar k$ before taking $\hbar \to 0$ with $\ell = \hbar\alpha$. Semiclassical phase shift is:

$$K^{cl}_{p,p'} = \lim_{\hbar \to 0} K_{p/\hbar, p'/\hbar} = \ell\delta(p - p').$$

- Need to cancel entropic factors in free energy as $\hbar \to 0$:

$$\epsilon^{cl}_p = \epsilon_{k=p/\hbar} + \frac{1}{\beta}\ln \hbar/\ell, \quad \mu^{cl} = \mu - \frac{1}{\beta}\ln \hbar/\ell\,.$$

- For experts: effective semiclassical “Fermi factor”, “backflow function”, “density of states” given by

$$\widetilde{\theta}_p = \theta_k/\hbar, \quad \partial_{p'}\widetilde{F}(p,p') = \hbar\partial_{k'}F(k,k'), \quad \widetilde{\rho}_p = \rho_k/\hbar.$$

- Once dust settles, get classical free Boltzmann equation:

$$v_p = {\epsilon^{cl}}'_p / {p^{\mathrm{dr}}}'_p = p.$$

Does semiclassical kinetic theory work?

- **Yes** - test vs microscopic "two reservoir" initial condition:

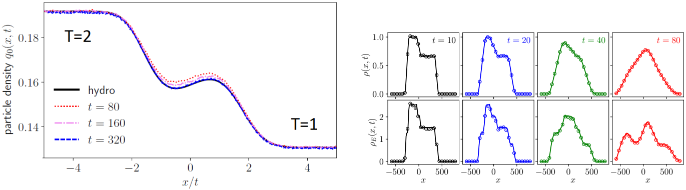


Figure: Left: dynamics of particle density in Toda, right: same for confined Calogero with temperatures reversed.

(related: semiclassical hydro of sinh-Gordon, *Bastianello, Doyon, Watts, Yoshimura, '18*, Toda hydro from classical arguments, *Spohn, '19, Doyon, '19,...*, see arXiv 2101.06528 (Spohn) for extensive review.)

Which degrees of freedom are streaming?

- Conjectured classical/quantum kinetic equations have same "streaming" form, e.g. in harmonic trap:

$$\partial_t \widetilde{\rho}_p + p\partial_x \widetilde{\rho}_p - \omega x \partial_p \widetilde{\rho}_p = 0.$$

- Looks like free particles but bare particles have nontrivial dynamics! What is doing the streaming?
- Clue from Toda: **classical quasiparticles** defined via "Bethe-Lax correspondence" (*VBB, Cao, Moore, '19*)

A Bethe-Lax correspondence

- Letting $X = \mathrm{diag}(x_1, x_2, \ldots, x_N)$, can write Calogero dynamics with matrices $L(x, \dot{x})$, $A(x)$ such that

$$\dot{X} + i[X, A] = L\,,$$
$$\dot{L} + i[L, A] = -V'(X).$$

- Idea: eigenvalue of L = "classical quasiparticle", spectral density of L = semiclassical DoS.
- With no trap, L has isospectral flow; thus

$$\langle \lambda_j | \dot{X} | \lambda_j \rangle = \lambda_j, \quad L|\lambda_j\rangle = \lambda_j |\lambda_j\rangle.$$

- So once we identify

$$\widetilde{\rho}_p = \lim_{\substack{N,L\to\infty,\\ N/L=\mathrm{const}}} \overline{\frac{1}{L}\sum_{j=1}^{N} \delta(p - \lambda_j)},$$

get **fully microscopic derivation** of non-interacting Boltzmann form $\widetilde{\rho}_p^J = p\widetilde{\rho}_p$.

Extension to traps

- Let's reconsider Lax equations:

$$\dot{X} + i[X, A] = L\,,$$
$$\dot{L} + i[L, A] = -V'(X).$$

- Known to be integrable for $V(x) = ax^4 + bx^3 + cx^2 + d$. But higher order V are less tractable.
- Leanest way to generalize to arbitrary traps is take "wavefunction" interpretation of $|\lambda_j\rangle$ literally:

$$\rho_p^{\text{emp}}(x) := \sum_{a=1}^{N} \sum_{j=1}^{N} |\langle x_a|\lambda_j\rangle|^2 \delta(x - x_a)\delta(p - \lambda_j) \qquad (2)$$

- Intuition: weight of "classical quasiparticle" j distributed over all positions.

Anderson localization of quasiparticles

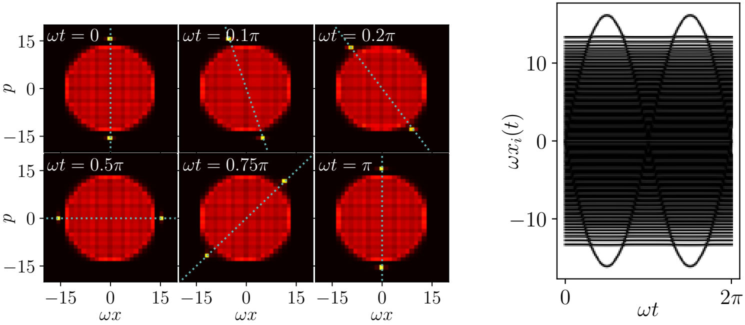
- At non-zero temperature, L is a random matrix with particular structure. Eigenvectors $|\lambda_j\rangle$ subject to “Anderson localization” in site index j.
- e.g. for Toda, Lax matrix is nearest neighbour, for Calogero it is long-ranged:

$$L_{ij}^{\mathrm{Calogero}} = \delta_{ij} p_i + (1 - \delta_{ij}) i\ell x_{ij}^{-1}.$$

- In both cases, see localization at $T > 0$ (resp. exponential and power law)
- Corrections to Boltzmann equation for $\rho_p^{\mathsf{emp}}(x)$ scale as $\mathcal{O}([X, L])$ - **valid insofar as quasiparticles localized**!

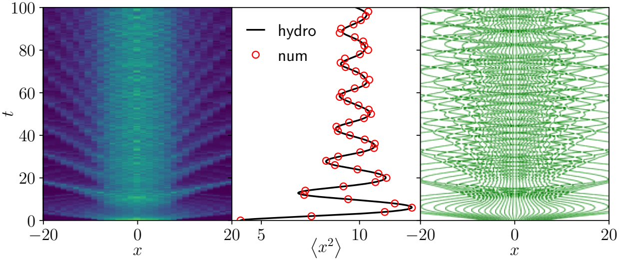
Application 1: solitons in a harmonic trap

- Known that classical Calogero in a harmonic trap has soliton solutions (*Abanov, Gromov, Kulkarni, '11*). Consider two-soliton initial condition.
- Using proposed hydro variable $\rho_p^{\text{emp}}(x)$, solitons appear as stable peaks in $x - p$ space:



Application 2: dynamics in a (presumably) non-integrable trap

- Next consider preparing a $T = 0$ cloud in a harmonic potential $V(x) = x^2/2$, before quenching to an anharmonic potential, $V(x) = 2\sqrt{1 + x^2}$.
- No breakdown of hydro of $\rho_p^{\text{emp}}(x)$ on accessible timescales (unlike for hard rods - see *Cao, VBB, Moore, '18*).
- **When/how does hydro break down?** Apparently no diffusion in Navier-Stokes limit!



Some open questions

- What is mechanism of thermalization here? Clearer in compact Calogero models?
- Microscopic expression for ρ_p^{emp} allows study of corrections to all orders, in principle. Do these match collective field theory at $T = 0$? (*Abanov, Wiegmann, '05*)
- Striking analytical derivation of thermodynamic Bethe ansatz for classical Toda (*Spohn, '19*). Analogue for Calogero?

Thanks for listening!