Twisted S-duality

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Introduction and Plan

Any original result I will discuss today is based on a joint work (in part in progress) with *Surya Raghavendran*.

By-products of twisted S-duality include

- ○ *obtaining a purely algebraic way of describing physics for de Rham geometric Langlands correspondence;*
- ○ *finding infinite new pairs of deformations of 4d $\mathcal{N} = 4$ super Yang–Mills theory that are S-dual to each other.*

Today mainly on translation for parts of string theory aimed at mathematicians: (P) for physics and (M) for mathematics

1. Type IIB Superstring theory and Topological String Theory
2. Twisted S-duality
3. Applications

Summary: What have we done?

- (P) mathematical understanding of S-duality of (a part of) massless sector of type IIB supergravity (no axio-dilaton);
- (M) recovering old conjectures and formulating new conjectures in geometric representation theory;
- (!) easy calculation of how S-duality acts on further deformations of twists of supersymmetric gauge theory;
- (!) setting up a framework that can be useful for future works;

Disclaimer: I am a string theory newbie!

Ⓟ Type IIB Superstring Theory

- Type IIB superstring theory on a 10-manifold M^{10}; need to consider the moduli spaces of Riemann surfaces;
- D-brane gauge theory for D_{2k-1}-branes wrapping on $N^{2k} \subset M^{10}$ is a $2k$-dimensional field theory; e.g.,
  - D3 branes on $\mathbb{R}^4 \subset \mathbb{R}^{10}$ yield 4d $\mathcal{N} = 4$ SYM theory;
  - D5 branes on $\mathbb{R}^6 \subset \mathbb{R}^{10}$ yield 6d $\mathcal{N} = (1, 1)$ SYM theory;
- Closed string field theory on M^{10} is a field theory on M^{10} describing string theory;
- Type IIB supergravity theory on a 10-manifold M^{10} is a low-energy limit of closed string field theory;
- Open-closed coupling; closed string state yields a deformation of D-brane gauge theory;
- Existence of $\mathrm{SL}_2(\mathbb{Z})$ symmetry or S-duality

Question: How much can we capture mathematically?
Answer: Most of it, for topological string theory.

Topological Quantum Field Theory

Definition

A *d-dimensional TQFT* is a symmetric monoidal functor

$$Z\colon (\underline{\mathrm{Bord}}_d, \amalg) \to (\mathsf{Vect}_{\mathbb{C}}, \otimes)$$

Here $(\mathsf{Vect}_{\mathbb{C}}, \otimes)$ is a symmetric monoidal category of $\mathbb{C}$-vector spaces and $\underline{\mathrm{Bord}}_d$ is a category where

- an object is a closed $(d-1)$-manifold;
- a morphism is a cobordism up to diffeomorphism;
- the composition is a gluing of cobordisms;
- the monoidal structure is a disjoint union $\amalg$.

2d TQFT

Theorem

A 2d TQFT Z is determined by a commutative Frobenius algebra $Z(S^1) = A$.

morphism in $\underline{\mathrm{Bord}}_2$	morphism in $\mathrm{Vect}_{\mathbb{C}}$
$\emptyset \to S^1$	$u\colon \mathbb{C} \to A$
$S^1 \to \emptyset$	$\mathrm{Tr}\colon A \to \mathbb{C}$
$S^1 \amalg S^1 \to S^1$	$m\colon A \otimes A \to A$
$S^1 \to S^1 \amalg S^1$	$\Delta\colon A \to A \otimes A$

This is a (baby) (topological) string theory, where

$$Z(S^1) = A = (\text{the space of } \textit{closed} \text{ string states})$$

Q. Can we see an open string (interval) as well?

Extended 2d TQFT

Roughly, an *extended 2d TQFT* is a symmetric monoidal functor

$$Z\colon (\mathrm{Bord}_2, \amalg) \to (\mathsf{DGCat}_{\mathbb{C}}, \otimes)$$

Bord_2	$\mathsf{DGCat}_{\mathbb{C}}$
closed 2-manifold	complex number
closed 1-manifold cobordism of 1-manifolds	$\mathbb{C}$-vector space $\mathbb{C}$-linear map
closed 0-manifold cobordism of 0-manifolds	$\mathbb{C}$-linear category $\mathbb{C}$-linear functor

Theorem (Costello, Hopkins–Lurie, Lurie)

An extended 2d TQFT Z is determined by a Calabi–Yau category $Z(\mathsf{pt}) = \mathcal{C}$.

$$Z(\mathsf{pt}) = \mathcal{C} = \text{(the category of boundary conditions)};$$
$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}_1, \mathcal{B}_2) = \text{(the space of open string states from } \mathcal{B}_1 \text{ to } \mathcal{B}_2\text{)}.$$

Topological String Theory as 2d Extended TQFT

By topological string, we mean such a 2d extended TQFT determined by "CY 5-category".

For instance, let X be a CY 5-fold with a non-vanishing holomorphic volume form Ω_X. Two main examples are

	A-model	B-model
$Z(\mathrm{pt}) = \mathcal{C}$	$\mathrm{Fuk}(X)$	$\mathsf{Coh}(X)$
$Z(S^1) = \mathsf{HH}(\mathcal{C})$	$\mathrm{QH}(X)$	$\mathsf{PV}(X)$

Here $\mathsf{PV}(X) = \bigoplus \mathsf{PV}^{i,j}(X)$ where $\mathsf{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i T_X)$ with a differential $\overline{\partial}\colon \mathsf{PV}^{i,j} \to \mathsf{PV}^{i,j+1}$.

For future reference, note that using the isomorphism $(-) \vee \Omega_X\colon \mathsf{PV}^{i,j}(X) \cong \Omega^{d-i,j}(X)$, one has $\partial\colon \mathsf{PV}^{i,j} \to \mathsf{PV}^{i-1,j}$.

Type IIB string theory on $M^{10} \rightsquigarrow$ Calabi–Yau 5-category $\mathcal{C}$

Example

- $\mathcal{C} = \mathsf{Coh}(X^5)$ for a CY 5-fold X
- $\mathcal{C} = \mathrm{Fuk}(T^*N) \otimes \mathsf{Coh}(X^3)$ for a smooth 2-manifold N

Ⓜ Classical Field Theory and BV Formalism

A d-dimensional classical field theory is described by

- a spacetime manifold $M = M^d$;
- a space of fields $\mathcal{F}$;
- an action functional $S\colon \mathcal{F} \to \mathbb{C}$.

BV formalism encodes $S(\phi) = \int_M \frac{1}{2}\langle \phi, Q\phi \rangle + \frac{1}{6}\langle \phi, \ell_2(\phi, \phi)\rangle$ as

(1) (-1)-shifted "symplectic" space $\mathcal{E}$;
(2) differential Q;
(3) a Lie bracket ℓ_2.

Example

- Free scalar theory $S(\phi) = \int_M \frac{1}{2}\langle \phi, \Delta_g \phi \rangle$ on (M, g) is

$$\mathcal{E} = C^\infty(M) \oplus C^\infty(M)[-1], \qquad Q = \Delta_g, \qquad \ell_2 = 0.$$

- CS theory $S(A) = \int_M \frac{1}{2}\langle A, dA \rangle + \frac{1}{6}\langle A, [A, A]\rangle$ on M is

$$\mathcal{E} = \Omega^\bullet(M^3) \otimes \mathfrak{g}[1], \qquad Q = d, \qquad \ell_2 = \wedge \otimes [-,-].$$

We may work in a $\mathbb{Z}/2$-graded setting.

D-brane Gauge Theory of String Theory ($\mathcal{C}$ fixed)

(P) Open strings ending on branes $\mathcal{B}$ yield D-brane gauge theory.

(M) [Brav–Dyckerhoff] The moduli $\mathcal{M}_{\mathcal{C}}$ of objects is $(2-d)$ shifted symplectic and $\mathbb{T}_{\mathcal{B}}[-1]\mathcal{M}_{\mathcal{C}} \cong \mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})$ for $\mathcal{B} \in \mathcal{C}$.

D-brane gauge theory on $N^{2k} \subset M^{10} \rightsquigarrow \mathcal{E} = \mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})[1]$

$\mathcal{C}$ a DG category $\rightsquigarrow$ associative and hence Lie on $\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})$
$\mathcal{C}$ a CY category $\rightsquigarrow$ a shifted symplectic structure on $\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})[1]$

Example

$\mathcal{C}$	$\mathrm{Coh}(\mathbb{C}^5)$	$\mathrm{Fuk}(\mathbb{R}^4) \otimes \mathrm{Coh}(\mathbb{C}^3)$
branes	D3's on $\mathbb{C}^2 \subset \mathbb{C}^5$	D3's on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4 \times \mathbb{C}^3$
$\mathcal{E}$	$\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D3}}(\mathbb{C}^2) :=$ $\Omega^{0,\bullet}(\mathbb{C}^2)[\varepsilon_1, \varepsilon_2, \varepsilon_3][1]$	$\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}) :=$ $\Omega^{\bullet}(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2][1]$
name	holomorphic twist of 4d $\mathcal{N} = 4$ theory	holomorphic-topological twist of 4d $\mathcal{N} = 4$ theory

Closed String Field Theory of String Theory ($\mathcal{C}$ fixed)

Recall $Z(S^1)$ is the space of closed string states, but note that

- (P) The worldsheet theory, being coupled with gravity theory, should be invariant under $\mathrm{Diff}(S^1)$. This motivates $Z(S^1)^{S^1}$.
- (M) Here $Z(S^1) = \mathsf{HH}(\mathcal{C})$ admits an S^1-action which corresponds to so-called Connes' B operator, so $Z(S^1)^{S^1} = \mathrm{Cyc}(\mathcal{C})$.
- (M) [Brav–Rozenblyum] $\mathbb{T}_{\mathcal{C}}[-1]\mathcal{M}_{\mathrm{CY}} \cong \mathrm{Cyc}^{\bullet}(\mathcal{C})[1]$ where $\mathcal{M}_{\mathrm{CY}}$ is the moduli space of Calabi–Yau categories.

Closed string field theory on $M^{10} \rightsquigarrow \mathcal{E} = \mathrm{Cyc}^{\bullet}(\mathcal{C})[2]$
where $\mathcal{E}$ is understood in the framework of [Butson–Y.].

Example (Bershadsky–Cecotti–Ooguri–Vafa, Costello–Li)

If $\mathcal{C} = \mathsf{Coh}(X^5)$, then $Z(S^1) \cong \mathsf{PV}(X)$ and $B = \partial$. Hence the corresponding closed string field theory is given by
$(\ker \partial \subset \mathsf{PV}(X)[2], \overline{\partial})$ or $\mathcal{E}_{\mathsf{BCOV}}(X) = (\mathsf{PV}(X)[\![t]\!][2], \overline{\partial} + t\partial)$.

Supergravity ($\mathcal{C}$ fixed)

Ⓟ Supergravity is a low-energy limit of closed SFT with neither non-perturbative effects nor non-propagating fields.

$$\text{SUGRA on } M^{10} \rightsquigarrow \text{propagating/dynamic part of } \mathrm{Cyc}^\bullet(\mathcal{C})[2]$$

The dynamic fields of BCOV theory can be identified:

Definition

Let (X, Ω_X) be a Calabi–Yau d-fold. A minimal BCOV theory is $\mathcal{E}_{\mathrm{m}}(X) = \mathcal{E}_{\mathrm{mBCOV}}(X) = \bigoplus_{i+k\leq d-1} t^k \,\mathrm{PV}^{i,\bullet}(X)$.

Example

If $\mathcal{C} = \mathrm{Coh}(X^3)$ (or $\mathcal{C} = \mathrm{Fuk}(\mathbb{R}^4) \otimes \mathrm{Coh}(X^3)$), then it is $\mathcal{E}_{\mathrm{m}}(X^3)$ (or $\Omega^\bullet(\mathbb{R}^4) \otimes \mathcal{E}_{\mathrm{m}}(X^3)$), where $\mathcal{E}_{\mathrm{m}}(X^3)$ is

$\underline{-2}$	$\underline{-1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\mathrm{PV}^{0,\bullet}$				
	$\mathrm{PV}^{1,\bullet}$	$\to t\,\mathrm{PV}^{0,\bullet}$		
		$\mathrm{PV}^{2,\bullet}$	$\to t\,\mathrm{PV}^{1,\bullet}$	$\to t^2\,\mathrm{PV}^{0,\bullet}$

Coupling of Open and Closed Sectors ($\mathcal{C}$ fixed)

Coupling of closed string field theory and D-brane gauge theory
$\rightsquigarrow$ closed-open map $\mathsf{CO}\colon \mathrm{Cyc}^\bullet(\mathcal{C})[1] \dashrightarrow \mathrm{Cyc}^\bullet(\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B}))[1]$

Physically, a closed string state $\alpha \in \mathrm{Cyc}^\bullet(\mathcal{C})$ gives a deformation of D-brane gauge theory given by $\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})$.

Example

$\mathcal{C}$	$\mathrm{Coh}(\mathbb{C}^5)$	$\mathrm{Fuk}(\mathbb{R}^4) \otimes \mathrm{Coh}(\mathbb{C}^3)$
$\mathcal{B}$	$\mathbb{C}^2_{z_i} \subset \mathbb{C}^5_{z_i,w_j}$	$\mathbb{R}^2 \times \mathbb{C}_z \subset \mathbb{R}^4 \times \mathbb{C}^3_{z,w_j}$
$\mathcal{E}$	$\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D3}}(\mathbb{C}^2_{z_i}) =$ $\Omega^{0,\bullet}(\mathbb{C}^2_{z_i})[\varepsilon_1,\varepsilon_2,\varepsilon_3][1]$ $\cong \mathcal{O}(\mathbb{C}^2_{z_i})[\varepsilon_1,\varepsilon_2,\varepsilon_3][1]$	$\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z) =$ $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}_z)[\varepsilon_1,\varepsilon_2][1]$ $\cong \mathcal{O}(\mathbb{C}_z)[\varepsilon_1,\varepsilon_2][1]$
CO	$\mathrm{PV}(\mathbb{C}^5_{z_i,w_j}) \to \mathrm{HH}(\mathcal{O}(\mathbb{C}^2_{z_i})[\varepsilon_j])$ $z_i, \partial_{z_i}, w_j, \partial_{w_j} \mapsto z_i, \partial_{z_i}, \partial_{\varepsilon_j}, \varepsilon_j$	$\mathrm{PV}(\mathbb{C}^3_{z,w_j}) \to \mathrm{HH}(\mathcal{O}(\mathbb{C}_z)[\varepsilon_j])$ $z, \partial_z, w_j, \partial_{w_j} \mapsto z, \partial_z, \partial_{\varepsilon_j}, \varepsilon_j$

Modification of BCOV Theory

Definition

Minimal BCOV theory with potential $\widetilde{\mathcal{E}}_{\mathrm{m}}(X)$ is a cochain complex

$$\begin{array}{ccccc} \underline{-2} & \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\ \mathrm{PV}^{0,\bullet} & & & & \\ & \mathrm{PV}^{1,\bullet} \rightarrow & t\,\mathrm{PV}^{0,\bullet} & & \\ & & \mathrm{PV}^{3,\bullet} & & \end{array}$$

with additional structures.

- (M) There is a "map" $\Phi\colon \widetilde{\mathcal{E}}_{\mathrm{m}} \to \mathcal{E}_{\mathrm{m}}$ that has $\partial\colon \mathrm{PV}^{3,\bullet} \to \mathrm{PV}^{2,\bullet}$, respecting structures of interest.
- (P) The modification amounts to introducing Ramond–Ramond forms as a potential for Ramond–Ramond field strengths.

Definition of S-duality

Recall

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \,\middle|\, S^4 = 1,\ (ST)^3 = S^2 \right\rangle .$$

Definition/Theorem (Raghavendran–Y.)

Let (X, Ω_X) be a CY 3-fold. There exists an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\widetilde{\mathcal{E}}_{\mathrm{m}}(X) = \mathrm{PV}^{0,\bullet}(X)[2] \oplus \left(\mathrm{PV}^{1,\bullet}(X)[1] \to t\,\mathrm{PV}^{0,\bullet}(X)\right) \oplus \mathrm{PV}^{3,\bullet}(X)$.

At the level of cohomology, it is

$$S \mapsto \begin{pmatrix} & & -(-) \vee \Omega_X \\ & \mathrm{Id} & \\ (-) \wedge \Omega_X^{-1} & & \end{pmatrix}, \quad T \mapsto \begin{pmatrix} \mathrm{Id} & & (-) \vee \Omega_X \\ & \mathrm{Id} & \\ & & \mathrm{Id} \end{pmatrix}$$

For instance,

$$\alpha \in \mathrm{PV}^0_{\mathrm{hol}}(X) \rightsquigarrow S(\alpha) = \alpha \wedge \Omega_X^{-1} \in \mathrm{PV}^3_{\mathrm{hol}}(X)$$
$$\gamma \in \mathrm{PV}^3_{\mathrm{hol}}(X) \rightsquigarrow S(\gamma) = -\gamma \vee \Omega_X \in \mathrm{PV}^0_{\mathrm{hol}}(X)$$

Ⓟ Where does S-duality come from?

S-duality is an action of $S \in \mathrm{SL}_2(\mathbb{Z})$ on type IIB string theory compatible with the following diagram

$$\overset{\mathrm{SL}_2(\mathbb{Z})}{\circlearrowright} \qquad\qquad\qquad\qquad \overset{\mathrm{SL}_2(\mathbb{Z})}{\circlearrowright}$$
$$\mathrm{M}[S^1_{\mathrm{M}} \times S^1_r \times M^9] \xrightarrow[\simeq]{\mathrm{red}_M} \mathrm{IIA}[S^1_r \times M^9] \xrightarrow[\simeq]{\mathbf{T}} \mathrm{IIB}[S^1_{1/r} \times M^9]$$

- M stands for M-theory;
- IIA stands for type IIA string theory;
- red_M is an equivalence from the "fact" that a circle reduction of M-theory is equivalent to type IIA theory;
- T-duality $\mathbf{T}$ is an equivalence between type II string theories;
- $\mathrm{SL}_2(\mathbb{Z})$-action on M-theory is on $S^1_M \times S^1_r$;
- $\mathrm{SL}_2(\mathbb{Z})$-action on IIB string theory is transferred from the $\mathrm{SL}_2(\mathbb{Z})$-action on M-theory through equivalences.

Theorem (Raghavendran–Y.)

There exists a corresponding diagram in the "twisted" setting. In particular, the $\mathrm{SL}_2(\mathbb{Z})$-action on twisted type IIB superstring is compatible with the $\mathrm{SL}_2(\mathbb{Z})$-action on T^2 of twisted M-theory.

Summary

- S-duality is a duality of type IIB string theory.
- We construct S-duality operation on a sector of supergravity theory, that is, a version of BCOV theory.
- Our interest is duality between D-brane gauge theories, or more precisely, deformations of D-brane gauge theory.
- Closed-open map + $\Phi\colon \widetilde{\mathcal{E}}_m \to \mathcal{E}_m$ yields S-duality of deformations of D-brane gauge theory.

From now on, we let $\mathcal{C} = \mathrm{Fuk}(\mathbb{R}^4) \otimes \mathsf{Coh}(\mathbb{C}^3)$. For instance, we may consider N D3 branes on $\mathbb{R}^2 \times \mathbb{C}_z \subset \mathbb{R}^4 \times \mathbb{C}^3_{z,w_1,w_2}$ to get $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z) = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$. Then for

$$\overset{S}{\circlearrowright}\,\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\ \Phi\ } \mathcal{E}_{\mathrm{m}} \xrightarrow{\ \mathsf{CO}\ } \mathsf{Cyc}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z))$$

we compare deformations of HT twist by S-dual elements.

S-duality gives Geometric Langlands: $F = w_1$

Based on [Elliott–Y.]

$$\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\ \Phi\ } \mathcal{E}_{\mathrm{m}} \xrightarrow{\ \mathsf{CO}\ } \mathsf{Cyc}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z))$$

$$\begin{array}{ccccc} w_1 & \longmapsto & w_1 & \longmapsto & \partial_{\varepsilon_1} \\ \Downarrow S & & & & \\ w_1 \partial_z \partial_{w_1} \partial_{w_2} & \longmapsto & \partial_{w_2} \wedge \partial_z & \longmapsto & \varepsilon_2 \partial_z \end{array}$$

Recall $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z) = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$. Globalizing with replacing $\mathbb{R}^2 \times \mathbb{C}$ by $\Sigma \times C$, one has $\mathrm{EOM}^{\mathrm{HT}}_{\mathrm{D3}}(\Sigma \times C) = \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, T^*[1]\,\mathsf{Higgs}_G(C))$, aka B-model with target Hitchin moduli. Here ε_1 is responsible for $T^*[1]$ and ε_2 makes C into C_{Dol}. Hence we have the following deformations

$$(\mathrm{B}, \mathsf{Higgs}_G(C)) \xrightarrow{\ \partial_{\varepsilon_1}\ } (\mathrm{B}, \mathsf{Bun}_G(C)_{\mathrm{dR}}), \qquad (\mathrm{B}, \mathsf{Higgs}_G(C)) \xrightarrow{\ \varepsilon_2 \partial_z\ } (\mathrm{B}, \mathsf{Flat}_G(C))$$

S-duality would yield $\mathsf{D}(\mathsf{Bun}_G(C)) \simeq \mathsf{QCoh}(\mathsf{Flat}_G(C))$ for $G = \mathsf{GL}_N$. Moreover, one also obtains quantum geometric Langlands conjecture in a simple way.

S-duality between Superconformal Deformations: $F = zw_2$

	0	1	2	3	4	5	6	7	8	9
	u		v		z		w_1		w_2	
K D5		$\times$	$\times$		$\times$	$\times$	$\times$	$\times$		
N D3	$\times$	$\times$			$\times$	$\times$				

$$\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\Phi} \mathcal{E}_{\mathrm{m}} \xrightarrow{\mathsf{CO}} \mathrm{Cyc}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D5}}(\mathbb{R}^2 \times \mathbb{C}^2_{z,w_1}))$$

$$zw_2 \mapsto zw_2 \longmapsto z\partial_{\varepsilon_2}$$

The deformation $z\partial_{\varepsilon_2}$ turns HT twist of 6d $\mathcal{N} = (1,1)$ theory to 4d CS theory on $\mathbb{R}^2 \times \mathbb{C}_{w_1}$ [Costello–Yagi]: it follows from

$$\Omega^{0,\bullet}(\mathbb{C}_{w_1}) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}_z)\varepsilon_2 \xrightarrow{z\partial_{\varepsilon_2}} \Omega^{0,\bullet}(\mathbb{C}_z)\right) \cong \Omega^{0,\bullet}(\mathbb{C}_{w_1})$$

The appearance of (truncated) Yangian of $\mathfrak{gl}(K)$ on the 1d defect can be understood as its S-dual 3d $\mathcal{N} = 4$ theory configuration, deformed by $S(zw_2) = \partial_{w_1}(w_2\partial_{w_2} - z\partial_z)$, where the Yangian is understood as the quantized Coulomb branch algebra.

New Examples of S-dual Theories: $F = w_1 w_2$

$$
\begin{array}{ccccc}
\widetilde{\mathcal{E}}_{\mathrm{m}} & \xrightarrow{\Phi} & \mathcal{E}_{\mathrm{m}} & \xrightarrow{\mathsf{CO}} & \mathrm{Cyc}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z)) \\
w_1 w_2 & \longmapsto & w_1 w_2 & \longmapsto & \partial_{\varepsilon_1}\partial_{\varepsilon_2} \\
\Downarrow S & & & & \\
w_1 w_2 \partial_z \partial_{w_1}\partial_{w_2} & \longmapsto & w_1\partial_z\partial_{w_1} - w_2\partial_z\partial_{w_2} & \longmapsto & \pi = \partial_{\varepsilon_1}\partial_z\varepsilon_1 - \partial_{\epsilon_2}\partial_z\varepsilon_2
\end{array}
$$

- As $(\mathbb{C}[\varepsilon_1, \varepsilon_2], \partial_{\varepsilon_1}\partial_{\varepsilon_2})$ is Clifford algebra $\mathrm{Cl}(\mathbb{C}^2) \cong \mathrm{End}(\mathbb{C}^{1|1})$, the element $\partial_{\varepsilon_1}\partial_{\varepsilon_2}$ deforms $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$ into $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{gl}_{N|N}[1]$, 4d CS theory with $\mathrm{GL}_{N|N}$.
- The category of line defects of 4d CS theory is known, in terms of modules over Yangian, quantum affine algebras, and elliptic quantum groups for $C = \mathbb{C}$, $\mathbb{C}^\times$, and E.
- The element π gives a particular deformation $\mathrm{Coh}(\mathrm{Higgs}_G(C), \pi)$ of $\mathrm{Coh}(\mathrm{Higgs}_G(C))$ in terms of difference modules as a category of boundary conditions.
- There should be an action of monoidal category of line defects on category of boundary conditions.

Thanks for your attention!