Non-equilibrium and periodically driven quantum systems

Tutorial

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Outline

- I. Eigenvalue statistics
- II. Forward scattering approximation.

I. Eigenvalue statistics

Preliminaries : Symmetries of a Hamiltonian

Consider a 1 d spin-1/2 interacting system of N sites described by a Hamiltonian like

$$H = \sum_{i=1}^{L-1} (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) + h \sum_{i=1}^{L} S_i^z$$

the symmetry of such Hamiltonian includes translation, inversion, magnetization ($M_z = \sum_{i=1}^{L} S_i^z$) etc.

- **Translation :** The action of translation operator (T) is

$$T|S_0^z, S_1^z, S_2^z, \cdots, S_{N-1}^z\rangle = |S_{N-1}^z, S_1^z, S_2^z, \cdots, S_{N-2}^z\rangle \quad ; \quad T^N = I$$

- The eigenstates of T can be choosen as $T|\Psi(k)\rangle = e^{ik}|\Psi(k)\rangle$ where $k = \frac{2n\pi}{N}$, $n = 0, \cdots, N-1$.
- If, the system has translation symmetry : $[H, T] = 0$.
- H is block diagonalized in momentum basis as $\langle\Psi(k)|H|\Psi(k')\rangle = 0$ if $k \neq k'$.

- How to construct the $|\Psi(k)\rangle$ s ?

$$|\Psi^a(k)\rangle = \frac{1}{\sqrt{N_a}} \sum_{r=0}^{N-1} e^{-ikr} T^r |a\rangle$$

- Example : for $N = 6$ and $k = 0$

$$|\Psi^0(0)\rangle = |000000\rangle$$

$$|\Psi^1(0)\rangle = \frac{1}{\sqrt{6}}(|100000\rangle + |010000\rangle + \cdots + |000001\rangle)$$

$$|\Psi^3(0)\rangle = \frac{1}{\sqrt{6}}(|110000\rangle + |011000\rangle + \cdots + |100001\rangle)$$

$$\cdots$$

$$|\Psi^9(0)\rangle = \frac{1}{\sqrt{3}}(|100100\rangle + |010010\rangle + |001001\rangle)$$

$$\cdots$$

$$|\Psi^{21}(0)\rangle = \frac{1}{\sqrt{2}}(|101010\rangle + |010101\rangle)$$

- $k = 0$ sector is the largest one.

Preliminaries : Symmetries of a Hamiltonian

- **Inversion :** Inversion operator (P) acts as

$$P|S_0^z, S_1^z, S_2^z, \cdots, S_{N-1}^z\rangle = |S_{N-1}^z, \cdots, S_2^z, S_1^z, S_0^z\rangle \quad P^2 = I$$

- The eigenstates of P can be choosen as $P|\Phi(p)\rangle = p|\Phi(p)\rangle$ where $p = \pm 1$.

- In general $[T, P] \neq 0$ but for $k = 0, \pi$ they commute. So, $k = 0$ and π sector can be further block diagonalized into two part using inversion symmetry.

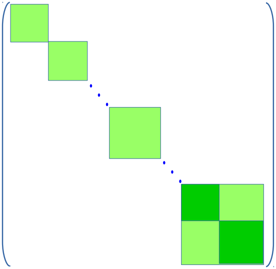
- **Example :** Consider the following two states

$$|\Psi^{11}(0)\rangle = \frac{1}{\sqrt{6}}(|110100\rangle + |011010\rangle + \cdots + |101001\rangle$$
$$|\Psi^{19}(0)\rangle = \frac{1}{\sqrt{6}}(|001011\rangle + |100101\rangle + \cdots + |010110\rangle$$

- We can construct

$$|\Theta(k=0,p=+1) = \frac{1}{\sqrt{2}}(|\Psi^{11}(0)\rangle + |\Psi^{19}(0)\rangle)$$
$$|\Theta(k=0,p=-1) = \frac{1}{\sqrt{2}}(|\Psi^{11}(0)\rangle - |\Psi^{19}(0)\rangle)$$

- One can check $\langle\Theta(k=0,p=+1)|H|\Theta(k=0,p=-1)\rangle = 0$. This means we can further reduce the number of states in $k=0$ sector by working in a particular inversion symmetry sector.



- Some numbers : In a 2D spin-1/2 Heisenberg model, one can deal with 6×6 (HSD=2^{36}) systems by using 7 such conserved quantities $(M_z, Z, k_x, k_y, p_x, p_y, p_d)$. The size of $(M_z = 0, Z = +1, k_x = 0, k_y = 0, p_x = +1, p_y = +1, p_d = +1)$ sector is 15804955 ($\approx$ 4348 times reduction).

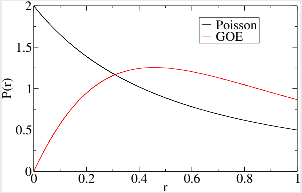
- Ref : "Computational Studies of Quantum Spin Systems ", A. W. Sandvik, AIP Conf.Proc.1297:135,2010.

I. Eigenvalue Statistics : Measure of quantum chaos

- Store the eigenvalues (e) in increasing order and calculate the gaps (Δ) : $\Delta(i) = e(i+1) - e(i)$. Then, study (average, distribution etc) the following quantity

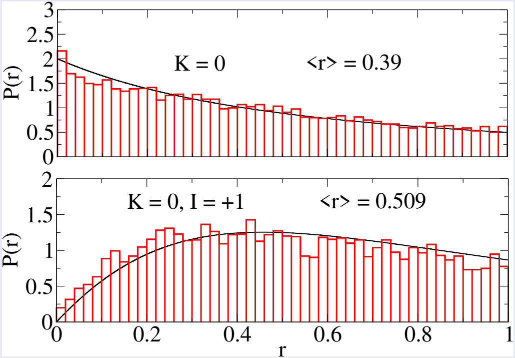
$$r = \frac{min(\Delta(i), \Delta(i+1))}{max(\Delta(i), \Delta(i+1))}$$

- For integrable systems, r shows Poissonian distribution with $P(r) = \frac{2}{(1+r)^2}$ and $\langle r \rangle \approx 0.386$. No, level repulsion.
- Non-integrable systems shows level repulsion, r shows *GOE* distribution ($P(r) = \frac{27}{4}\frac{r+r^2}{(1+r+r^2)^{2.5}}$) and $\langle r \rangle \approx 0.536$.



one subtle issue

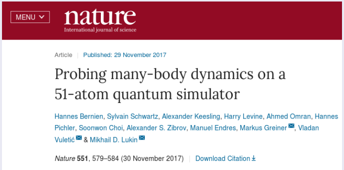
- One needs to resolve all the conserved quantities present in the system, otherwise you will get wrong result.



- Ref : “Quantum Signatures of Chaos ”, Fritz Haake.

I. Forward scattering approximation

Introduction : Experiment



MENU
nature
International journal of science

Article | Published: 29 November 2017

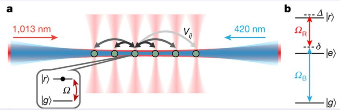
Probing many-body dynamics on a 51-atom quantum simulator

Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soonwon Choi, Alexander S. Zibrov, Manuel Endres, Markus Greiner, Vladan Vuletić & Mikhail D. Lukin

Nature **551**, 579–584 (30 November 2017) | Download Citation

Array of 51 ^{87}Rb atom described by

$$\frac{\mathcal{H}}{\hbar} = \sum_i \frac{\Omega_i}{2}\sigma_x^i - \sum_i \Delta_i n_i + \sum_{i<j} V_{ij} n_i n_j$$



$$\boxed{|r\rangle = |70S_{1/2}, J = 1/2, m_J = -1/2\rangle}$$

$$\boxed{|g\rangle = |5S_{1/2}, F = 2, m_F = -2\rangle}$$

Quantum many body scars

Turner et al, *Nature Physics* **14**, *745-749 (2018).*

$$HSD_L = \#|\underbrace{r, g, \dots\dots,}_{L-1\ sites} g\rangle + \#|\underbrace{r, g, \dots\dots,}_{L-2\ sites} g, r\rangle$$

$$= \quad HSD_{L-1} \quad + \quad HSD_{L-2}$$

with $HSD_1 = 2$ and $HSD_2 = 3$ we get $HSD_L^{OBC} = F_{L+2}$

similarly we get $HSD_L^{PBC} = F_{L-1} + F_{L+1} \sim \tau^L$ in large L limit.

τ is the golden ratio~ 1.62

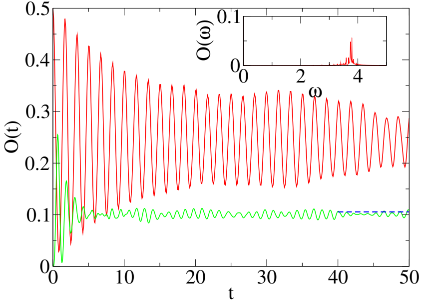
The effective Hamiltonian

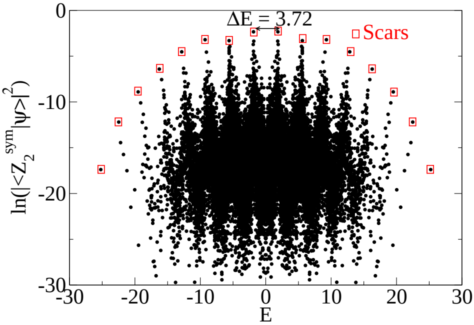
$$H = -\sum_i (\tilde{\sigma}_i^x + \lambda \sigma_i^z)$$

where $\tilde{\sigma}_i^x = P_{i-1}\sigma_i^x P_{i+1}$ and $P_i = \frac{1-\sigma_i^z}{2}$.

$$\psi(t=0) = \mathbb{Z}_2^{sym} = \frac{|0,1,0,1...\rangle + |1,0,1,0...\rangle}{\sqrt{2}}$$

$$N = 30, HSD = 1860498, HSD(K=0, P=+1) = 31836$$





II A. Forward scattering approximation (from $|\mathbb{Z}_2\rangle$)

Preliminaries 1 : Gram-Schmidt orthogonalization

- Let, $Z = \{z_1, z_2, z_3 \cdots, z_n\}$ is a set of linearly independent vectors (non-orthogonal in general).
- Gram-Schmidt process generates a corresponding orthogonal set of vectors $Q = \{q_1, q_2, \cdots, q_n\}$ such that $sp\{Q\} = sp\{Z\}$.
- Q is given by

$$
\begin{aligned}
q_1 &= z_1 \\
q_2 &= z_2 - \text{proj}_{q_1}(z_2) \\
q_3 &= z_3 - \text{proj}_{q_1}(z_3) - \text{proj}_{q_2}(z_3) \\
&\vdots \\
q_k &= z_k - \sum_{i=1}^{k-1} \text{proj}_{q_i} z_k
\end{aligned}
$$

  where $\text{proj}_q(z) = \frac{\langle q|z\rangle}{\langle q|q\rangle} q$.
- You can orthonormalize Q if you wish.

Preliminaries 2: Lanczos algorithm

- **Motivation** : Calculation of ground and a few low lying excited states of large (Hilbert space dimension D) quantum systems described by a Hamiltonian H.
- **Krylov subspace :** $\mathcal{H}_k \equiv \{v_0, Hv_0, H^2 v_0, H^3 v_0, \cdots, H^{k-1} v_0\}$ where v_0 is a initial choice which should have nonzero overlap with the ground state. In many cases ($k \ll D$).
- orthonormalize the Krylov subspace by the following iteration

$$\beta_1 v_1 = w_0 = Hv_0 - \alpha_0 v_0$$
$$\beta_2 v_2 = w_1 = Hv_1 - \alpha_1 v_1 - \beta_1 v_0$$
$$\vdots$$
$$\beta_{i+1} v_{i+1} = w_i = Hv_i - \alpha_i v_i - \beta_i v_{i-1}$$

where $\alpha_i = v_i^T H v_i$ and $\beta_i = ||w_{i-1}|| = v_{i-1}^T H v_i$.

Preliminaries 2: Lanczos algorithm

- $V = \{v_1, v_2, \cdots, v_n\}$ are known as Lanczos vectors each of size $(D \times 1)$.
- Note that we do orthogonalization w.r.t only previous two vectors. But, the beauty of Lanczos algorithm is that this guarantees global orthonormality (in exact arithmetic). Therefore, $V^T V = I$.
- This gives the following tridiagonal matrix

$$H^{Lanczos} = V^T H V = \begin{pmatrix} \alpha_0 & \beta_1 & & & \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \beta_2 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \beta_{k-1} \\ & & & \beta_{k-1} & \alpha_{k-1} \end{pmatrix}_{(k \times k)}$$

- Diagonalization of $H_{Lanczos} \implies$ low lying eigenvalues of H.

Preliminaries 2: Lanczos algorithm

- Eigenvectors : one need to store all the Lanczos vectors i.e the matrix V.

- If $H_{Lanczos}\psi^0 = E_0\psi^0$ then $\Psi = V_{D\times k}\psi^0_{k\times 1}$ gives an approximate eigenstate of H of eigenvalue E_0.

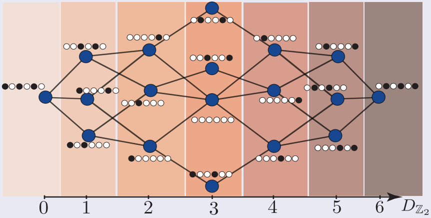
- **Stability :** very prone to numerical instability. Loss of orthogonality is the main issue. Reorthogonalization should be done whenever necessary.

- some numbers : Ground and low lying excited states of systems with Hilbert space dimension ($D \sim 30000000$) can be be obtained with k as small as 1000.

- Ref : "Lanczos Algorithms for Large Symmetric Eigenvalue Computation "by Cullum & Willoughby.

FSA (from $\mathbb{Z}_2$)

- Lanczos calculation starting from $|v_0\rangle = \mathbb{Z}_2$ with Krylov space dimension (k) $L+1$ is sufficient to capture the scars in PXP model.



- Lets do the following decomposition of PXP model

$$H = -\sum_i \tilde{\sigma}_i^x = H^+ + H^- \quad \text{where } H^{\pm} = -\sum_{i\in even} \tilde{\sigma}_i^{\pm} - \sum_{i\in odd} \tilde{\sigma}_i^{\mp}$$

- Note that $H^-\mathbb{Z}_2 = 0$ and $H^+\bar{\mathbb{Z}}_2 = 0$. So, if we take $v_0 = \mathbb{Z}_2$, then from Lanczos algorithm

FSA (from $\mathbb{Z}_2$)

$$\beta_1 v_1 = Hv_0 - \alpha_0 v_0 = H^+ v_0 \quad \text{as } \alpha_0 = v_0^T H v_0 = 0$$

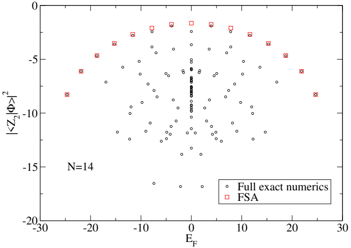
$$\beta_2 v_2 = Hv_1 - \alpha_1 v_1 - \beta_1 v_0 = H^+ v_1 + (H^- v_1 - \beta_1 v_0) \quad \text{as } \alpha_1 = 0$$

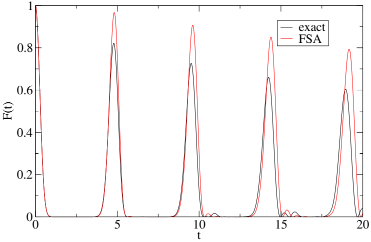
- $\alpha_i = 0, \forall i$.
- note that $H^+ v_0 = -\sum_{i \in odd} |0_i 01010 \cdots\rangle$. So, $\beta_1 = ||H^+ v_0|| = \sqrt{\frac{L}{2}}$.
- $H^- v_1 = \frac{L}{2}\sqrt{\frac{2}{L}}|1010\cdots\rangle = \beta_1 v_0$. $\therefore (H^- v_1 - \beta_1 v_0) = 0$.
- Similarly, $H^- v_2 - \beta_2 v_1 = 0$.
- Let's write (though not true in general)

$$H^+ v_i = \beta_{i+1} v_{i+1} \quad \textbf{forward scattering}$$
$$H^- v_i = \beta_i v_{i-1} \quad \textbf{backward scattering}$$

- Exact at all j, only for free paramagnet ($H = -\sum_i \sigma_i^x$).

$$H_{FSA}^{PXP} = V^T H V = \begin{pmatrix} 0 & \beta_1 & & & \\ \beta_1 & 0 & \beta_2 & & \\ & \beta_2 & 0 & \ddots & \\ & & \ddots & \ddots & \beta_{k-1} \\ & & & \beta_{k-1} & 0 \end{pmatrix}_{((L+1)\times(L+1))}$$





- FSA gives good results at very small ($\sim L$) computational cost.
- To calculate observables $\Rightarrow$ store FSA vectors ($\sim L\phi^L$).

Perfect scars and perfect oscillations

- Why the oscillations decay ? System leak outside the FSA manifold.
- Quantify the FSA erros :

$$\delta_j = ||H^{-}|v_j\rangle - \beta_j|v_{j-1}\rangle||$$

- $\delta_j \neq 0$ for $j > 2$ in PXP model.
- Can the oscillation be enhanced and made nearly perfect ?
- Note that, FSA errors $\equiv$ damping force. Therefore,

  reduction of FSA errors $\Rightarrow$ enhancement of oscillations.

- What term can be added to PXP model to fulfill this dream ??

$$H_{perturb} = -h_{xz} \sum_i \tilde{\sigma}_i^x (\sigma_{i-2}^z + \sigma_{i+2}^z)$$

FSA with $H_B = H^{PXP} + H_{perturb}$

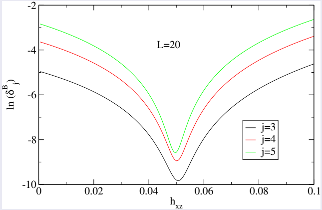
Redefine the decomposition $\Rightarrow H_B = H_B^+ + H_B^-$ with

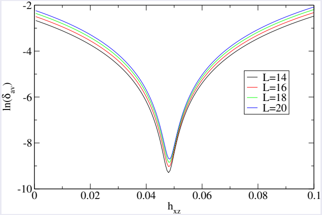
$$H_B^{\pm} = -\sum_{i \in even} \tilde{\sigma}_i^{\pm} W_i - \sum_{i \in odd} \tilde{\sigma}_i^{\mp} W_i$$

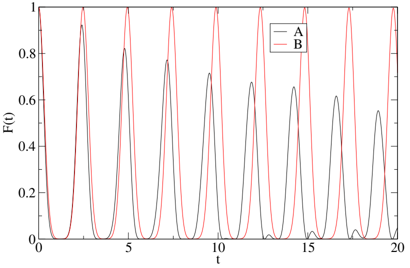
where $W_i = \mathbb{I} + h_{xz}(\sigma^z_{i-2} + \sigma^z_{i+2})$.

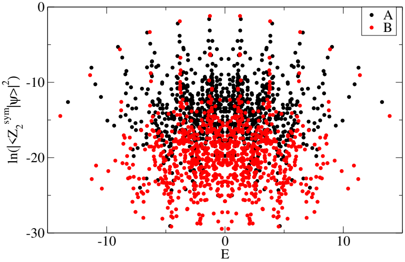
the new FSA errors : $\delta_j^B = ||H_B^- |v_j^B\rangle - \beta_j^B |v_{j-1}^B\rangle||$.

Again, $\delta_1^B = \delta_2^B = 0$.









further

$$\delta H_R = \sum_i \sum_{d=2}^{R} h_d(\sigma^z_{i-d} + \sigma^z_{i+d})$$

with

$$h_d = h_0(\tau^{d-1} - \tau^{-(d-1)})^{-2}$$

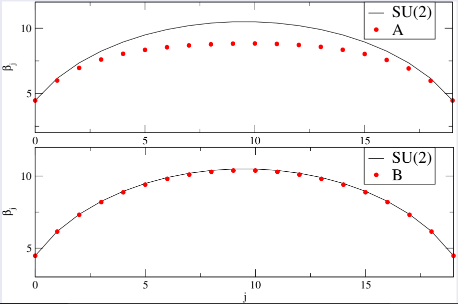
gives 99.9999% revival for N=32 !!!

- Rest of the spectrum becomes strongly ergodic.

Emergent SU(2) algebra

- When all FSA errors are cancelled, FSA vectors become eigenstate of $H_B^z = [H_B^+, H_B^-]$.
- $\{H_B^+, H_B^-, H_B^z\}$ plays the role of $\{S^+, S^-, S^z\}$ within $\mathcal{K}_{L+1}$ and forms an $s = L/2$ representation of $SU(2)$ algebra.

$$\begin{aligned} S^-|s,j\rangle &= \sqrt{(s+j)(s-j+1)}|s,j-1\rangle \\ H_B^-|v_j^B\rangle &= \beta_j|v_{j-1}^B\rangle \end{aligned}$$



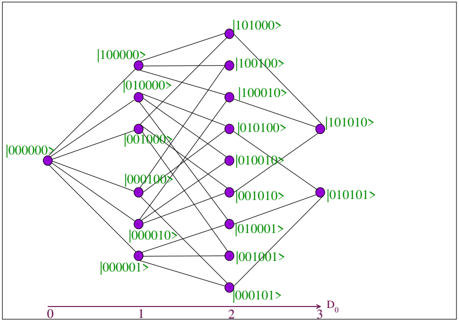
II B. Forward scattering approximation (from $|0\rangle$).

Decompose the PXP Hamiltonian

$$H = -\sum_i \tilde{\sigma}_i^x = H^+ + H^-$$

where

$$H^{\pm} = -\sum_i \tilde{\sigma}_i^{\pm}$$



- # FSA vectors $= \frac{L}{2} + 1$.

• We find, in this case also, the first two step is exact i.e $\delta_1 = \delta_2 = 0$ and $\delta_i \neq 0$ for $i \geq 3$.

• We find $\sum_{i=3}^{L/2+1} \delta_i = 17.54$ for $|0\rangle$ where as $\sum_{i=3}^{L+1} \delta_i = 1.85$ for $\mathbb{Z}_2$ in bare PXP model (Model-A) at L=20. This causes rapid thermalization from $|0\rangle$.

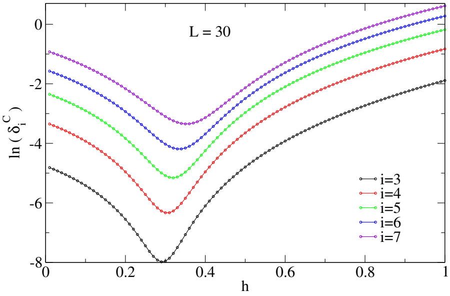
• Can we come up with terms that reduces the FSA errors for $|0\rangle$?? (Again the most challenging part).

$$\textbf{Model-C :} \; H_C = -\sum_i \tilde{\sigma}_i^x - (h \sum_i \tilde{\sigma}_i^+ \tilde{\sigma}_{i-1}^- \tilde{\sigma}_{i+1}^- + h.c)$$

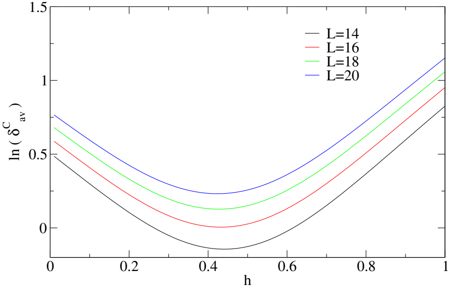
the additional term connects $|\cdots 01010 \cdots\rangle$ to $|\cdots 00100 \cdots\rangle$.

• Again we find $\delta_1^C = \delta_2^C = 0$ though the FSA vector changes.

- Higher order FSA errors must also be minimized. But unlike $\mathbb{Z}_2$, both $\delta_j^{C\ min}$ and corresponding h^{min} increases steadily with i.



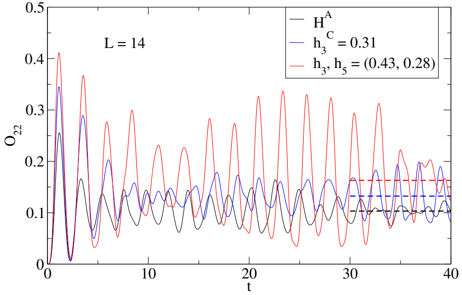
- Average error per step increases with L, much more rapidly compared to $\mathbb{Z}_2$.

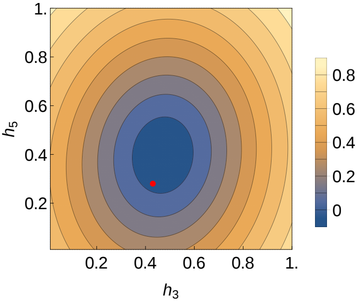


- **Solution :** Just like the $\mathbb{Z}_2$ case, add longer range term of similar nature.

$$H = H^C + h_5 \sum_i (\tilde{\sigma}^+_{i-1}\tilde{\sigma}^+_{i+1}\tilde{\sigma}^-_{i+2}\tilde{\sigma}^-_{i}\tilde{\sigma}^-_{i-2} + h.c)$$

- these parameters must be optimized together.





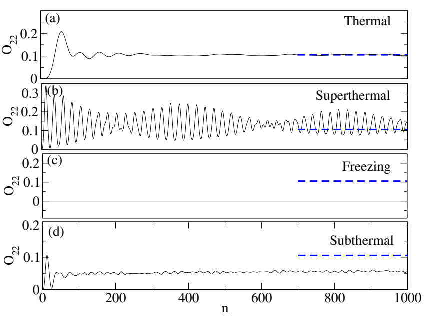
Physical realization

- Periodic dynamics of vacuum state ($|0\rangle$).
- Protocol :

$$\lambda(t) = +\lambda \; ; \; 0 < t \leq T/2$$
$$\quad = -\lambda \; ; \; T/2 < t \leq T.$$

$$U(T,0) = e^{-iH[\lambda]T/2\hbar} e^{-iH[-\lambda]T/2\hbar}$$
$$\quad = e^{-iH_F T}$$

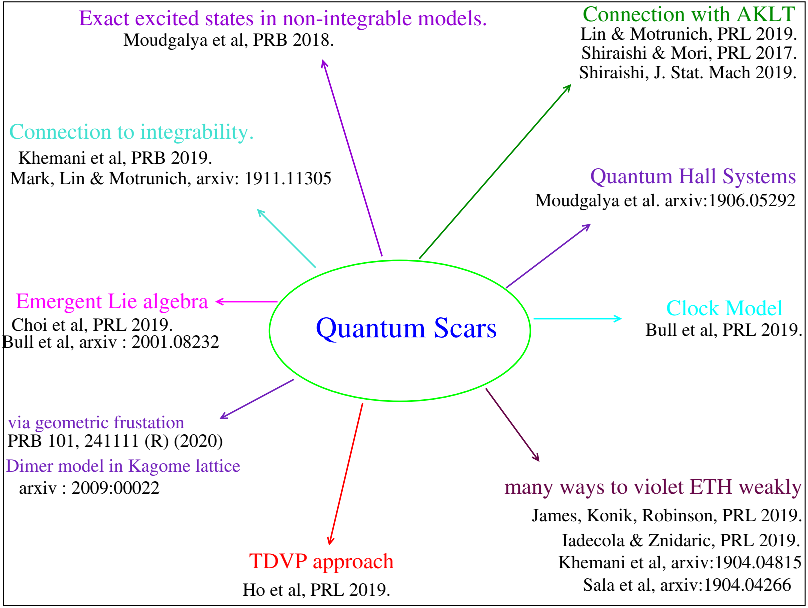
- Variety of non-thermal phases.
  - Superthermal. $\Longrightarrow$ FSA
  - Freezing of wavefunction.
  - Subthermal.



$$H_F^{eff} = \sum_i (C_1(\lambda,T)\tilde{\sigma}_i^x + C_2(\lambda,T)\tilde{\sigma}_i^y) + C_3(\lambda,T)\sum_i(\tilde{\sigma}_i^+\tilde{\sigma}_{i-1}^-\tilde{\sigma}_{i+1}^- + h.c)$$

$$\left.\frac{|C_3|}{\sqrt{|C_1|^2+|C_2|^2}}\right|_{\lambda^{super},T^{super}} = 0.35$$

Mukherjee et al, Phys. Rev. B 102, 075123 (2020)


Quantum Scars
Exact excited states in non-integrable models.
Moudgalya et al, PRB 2018.
Connection with AKLT
Lin & Motrunich, PRL 2019.
Shiraishi & Mori, PRL 2017.
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Emergent Lie algebra
Choi et al, PRL 2019.
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Clock Model
Bull et al, PRL 2019.
via geometric frustation
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Dimer model in Kagome lattice
arxiv : 2009:00022
many ways to violet ETH weakly
James, Konik, Robinson, PRL 2019.
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TDVP approach
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