Branching Random Walks: Two Conjectures and a Theorem

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Joint work with Ayan Bhattacharya and Rajat Subhra Hazra

Indian Statistical Institute

June 6, 2022

We shall start with an icebreaker

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What is an ice-breaker?

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According to *Merriam-Webster:* it means

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According to *Merriam-Webster:* it means

1. a ship equipped (as with a reinforced bow) to make and maintain a channel through ice (see the picture above);
2. something that is done or said to get through the first difficulties in starting a conversation or discussion.

An icebreaking question

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About a decade ago, I was travelling by a train on a lazy Sunday afternoon. I met two school-students who asked me what mathematicians do.

What do we do?

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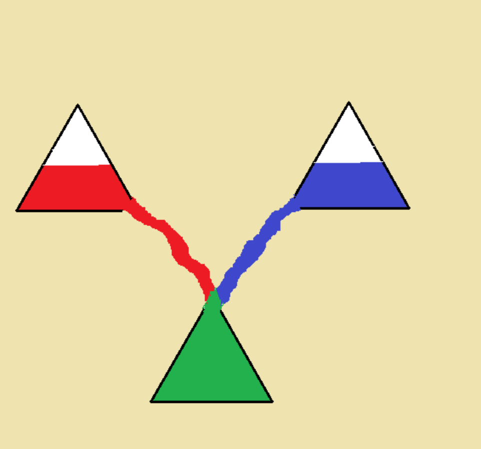
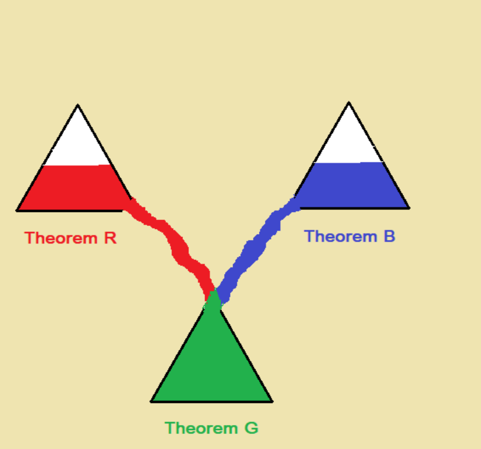


What the heck do we do?



A few things before we start

- Will go very very slowly.
- Please feel free to ask questions in the chat. The moderator will ask the questions on your behalf.
- After each 15 minutes (approximately), we will take a break and allow questions + comments using your microphone.
- In the end, we will also have a detailed discussion session.
- Please ask questions. Really want the talk to be interactive.

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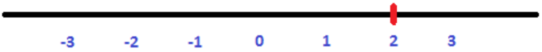

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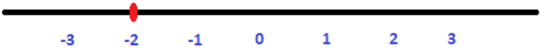

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5. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
6. Repeat 4 and 5 again and again.


QUESTIONS?
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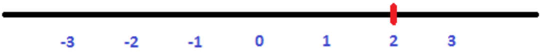

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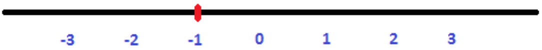

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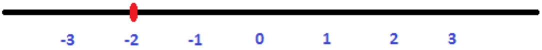

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In case of simple random walk, $p_1 = p_{-1} = \frac{1}{2}$ and all other p_i's are 0.

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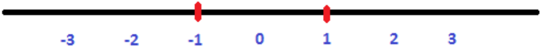

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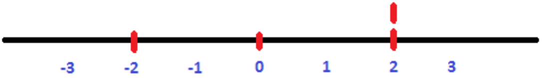

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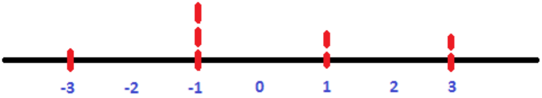

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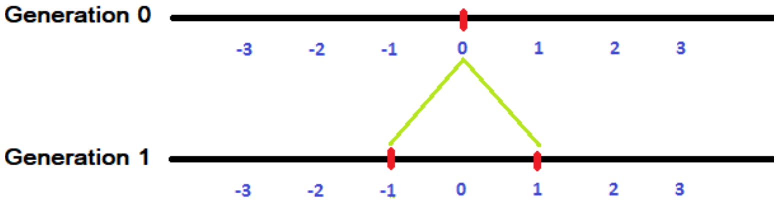

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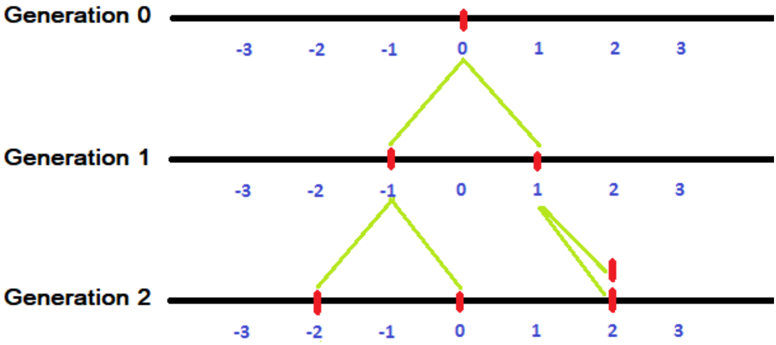

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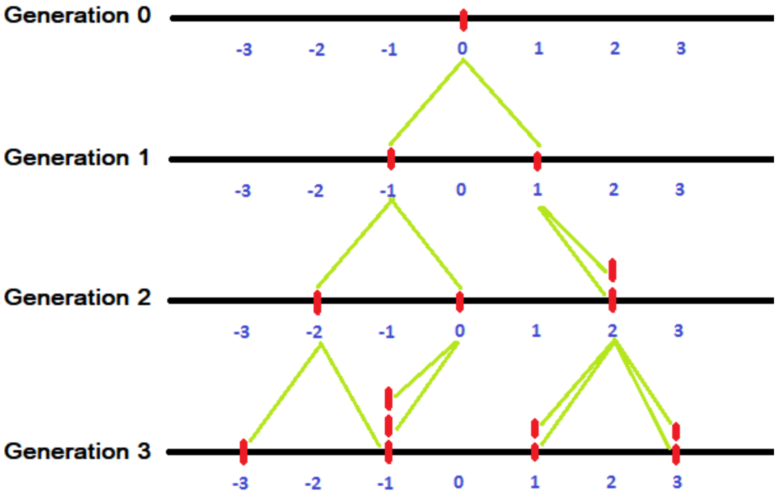

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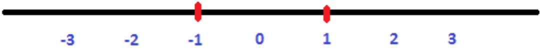

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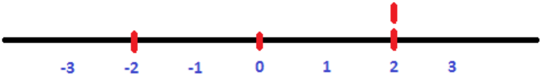

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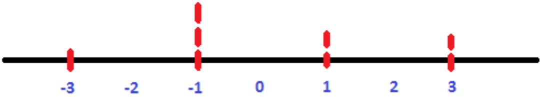


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This model was introduced by Hammerseley (1974), Kingman (1975) and Biggins (1976).

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- In our rather simplified model, we only allow the particles to move along a single line (or even integers).
- More complicated models can also be considered where particles move in a plane or in a box.
- For the purpose of this talk, we shall restrict ourselves to the simple model and talk about the *long run configuration* of the positions of particles.

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- The *long run configuration* is of great importance in statistical physics, mathematical biology and probability theory.

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Key Question: What if the conditions are not satisfied?

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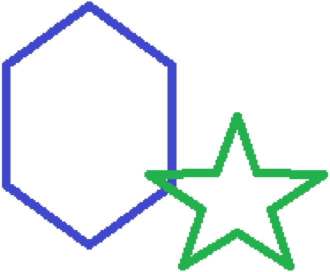

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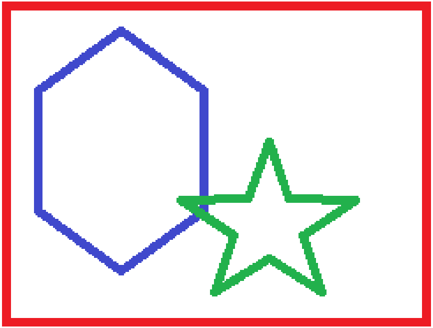

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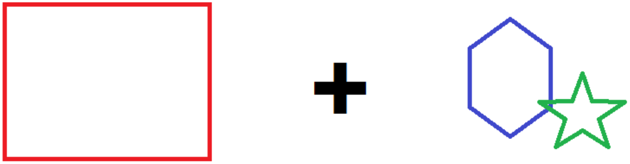

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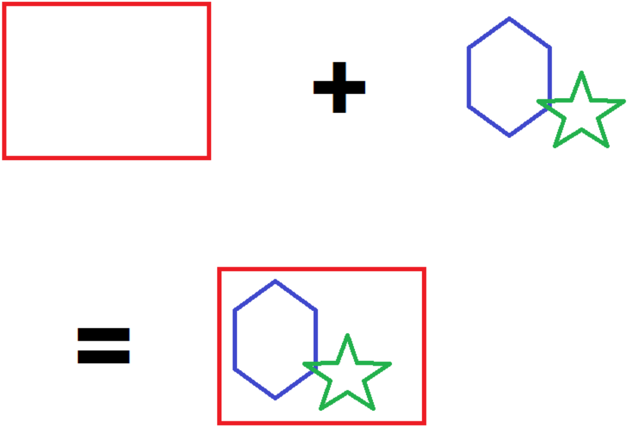

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- We let both of them run the dynamics for many many generations and get two pictures (or snapshots).

- Superposability means if we superpose these two pictures, we won't see any qualitative difference with either of them.


QUESTIONS?
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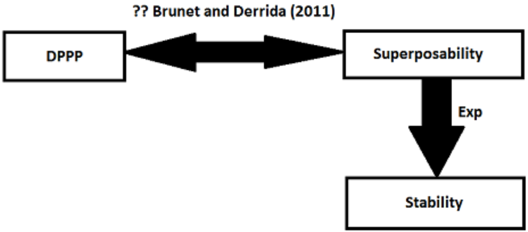

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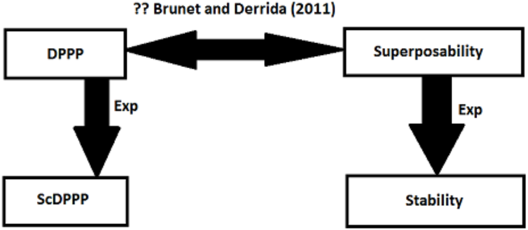

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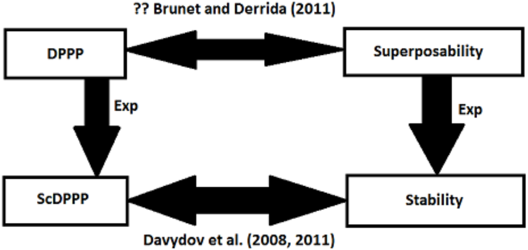

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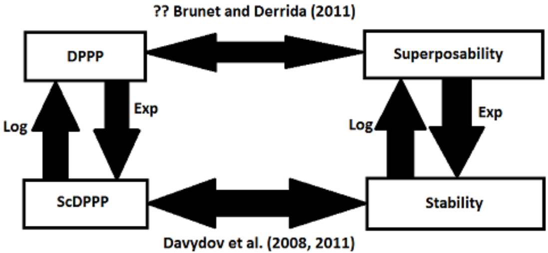

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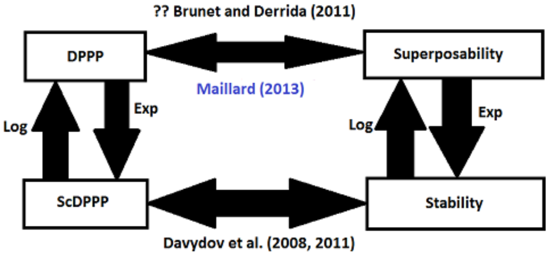

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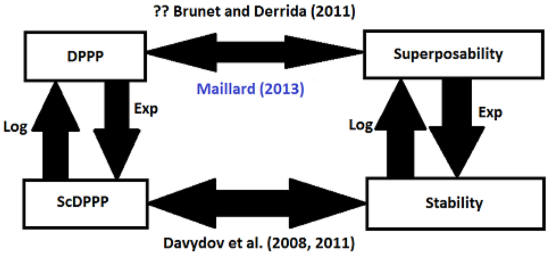

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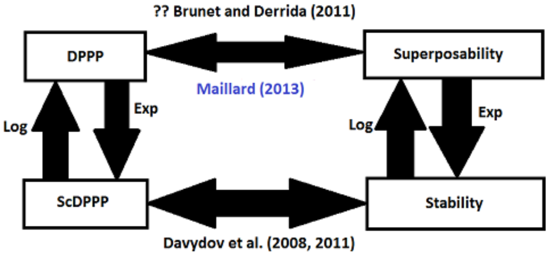

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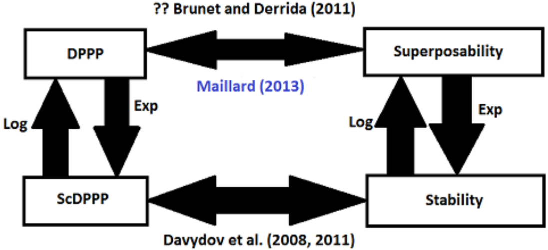

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Bhattacharya, Hazra and R. (2017, 2018): (1) Brunet-Derrida conjectures hold. (2) Long run configuration has been explicitly computed (missing when step-sizes are small).

My collaborators . . .

Ayan Bhattacharya
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Dept of Mathematics
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India

Rajat Subhra Hazra
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