# NNLO QCD corrections to SIDIS

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INTERNATIONAL SCHOOL AND WORKSHOP ON PROBING HADRON STRUCTURE AT THE ELECTRON-ION COLLIDER

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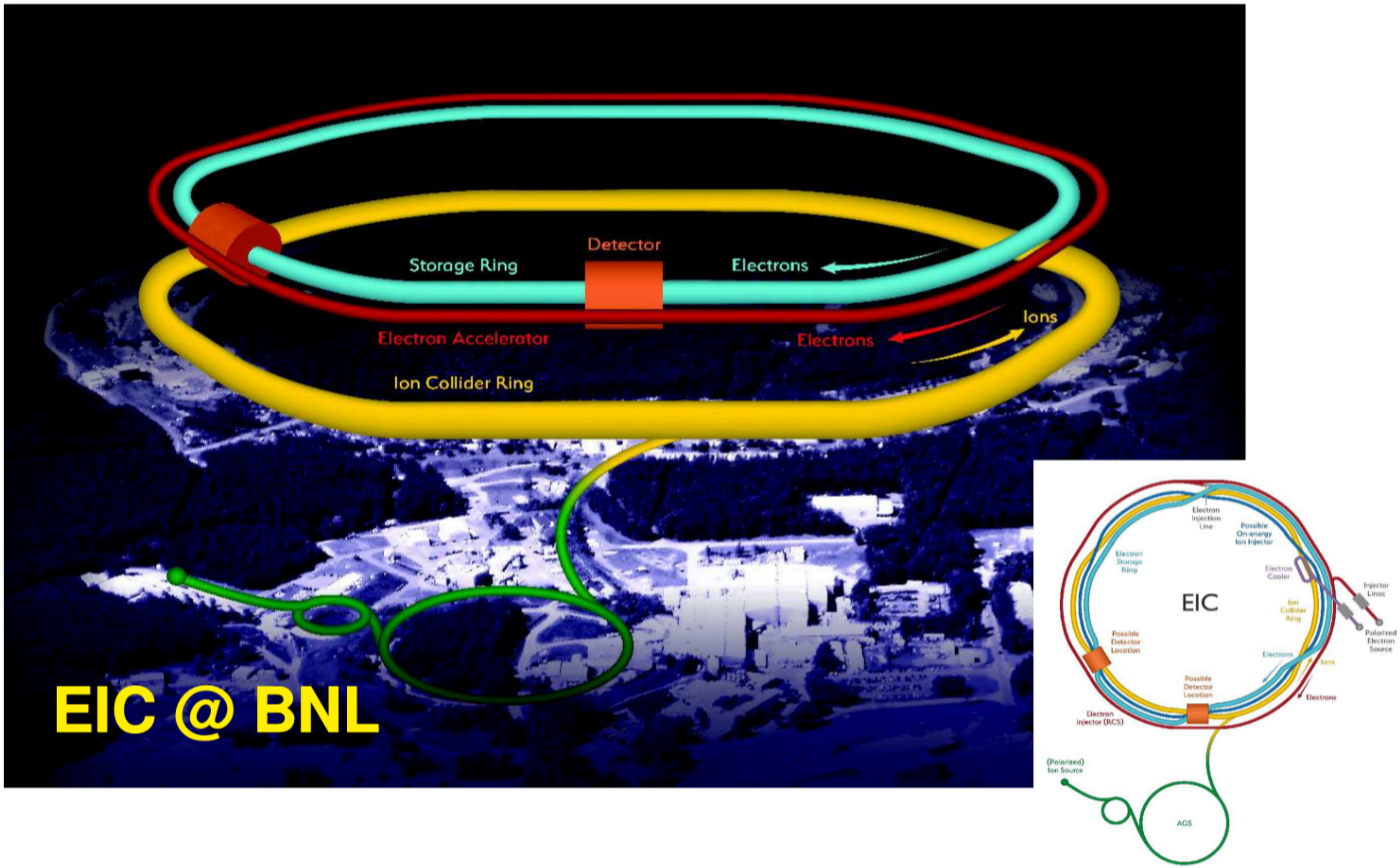
- Electron-Ion Collider

  *A machine that will unlock the secrets of the strongest force in Nature*



- Introduction to SIDIS
- Kinematics
- Hadronic Cross Section
- QCD improved Parton Model
- NNLO QCD effects
- Checks on our results
- Conclusion

# what is SIDIS?

In inclusive DIS, on sums up all the particles in the final state, except the scattered lepton

In SIDIS, in addition to the scattered lepton, one tags one of the final-state hadron.

Hence, the SIDIS depends on Parton Distribution Function (PDF) of the incoming hadron and Parton Fragmentation (FF) of the final state hadron.

# Predictions for SIDIS

Semi Inclusive Deep Inelastic Scattering ( SIDIS ) helps to study hadron structure both incoming hadron as well as the hadron that fragments in the final state.

Perturbative QCD provides framework to compute them order by order in strong coupling constant.

Leading order results are sensitive to theoretical uncertainty

1. Factorisation scale dependence from PDFs and FFs
2. Choice of PDFs and FFs

Higher order predictions are essential to resolve them

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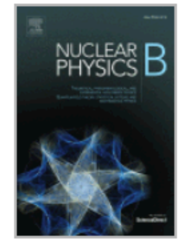



# Processes involving fragmentation functions beyond the leading order in QCD ☆

G. Altarelli, R.K. Ellis, G. Martinelli, So-Young Pi

NLO Altarelli et al 1979

**Threshold resummation at N$^3$LL accuracy and approximate N$^3$LO corrections to semi-inclusive DIS**

March 16, 2022

Maurizio Abele[a], Daniel de Florian[b], Werner Vogelsang[a]

**Approximate NNLO QCD corrections to semi-inclusive DIS**

March 16, 2022

S+V NNLO

Vogelsang et al 2022

# Predictions for SIDIS

First NLO prediction came out more than four decades back from Altarelli et al (1979).

There are only few attempts (Borsa et.al., Vogelsang et al), to obtain  results beyond  next-to leading order.

These results are valid in the hadronic thresholds (x -> 1, z -> 1)

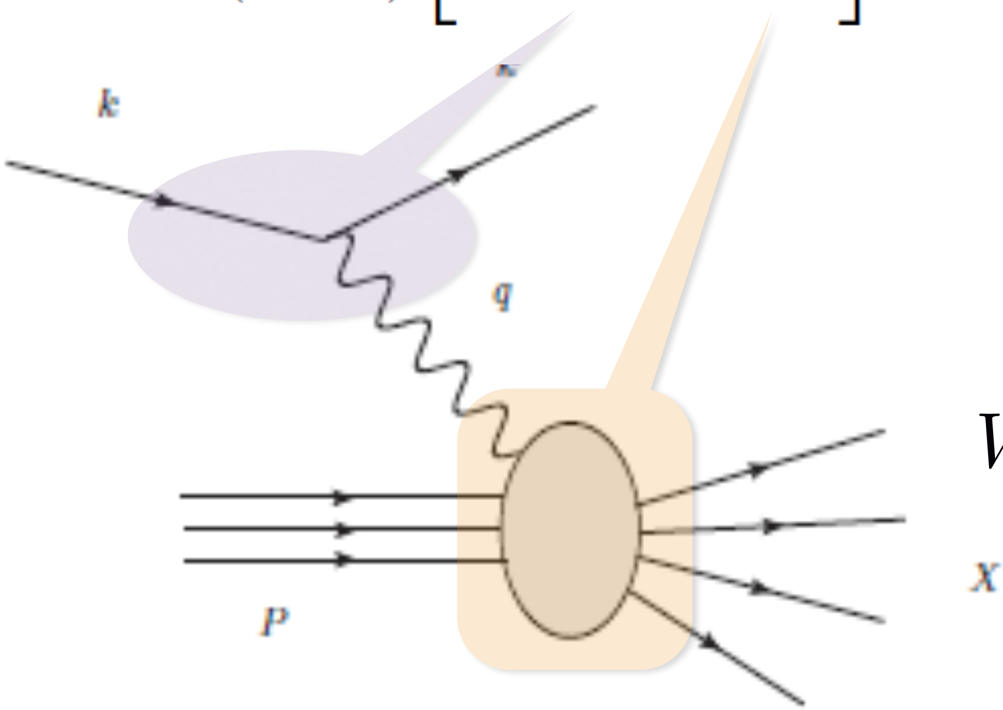
We present here the first results on NNLO corrections in QCD strong coupling constant for SIDIS process using Feynman diagrammatic approach.

- HERA: deep structure of proton at highest $Q^2$ and smallest $x$

# Structure Functions from DIS

## Inelastic Scattering Factorises

$$d\sigma = \frac{1}{4(k\cdot P)}\left[\frac{4\pi e^4}{q^4} L_{\mu\nu} W^{\mu\nu}\right]\frac{d^3k'}{2E'(2\pi)^3}$$




$x = \frac{Q^2}{2p.q}$; $y = \frac{p.q}{p.k}$; $Q^2 = xys$

### Leptonic Tensor

$$L_{\mu\nu} = 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \frac{Q^2}{2} g_{\mu\nu}\right]$$

### Hadronic Tensor

$$W^{\mu\nu} = \left(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\right) W_1 + \left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right) W_2$$

$W_i(\nu, Q^2) \quad i = 1, 2$ Structure Function

**Not calculable**

Inclusive Cross section

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\theta}{2}}\left(W_2(\nu, Q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, Q^2)\sin^2\frac{\theta}{2}\right)$$

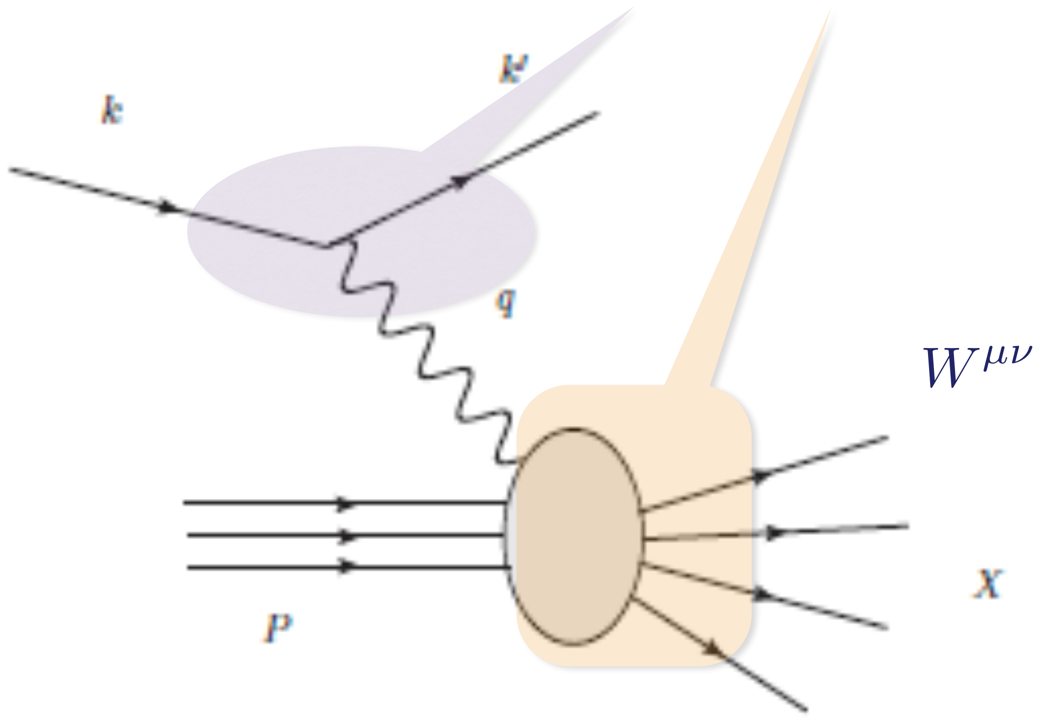
$$m_p W_1(\nu, Q^2) \to F_1(x) \qquad \nu W_2(\nu, Q^2) \to F_2(x)$$

# Semi-inclusive DIS

SIDIS: $l(k_l) + H(P) \to l(k_l') + H'(P')X$

## SIDIS Factorises

$$\frac{d^2\sigma_{e-H}}{dE_l' d\Omega dz} = \frac{E_l'}{E_l}\frac{\alpha_e^2}{Q^4} L^{\mu\nu}(k_l, k'_l, q) W_{\mu\nu}(q, P, P_H).$$




$$x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys$$

## Leptonic Tensor

$$L_{\mu\nu} = 2\Big[k_\mu k_\nu' + k_\mu' k_\nu - \frac{Q^2}{2} g_{\mu\nu}\Big]$$

## Hadronic Tensor

$$W^{\mu\nu} = F_1\Big[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big] + F_2\Big[\frac{1}{P.q}(P^\mu - \frac{P.q}{q^2}q^\mu)(P^\nu - \frac{P.q}{q^2}q^\nu)\Big]$$

**Structure Function**

**Not calculable**

Partonic contributions to Semi-Inclusive Cross section are calculable

## Scattering Process:

$$e^-(k_l) + H(P) \to e^-(k_l^{'}) + H'(P_H) + X'$$

## Factorises as

$$\frac{d^2\sigma_{e-H}}{dE_l' d\Omega dz} = \frac{E_l'}{E_l}\frac{\alpha_e^2}{Q^4} L^{\mu\nu}(k_l, k'_l, q) W_{\mu\nu}(q, P, P_H).$$

### Leptonic Tensor

$$L_{\mu\nu} = 2\Big[k_\mu k_\nu' + k_\mu' k_\nu - \frac{Q^2}{2} g_{\mu\nu}\Big]$$

### Hadronic Tensor

$$W^{\mu\nu} = F_1\Big[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big] + F_2\Big[\frac{1}{P.q}(P^\mu - \frac{P.q}{q^2}q^\mu)(P^\nu - \frac{P.q}{q^2}q^\nu)\Big]$$

# Structure function of SIDIS

$$W^{\mu\nu} = F_1 \underbrace{\Big[ - g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \Big]}_{T_1^{\mu\nu}} + F_2 \underbrace{\Big[ \frac{1}{P.q} (P^\mu - \frac{P.q}{q^2} q^\mu)(P^\nu - \frac{P.q}{q^2} q^\nu) \Big]}_{T_2^{\mu\nu}}$$

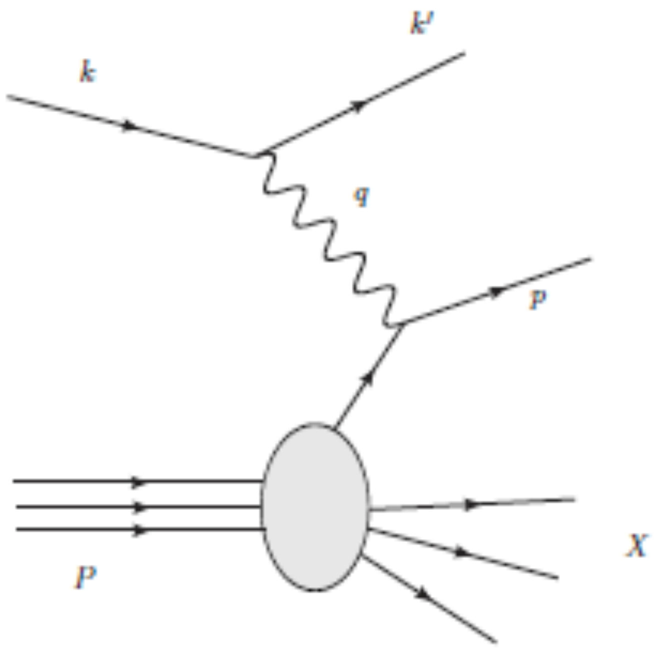
These functions Fi (i=1,2) are dimentionless,

Lorentz invariants $Fi(x,z,Q^2)$

Not calculable in perturbation theory.

We'll use Parton model to express them in terms pertubratively calculable parsonic cross section and non-perturbative PDFs and FFs

# Naive Parton Model




$$d\sigma^{DIS}(P,q) = \sum_i \int_x^1 dz\, f_i(z)\, d\hat{\sigma}_i(zP,q)$$

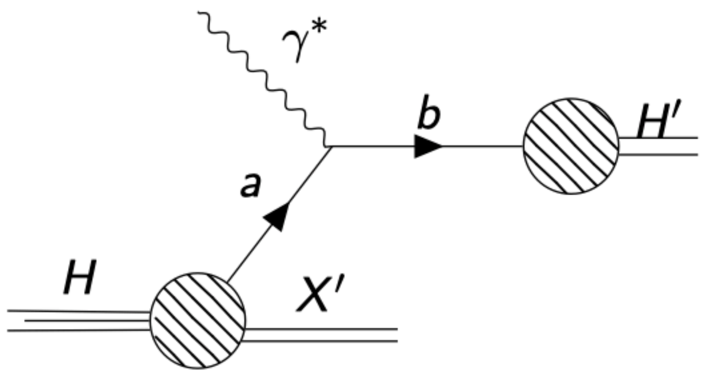
- Elastic scattering cross section with i-th parton
- Does not depend on the details of the target proton - Target Independent

$f_i(z)$ Parton Distribution Function (PDF)

- Probability of finding i-th parton with momentum fraction z of proton

- Does not depend on the future course of action of the i-th parton - Process Independent

# Paron Model for SIDIS




$$F_l = x^{l-1} \sum_{a,b} \int_x^1 \frac{dx_1}{x_1} f_a(x_1, \mu_F^2) \int_z^1 \frac{dz_1}{z_1} D_b(z_1, \mu_F^2)$$

$$\times \mathcal{F}_{l,ab}\left(\frac{x}{x_1}, \frac{z}{z_1}, Q^2, \mu_F^2\right).$$

# Paron Model for SIDIS

$$F_I = x^{I-1} \sum_{a,b} \int_x^1 \frac{dx_1}{x_1} f_a(x_1, \mu_F^2) \int_z^1 \frac{dz_1}{z_1} D_b(z_1, \mu_F^2) \times \mathcal{F}_{I,ab}(\frac{x}{x_1}, \frac{z}{z_1}, Q^2, \mu_F^2).$$

- $f_a dx_1$: The probability of finding a parton of type 'a' which carries a momentum fraction $x_1$ of the parent hadron $H$.
- $D_b dz_1$: The probability that a parton of type 'b' will fragment into hadron $H'$ which carries a momentum fraction $z_1$ of the parton.
- $\mathcal{F}_{I,ab}$ are the finite coefficient functions (CFs) that can be computed perturbatively, it is related to partonic cross section.

## Hadronic Kinematics :

$P$ : Initial hadron momenta

$k_l$ : Initial $e^-$ momenta

$q$ : Virtual photon momenta

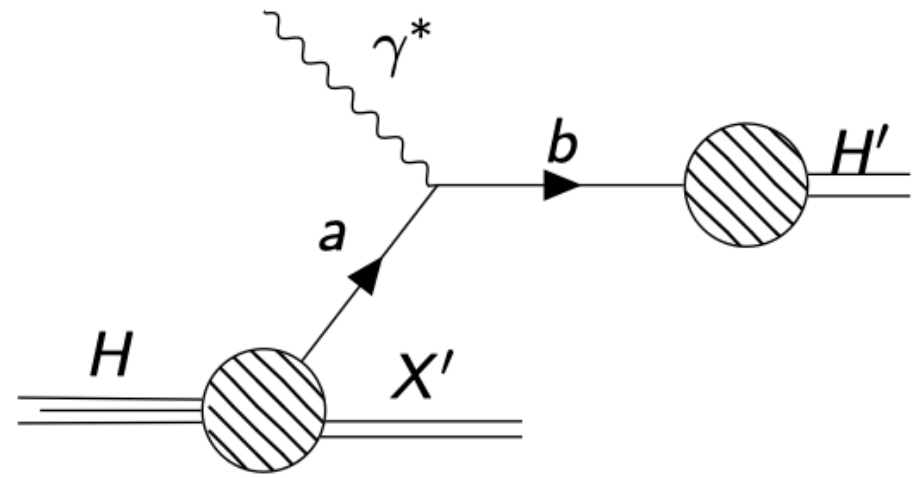
$q^2$ : $-Q^2 < 0$

$P_H$ : Momenta of hadron H'

$y$ : $\frac{P.q}{P.k_l}$, Fractional energy loss by $e$

$x$ : $\frac{Q^2}{2P.q}$, Bjorken-x,

$z$ : $\frac{P.P_H}{P.q}$




## Partonic Kinematics :

$p_a$ : parton ‘a’ momenta

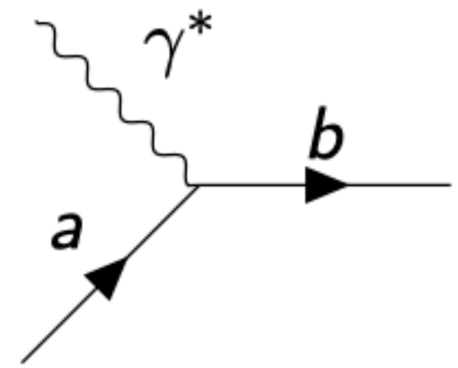
$p_b$ : Tagged parton momenta ‘b’

$x_1$ : $\frac{p_a}{P}$, $\qquad z_1$ : $\frac{P_H}{p_b}$

$x'$ : $\frac{x}{x_1} = \frac{Q^2}{2p_a.q}$, $\qquad z'$ : $\frac{z}{z_1} = \frac{p_a.p_b}{p_a.q}$

$k_i$ : Momentum of real radiations

# Partonic Cross sections




$$\hat{\sigma}_{I,ab} = \frac{\mathcal{P}_I^{\mu\nu}}{4\pi} \int \mathrm{dPS}_{X'+b} \overline{\Sigma} |M_{ab}|^2_{\mu\nu} \, \delta\Big( \frac{z}{z_1} - \frac{p_a \cdot p_b}{p_a \cdot q} \Big)$$

Fragmentaion

$|M_{ab}|^2$ is the squared amplitude for the process

$$a(p_a) + \gamma^*(q) \to \text{“}b\text{”}(p_b) + X'$$

$\mathcal{P}_I^{\mu\nu}$ are the projectors to project out CFs

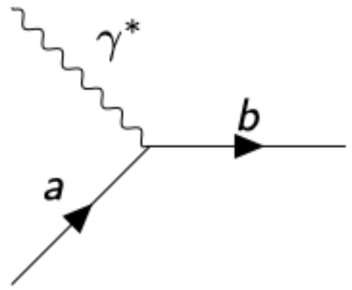
$$\mathcal{P}_1^{\mu\nu} = \frac{1}{(D-2)} \big( T_1^{\mu\nu} + 2x T_2^{\mu\nu} \big)$$

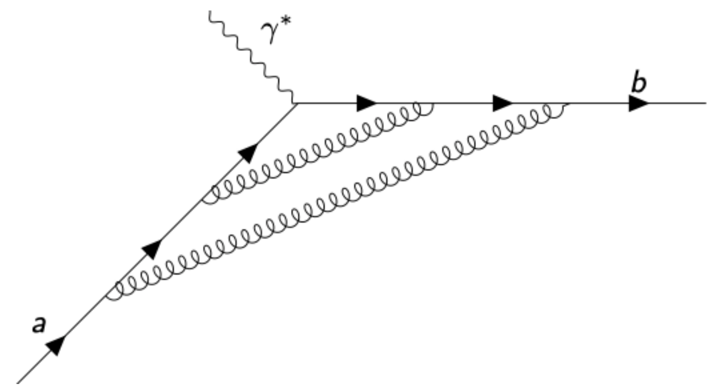
$$\mathcal{P}_2^{\mu\nu} = \frac{2x}{(D-2)x_1} \big( T_1^{\mu\nu} + 2x(D-1) T_2^{\mu\nu} \big).$$

# Partonic Subprocesses:

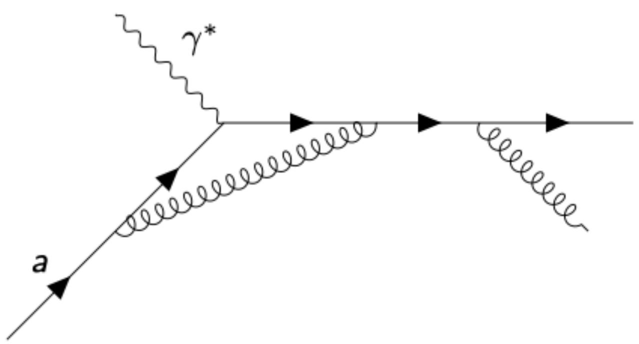
| | | |
|---|---|---|
| LO | | $\gamma^* q \to q$ |
| NLO | 1 Loop:(V) | $\gamma^* q \to q$ |
| | | $\gamma^* q \to q + g$ |
| | | $\gamma^* g \to q + \overline{q}$ |
| | 2 Loop:(VV) | $\gamma^* q \to q$ |
| NNLO | 1 Loop:(RV) | $\gamma^* q \to q + g$ |
| | | $\gamma^* q \to q + g + g$ |
| | | $\gamma^* q \to q + q_i + \overline{q}_i$ |
| | 1 Loop:(RV) | $\gamma^* g \to q + \overline{q}$ |
| | | $\gamma^* g \to q + \overline{q} + g$ |

# Partonic Subprocesses:



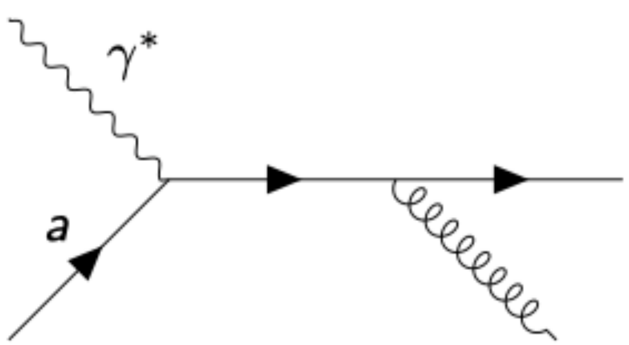



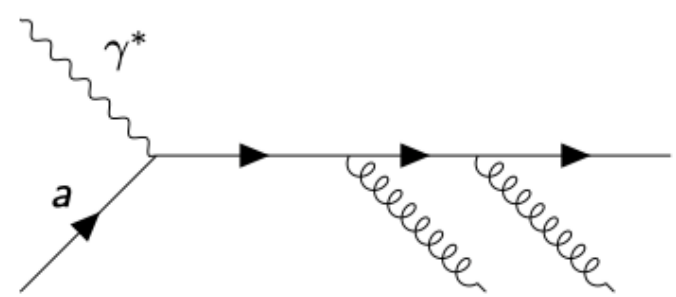

Pure Virtual



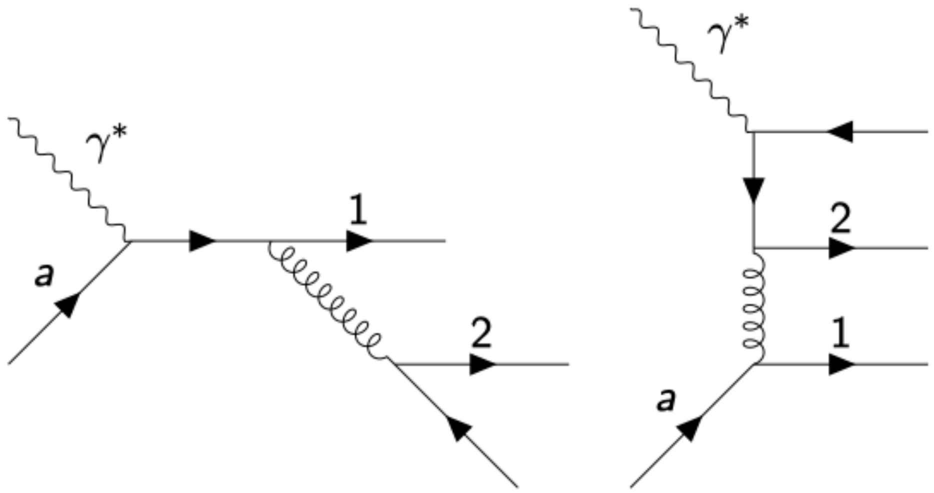

Pure Virtual

Pure Real emission



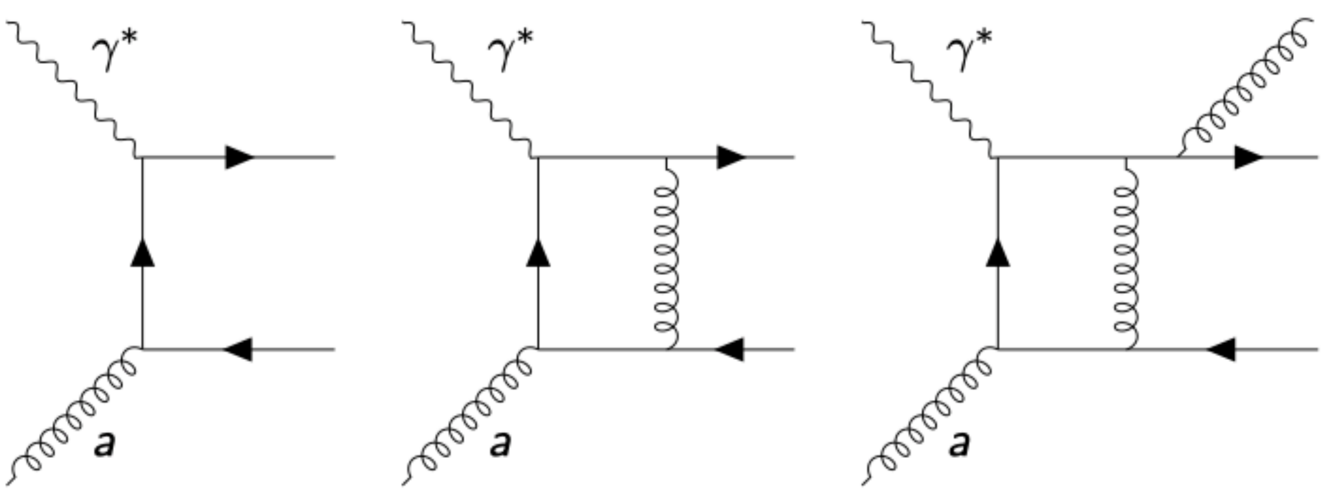



# Partonic Subprocesses:




Real Emission

Gluon initiated:



# Computation

Generation of set of Feynman diagrams using ` QGRAF',
which gives partonic level diagrams in symbolic form.

Using in-house code, we convert the out put of QGRAF to 'FORM' form

Using FORM, we performed Lorentz contractions, Dirac algebra,
tracing Gell-Mann matrices.

Pure Virtual: requires one and two loop integrals
Real - Virtual: requires one-loop integrals, two body phase-space integrals
Pure Real emission: requires three body phase space integralsT

Loop integrations and Phase space integrations are

both UV and IR divergent

and hence regularized in $D=4+\epsilon$, dimensions using the method of dimensional regularisation.

# Computation

We encounter a large number of loop integrals:

Integration-by-parts identities reduce them to fewer Master Integrals ( MIs ) .

$$\int d^D l \frac{\partial}{\partial l^\mu} \left[ \frac{l^\mu, p^\mu}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_n^{\nu_n}} \right] = 0$$

We choose a convenient set of families

Maping the loop integrals onto these Integral families is done by shifting of momenta ( 'Reduze' ) .

We used 'LiteRed' package  perform IBP reduction to obtain MIs

Finally solve the MIs

# Computation

3-Body Phase Space, $p_a + q \to$ "$p_b$" $+ k_1 + k_2$,

$$\int [dPS]_3 =$$

$$\frac{1}{(2\pi)^{2D-3}} \int d^D k_1 \int d^D k_2 \int d^D p_b \delta(k_1^2)\delta(k_2^2)\delta(p_b^2)\delta^D(p_a + q - p_b - k_1 - k_2)\delta(z' - \frac{p_a.p_b}{p_a.q})$$

**Reverse Unitarity method :**

phase-space integral into loop integral
performing reduction to get set of Master Integrals.

$$\delta(p^2 - m^2) \to \frac{i}{p^2 - m^2} - c.c.$$

can be almost forget

**We get total '21' MIs in phase space calculation.**

Next task: Solving the Master Integrals

# Master Integrals

## Generalization with set of MIs

$$\vec{I} = \left(I_1, I_2, \cdot, \cdot, \cdot, I_N\right)$$

$\{I_i(\vec{x})\}$ depend on Scaling variables

$$x_i = f_i\left(\frac{s_{ij}}{Q^2}\right)$$

$$\vec{x} = \left(x_1, x_2, \cdot, \cdot, \cdot, x_M\right)$$

## Differential equation:

$$d\vec{I} = \sum_{i=1}^{M} \mathbf{A}_i dx_i \, \vec{I}$$

$$\frac{\partial}{\partial x_i} \begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix} = \begin{bmatrix} \mathbf{A_{11}} & \cdots & \mathbf{A_{1N}} \\ \cdot & \cdots & \cdot \\ \mathbf{A_{N1}} & \cdots & \mathbf{A_{NN}} \end{bmatrix} \begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix}$$

$$i = 1, 2, \cdot, \cdot, \cdot M$$

# Master Integrals

Consider Diff equation:

$$s\frac{\partial}{\partial s}I(s,n) = A(s,n)I(s,n)$$

Expand around n = 4

$$I(s,n) = I^{(0)}(s) + (n-4)I^{(1)}(s) + \mathcal{O}((n-4)^2)$$

$$A(s,n) = A^{(0)}(s) + (n-4)A^{(1)}(s) + \mathcal{O}((n-4)^2)$$

$$s\frac{\partial}{\partial s}I^{(0)}(s) = A^{(0)}(s)I^{(0)}(s) \qquad \text{———(1)}$$

Solution

$$I^{(0)}(s) = I^{(0)}(s_0)e^{\int_{s_0}^{s}\frac{d\lambda}{\lambda}A^{(0)}(\lambda)}$$

$$s\frac{\partial}{\partial s}I^{(1)}(s) = A^{(0)}(s)I^{(1)}(s) + A^{(1)}(s)I^{(0)}(s) \qquad \text{———(2)}$$

# Canonical/Henn's Basis

Consider Diff equation:

$$d\vec{I}(\vec{x},n) = \sum_i \mathbf{A}_i(\vec{x},n)dx_i \; \vec{I}(\vec{x},n)$$

Choose U Transformation such that

$$U^{-1}\mathbf{A}(\vec{x},n)U - U^{-1}dU = (n-4)\;\overline{\mathbf{A}}(\vec{x})$$

Diff equation contains ‚n' independent A

$$d\vec{\bar{I}}(\vec{x},n) = (n-4)\sum_i \overline{\mathbf{A}}_i(\vec{x})dx_i \; \vec{\bar{I}}(\vec{x},n)$$

Solution

$$\vec{\bar{I}}(\vec{x},n) = \vec{\bar{I}}(\vec{x}_0,n)\;\mathbf{P}\exp\left((n-4)\int \frac{d\lambda}{\lambda}\overline{\mathbf{A}}(\lambda)\right)$$

$\mathbf{P}$ - Path Ordered exponential

# Canonical/Henn's Basis

Start with Henn's Diff equation:

$$s\frac{\partial}{\partial s}\overline{I}(s,n) = (n-4)\overline{A}(s)\;\overline{I}(s,n)$$

If $\overline{A}$ contains poses at $s_i$

$$\overline{A}(s) = \sum_i \frac{\tilde{A}_i(s_i)}{s - s_i}$$

$$\overline{I}(s,n) = \overline{I}^{(0)}(s_0) + (n-4)\sum_i \tilde{A}_i(s_i)\log\left(\frac{s-s_i}{s_0-s_i}\right)$$

$$+(n-4)^2\sum_i \tilde{\tilde{A}}(s_i)\mathcal{L}_i(s_0,s_i) + \cdots$$

Polylogarithms - Uniform transcendental terms

# Computation

**Dimensionally regulated integrals contain functions that require correct analytic continuation:**

$$(z' - x')^{a\epsilon - b}, (1 - z' - x')^{c\epsilon - d}$$

Feynman $+i\varepsilon$ prescription of propagators

$$x' \equiv x' - i\varepsilon \quad \text{and} \quad z' \equiv z' - i\varepsilon.$$

Partial fractioning and theta function to separate different sectors.

$$\left(\frac{z'-x'}{1-x'}\right)^{\varepsilon} = \left|\frac{z'-x'}{1-x'}\right|^{\varepsilon} \Big(\theta(z'-x') + (-1+i\varepsilon)^{\varepsilon}\theta(x'-z')\Big)$$

$$\left(\frac{1-z'}{1-z'-x'}\right)^{\varepsilon} = \left|\frac{1-z'}{1-z'-x'}\right|^{\varepsilon} \Big(\theta(1-z'-x') + (-1-i\varepsilon)^{\varepsilon}\theta(z'+x'-1)\Big)$$

# `+` Distributions

Dimensionally regulated integrals contain divergences as x ->1 and z->1

$$\frac{(1-x)^\epsilon}{1-x} = \frac{1}{\epsilon}\delta(1-x) + \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!}\left[\frac{\ln^i(1-x)}{1-x}\right]_+$$

$$= \frac{1}{\epsilon}\delta(1-x) + \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} D_i(x)$$

`+` distributions

$$\int_0^1 dx f(x)[g(x)]_+ = \int_0^1 dx \big(f(x) - f(1)\big) g(x)$$

Double distribution,

$$\int_0^1 dx \int_0^1 dz \frac{f(x,z)}{[1-x]_+[1-z]_+} =$$

$$\int_0^1 dx \int_0^1 dz \frac{f(x,z) - f(1,z) - f(x,1) + f(1,1)}{(1-x)(1-z)}$$

# Mass factorization

Soft divergences cancels among virtual and real emission processes,

The collinear divergences related to the a and bpartons in the initial state and the final fragmentation state remain.

These divergences can be factored out into Altarelli-Parisi ( AP ) kernels ( mass factorisation ) at factorisation scale,

$$\frac{\hat{\sigma}_{I,ab}(\epsilon)}{x'^{I-1}} = \Gamma_{c\leftarrow a}(\mu_F^2,\epsilon) \otimes \mathcal{F}_{I,cd}(\mu_F^2,\epsilon) \tilde{\otimes} \tilde{\Gamma}_{b\leftarrow d}(\mu_F^2,\epsilon),$$

$$[f \otimes g](x) = \int_x^1 \frac{dt}{t} f(t) g(\frac{x}{t})$$

# Mass Factorisation

Collinear singularities from initial parton from hadron and fragmenting parsons are removed by mass factorization

## NLO-level

$$
\begin{aligned}
\left(\frac{1}{\mu_F^2}\right)^{\frac{\epsilon}{2}} \hat{\sigma}^{(1)}_{1,qq} = {} & \delta(1-x') \otimes \mathcal{F}^{(1)}_{1,qq} \tilde{\otimes} \delta(1-z') \\
& + \delta(1-x') \otimes \mathcal{F}^{(0)}_{1,qq} \tilde{\otimes} \tilde{\Gamma}^{(1)}_{qq} + \Gamma^{(1)}_{qq} \otimes \mathcal{F}^{(0)}_{1,qq} \tilde{\otimes} \delta(1-z'),
\end{aligned}
$$

## NNLO-level

$$
\begin{aligned}
\left(\frac{1}{\mu_F^2}\right)^{\epsilon} \hat{\sigma}^{(2)}_{1,qq} + \frac{2\beta_0}{\epsilon}\left(\frac{1}{\mu_F^2}\right)^{\frac{\epsilon}{2}} \hat{\sigma}^{(1)}_{1,qq} = {} & \delta(1-x') \otimes \mathcal{F}^{(2)}_{1,qq} \tilde{\otimes} \delta(1-z') \\
& + \delta(1-x') \otimes \mathcal{F}^{(1)}_{1,qb'} \tilde{\otimes} \tilde{\Gamma}^{(1)}_{qb'} + \delta(1-x') \otimes \mathcal{F}^{(0)}_{1,qq} \tilde{\otimes} \tilde{\Gamma}^{(2)}_{qq} \\
& + \Gamma^{(1)}_{a'q} \otimes \mathcal{F}^{(0)}_{1,a'b'} \tilde{\otimes} \tilde{\Gamma}^{(1)}_{qb'} + \Gamma^{(1)}_{a'q} \otimes \mathcal{F}^{(1)}_{a'q} \tilde{\otimes} \delta(1-z') \\
& + \Gamma^{(2)}_{qq} \otimes \mathcal{F}^{(0)}_{1,qq} \tilde{\otimes} \delta(1-z').
\end{aligned}
$$

# What we obtain the dominant contr:

We considered only Non-singlet contributions from

1. Quark Initiated processes and
2. Quarks fragmenting to hadrons

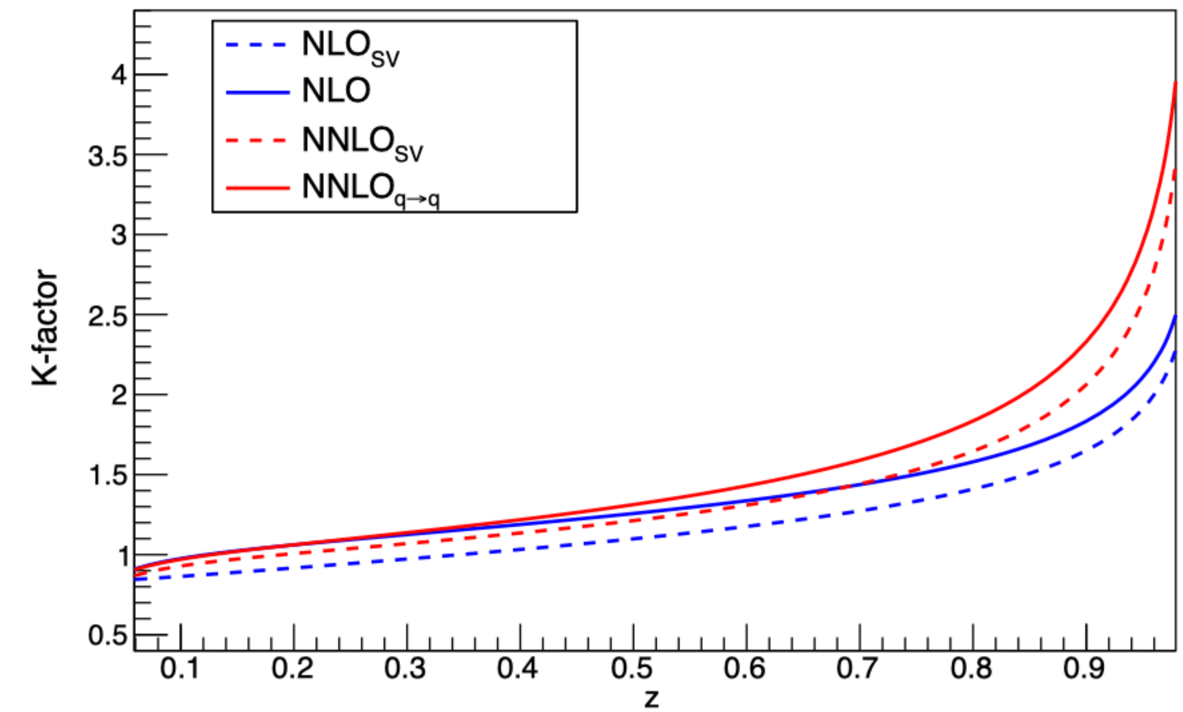
The MIs were computed using two different methods

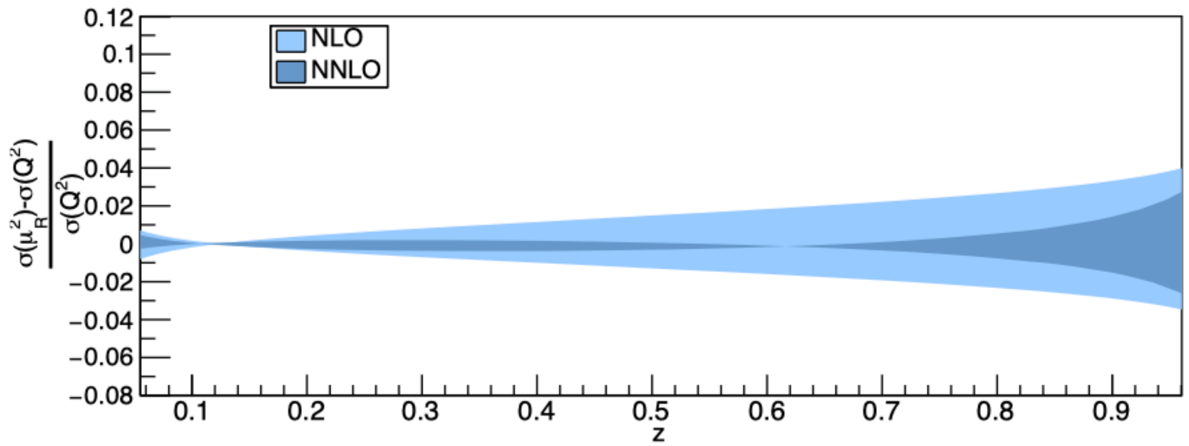
Initial and final collinear singularities were removed using mass factorization

We confirmed the + distributions and Dirac delta contributions, called Soft-Virtual part in the literature

We obtain regular contributions for the first time

# Numerical Impact:




$$xq(x,\mu_F^2) = 0.6x^{-0.3}(1-x)^{3.5}(1+5.0x^{0.8}),$$
$$xg(x,\mu_F^2) = 1.6x^{-0.3}(1-x)^{4.5}(1-0.6x^{0.3}).$$

[5] S. Moch et.al.{arXiv:0404111}

# Semi-inclusive deep-inelastic scattering at NNLO in QCD

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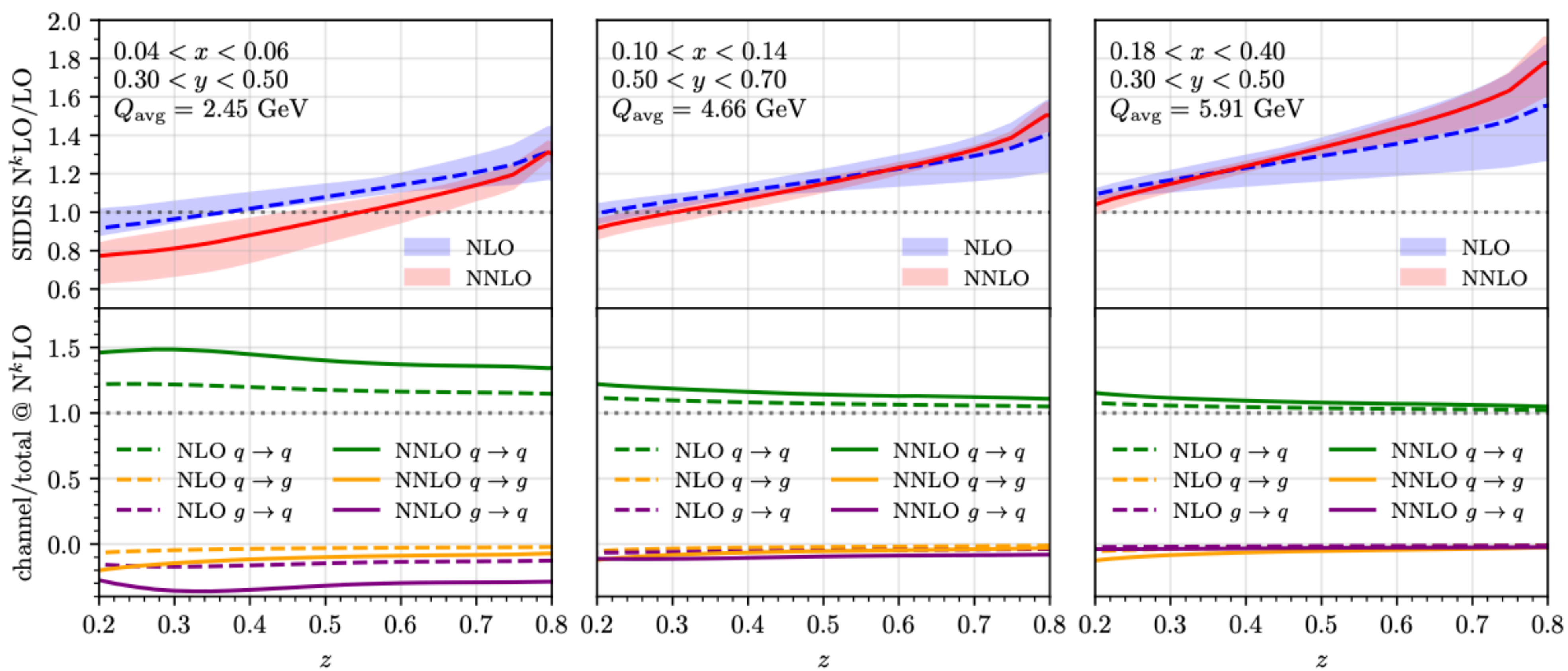
Semi-inclusive hadron production processes in deep-inelastic lepton-nucleon scattering are important probes of the quark flavour structure of the nucleon and of the fragmentation dynamics of quarks into hadrons. We compute the full next-to-next-to-leading order (NNLO) QCD corrections to the coefficient functions for semi-inclusive deep-inelastic scattering (SIDIS) in analytical form. The numerical impact of these corrections for precision physics is illustrated by a detailed comparison with data on single inclusive hadron spectra from the CERN COMPASS experiment.



# Conclusions

- First results on NNLO QCD corrections to Hadronic Cross Section is now available
- Used QCD improved Parton Model
- Technically challenging
- Checks on our results