I. FINITE DIMENSIONAL VECTOR SPACES: APPENDIX – A

1. Let $S = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ be a set with 4- vectors each of size 3. Verify that any subsets of three vectors are linearly independent. How many such subsets are there?

2. If X_1, X_2, X_3 are three linearly independent vector in R^n, then so are $(X_1 + X_2)$, $(X_1 + X_3)$ and $(X_2 + X_3)$. Prove or disprove.

3. Let $a = (a_1, a_2, a_3)^T$, $b = (b_1, b_2, b_3)^T$, $c = (c_1, c_2, c_3)^T$. Under what condition these are linearly independent?

4. Let

   $S_1 = \{ X = (x_1, x_2, x_3)^T | \, x_1 + x_2 = 1\}$
   $S_2 = \{ X = (x_1, x_2, x_3)^T | \, x_1 + x_2 = 0\}$
   $S_3 = \{ X = (x_1, x_2, x_3)^T | \, x_1 = 0\}$

   Which of these are subspaces?

5. A function $f : R^n \to R$ is called a functional. A functional in linear if

   $f(x_1 + x_2) = f(x_1) + f(x_2)$, $\boxed{x_1, x_2 \in R^n}$

   $f(a\,x) = a\, f(x)$ $\boxed{x \in R^n, a \in R}$

   Which of the following are linear functional?

   (a) $f(x) = \sum_{i=1}^{3} a_i\, x_i + b$
   (b) $f(x) = x_2$
   (c) $f(x) = x_1^2$,
   (d) $f(x) = -x_1 + 5\,x_2 + 7x_3$
   (e) $f(x) = x_1 - x_2$
   (f) $f(x) = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

6. Consider a Quadratic functional

   $Q : R^2 \to R$ given by

   $Q(x) = (x_1 \; x_2) \begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \lambda\, x_1^2 + x_2^2$

   Draw the contour plots of $Q(x)$ for

   $\lambda = 0.1$, $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$, $\lambda = 5$, comment on the effect of variation of λ on the shape of the contours.

7. Let

$$Q(x) = (x_1, x_2)\begin{pmatrix} 2 & a \\ a & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Draw the contours of $Q(x)$ for $a = \pm 0.5, \pm 1.0$. What happens when a is +ve and a is –ve ?

8. Verify : $\|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$ - Parallelogram law.

9. Draw the unit sphere in $\|x\|_2$, $\|x\|_1$, $\|x\|_\infty$, $\|x\|_A$

10. Cauchy – Schwartz inequality :

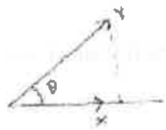
$$|X^T Y| = \|X\| \; \|Y\| \cos\theta$$

And conclude

$$|X^T Y| \leq \|X\| \; \|Y\|$$

11. Let $\|X\| = 1$ and Y be any vector.

$$X^T Y = \|Y\| \, \|X\| \cos\theta = \|Y\| \cos\theta$$
$$= \text{Projection of Y onto X}$$



II. MATRICES: APPENDIX – B

1. Prove $(AB)^T = B^T A^T$

   $\det(aA) = a^n \det(A)$

2. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   Compute $\det(A)$, $A^{-1}, AB, (AB)^{-1}$

3. Let

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   (a) Verify (i) $A^T A = A A^T = I$ (ii) $A^T = A^{-1}$

   (b) Let $x = \binom{1}{1}$ and $y = Ax$. Plot y for $\theta = 0, 15^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ$.

   (c) Explain the action of A on $X = \binom{1}{1}$

   (d) Find $\|x\|$, $\|y\|$ where $y = Ax$. Are they same? Why?

4. Find examples of two 2×2 matrices A and B such that $AB \neq BA$.

5. Let $P \in R^{n \times n}$, non-singular, $h \in R^n$. Then $hh^T \in R^{n \times n}$ and $(P + h\,h^T) \in R^{n \times n}$.

   Verify that $(P + hh^T)^{-1} = P^{-1} - \frac{P^{-1} h\, h^T P^{-1}}{1 + h^T P^{-1} h}$
   by multiplying $(P + h\,h^T)(P + h\,h^T)^{-1}$ and showing that the product is I_n.

6. Verify : $tr(AB) = tr(BA)$

   $tr(ABC) = tr(CAB) = tr(BCA)$

7. Let A and B be non-singular : verify that
   $(AB)^{-1} = B^{-1} A^{-1}$

8. Verify : Sherman – Morrison – Woodbury formula

   . $(I_n + cd^T)^{-1} = I_n - \dfrac{c\,d^T}{1 + d^T c}$ $\quad c, d \in R^n$

   . $(A + cd)^{-1} = A^{-1} - \dfrac{A^{-1} c\, d^T A^{-1}}{1 + d^T A^{-1} C}$ $\quad c, d \in R^n$

$. (A + CD)^{-1} = A^{-1} - A^{-1}C\,[I_k + D^T A^{-1} C]^{-1} D^T A^{-1}$

$A \epsilon R^{n \times n}$, $C \epsilon R^{n \times k}$, $D \epsilon R^{n \times k}$

$. (A + CBD)^{-1} = A^{-1} - A^{-1}\, C\,[B^{-1} + D^T A^{-1} C]^{-1} D^T A^{-1}$

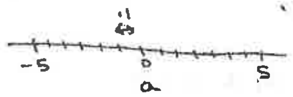
$A \epsilon R^{n \times n}$, $B \epsilon R^{k \times k}$, $C \epsilon R^{n \times k}$, $D \epsilon R^{\,n \times k}$

9. Find the eigenvalues and eigenvectors of

$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$

10. Compute the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

11. Plot the eigenvalues of



$A = \begin{bmatrix} 5 & a \\ a & 8 \end{bmatrix}$

for $a \,\epsilon\, [-5, 5]$ in steps of 0.1

12. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Rewrite

$X^T A\, X = a\, x_1^2 + 2\, b\, x_1 x_2 + \; c\, x_2^2$ as

$X^T A\, X = a \left(x_1 + \frac{b}{a}\, x_2 \right)^2 + \left(c - \frac{b^2}{a} \right) x_2^2$

Verify A is positive definite when $a > 0, c > 0 \; and \; b^2 < a\, c$

III. MULTIVARIATE CALCULUS: APPENDIX - C

1. Let $a \epsilon R^n$, $X \epsilon R^n$. Let $f(X) = a^T X = \sum_{i=1}^{n} a_i x_i$

$\nabla f(x) = a$ $\quad$ – $\boxed{Gradient\ of\ f(x)}$

2. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $f(x) = X^T A X$

$\nabla f(x) = 2AX$

3. Let $Q(X) = \frac{1}{2} X^T A X - b^T X$, $\quad$ $\boxed{A - symmentic\ \epsilon R^{n \times n} \quad b \epsilon R^n,\ X \epsilon R^n}$

$\nabla Q = AX - b$

4. $h : R^n \rightarrow R^m$ $h(x) = (h_1(x), h_2(x), \dots, h_m(x))^T$

$x = (x_1, x_2, \dots, x_n)^T$

Jacobian of $h(x) = D_h(x) \epsilon R^{m \times n}$

$$D_h(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial h_n} \end{bmatrix}$$

5. $f : R^n \rightarrow R$

Gradient: $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \dots \frac{\partial f}{\partial x_n} \right)^T \epsilon R^n$

$$\text{Hessian : } \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n^2} \end{bmatrix} \epsilon R^{n \times n}$$

Compute the Hessian of $f(x)$ in (2) above.

6. Let $h(t, x) = x_2 \exp\left\{ -\left(\frac{t + x_3}{x_4} \right)^2 \right\} + x_1$ where

$x = (x_1, x_2, x_3, x_4,)$ and t is time.

Compute $\frac{\partial h}{\partial t}$, $\nabla_x h(x)$, $\nabla_x^2 h(x)$

7. Let $x \epsilon R^n$, $h: R^n \to R^m$ and $h(x) = (h_1(x), h_2(x) \dots h_m(x))^T$. Let
$\emptyset(x) = \frac{1}{2} h^T(x)\, A\, h(x)$, $A - SPD$

Verify that

$\nabla\emptyset(x) = D_h^T(x)\, A\, h(x)$, $D_h(x) \epsilon R^{m\times n}$ is the Jacobian of h.

8. Let $f(x)$ be a smooth function. Taylor expansion for $f(x)$:
$$f(x+\alpha) = f(x) + \frac{df}{dx}.\alpha + \frac{1}{2}\frac{d^2 f}{dx^2}\alpha^2$$

. Let $f: R^n \to R$, $x \epsilon R^n$, $\alpha \epsilon R^n$

$$f(x+\alpha) = f(x) + \alpha^T \nabla f(x) + \frac{1}{2}\alpha^T \nabla^2 f(x)\, \alpha$$

9. First Variation $\delta f(x)$ resulting from variation δx in x.
$\delta f(x) = < \nabla f(x), \delta x >$

10. Let $\emptyset(x) = a^T x = < a_1 x >$. Then, $\delta\emptyset = a^T \delta x = < a, \delta x >$
Let $\emptyset(x) = \frac{1}{2} x^T A x$. Then, $\delta\emptyset(x) = < Ax, \delta x >$
Let $\emptyset(x) = \frac{1}{2}(z - Hx)^T(z - Hx)$. Then, $\delta\emptyset = < H^T(Hx - z), \delta x >$

11. Let $\emptyset: R^n \to R$ be twice continuously differentable. Then, $\emptyset$ is convex when the Hessian $\nabla^2\emptyset$ is non-negative definite and $\emptyset$ is strictly convex if $\nabla^2\emptyset$ is positive definite.
. Examine the convexity of $\emptyset(x) = x^2$, $\emptyset(x) = x^3$, $\emptyset(x) = (z - Hx)^T(z - Hx)$

IV. OPTIMIZATION: APPENDIX – D

1. $\emptyset : R^n \to R$. At the minimum :
   $\nabla\emptyset(x) = 0$
   $\nabla^2\emptyset(x)$ is SPD

2. Minimize $\emptyset(x)$ when $g(x) = b$ is an equality constrained minimization problem.

   Lagrangian formulation (Strong constraint)
   $$\min_{x,\lambda} L\,(x,\lambda) = \emptyset\,(x) + \lambda^T\,(g(x) - b)$$

   Penalty function method (Week constraint)
   $$\min_{x} P\ (x) = \emptyset\,(x) + \frac{\propto}{2}\ \|g(x) - \propto\|^2$$

   When $\propto > 0$ is called penalty constant.
   It turns out that the solution of the weak constraint problem tends to that of the strong constraint problem as $\propto \to \infty$

3. (i) Find the minimum of $\emptyset(x) = x_1^2 + x_2^2$ when $g(x) = x_1 + x_2$ and $b = 1$
   (ii) Minimize $x_1^2 + x_2^2$ when $x_1 + x_2 = 1$ by both formulation. Verify that the weak solution tends to strong solution as $\propto \to \infty$.

4. Let $A = ab$ when $a + b = \frac{L}{2}$
   Maximize A using strong and weak constrained formulation.

V. STATIC INVERSE PROBLEM: CHAPTER 5-8

1. $H \epsilon R^{m \times n}$, $m > n$. H be of Full Rank, i.e, Rank (H) = min {m , n} = n
   The generalized inverse of H is denoted by H^+ and is given by
   $H^+ = (H^T H)^{-1} H^T, \quad H \epsilon R^{m \times n}$ (1)
   Properties of H^+
   (a) $H H^+ H = H$ (c) $(H H^+)^T = H H^+$
   (b) $H^+ H H^+ = H^+$ (d) $(H^+ H)^T = H^+ H$

   - Verify properties $(a) - (d)$
   - Compute the generalized inverse of

   $(a)\ H = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (b)\ H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and verify these properties.

2. Define
   $P_H = H H^+ = H (H^T H)^{-1} H^T$ called the projection (orthogonal) matrix (2)

   - Compute P_H when $H = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
   - Compute $P_H z$ when $z = (1, 1, 1)^T$ for these values of H
   - Compute $e = z - P_H z$ and compute $< e , P_H z >$ and verify $e \perp P_H z$

3. Verify that P_H *in* (2) is such that
   a) $P_H = P_H^T$ - Symmetric
   b) $P_H^2 = P_H$ - idempotent

   P_H is called orthogonal projection matrix.

4. $\text{Min}_x f(x) = (z - Hx)^T w (z - Hx)$ where $z \epsilon R^m$, $H \epsilon R^{m \times n}$, $x \epsilon R^n$, $W \epsilon R^{m \times m}$ - SPD

   Verify that the minimizer is $\hat{x} = (H^T W H)^{-1} H^T W z$

5. Define
   $P_{H,W} = H (H^T W H)^{-1} H^T W$
   (i)Verify $P_{H,W}$ is not symmetric but is idempotent , that is
   (ii) $P_{H,W}^2 = P_{H,W}$
   $P_{H,W}$ is called an oblique projection matrix.

6. Let $h = \binom{1}{1}$, $z = \binom{2}{3}$, $W = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$ be the weight matrix.
   - Formulate and solve liner weighted least squares problem $z = hx$.

- Find x_{LS} and $e(x_{LS})$
- Find the angle between $e(x_{LS})$ and h, where $\alpha = 1,\ 1.5,\ 0.75$.

7. Define

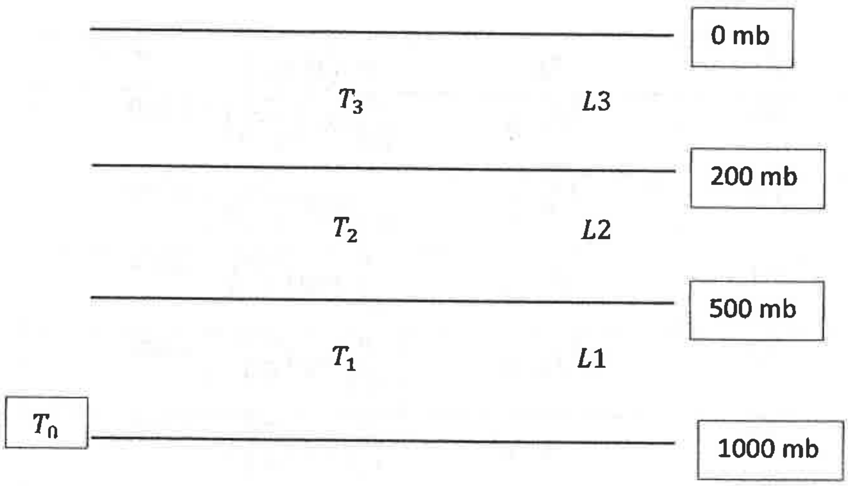
   $f(x) = (z - Hx)^T R^{-1}(z - Hx) + (x - x_B)^T B^{-1}(x - x_B)$

   Find the minimize x_{LS} of $f(x)$.

   Note - This is the so called 3-D VAR Problem.

8. Verify the computation in section 8.1 (Chapter 8 in LLD(2006)) leading to the recursive form of least squares solution.

9. PROBLEM 5.7 (Chapter 5, LLD(2006))Three layer atmosphere



.T_0 – temperature of earth's surface

. T_i – Average temperature of layer $i, 1 \le i \le 3$

Energy radiated R at frequency f is

$$R_f = e^{-\gamma f} + \int_0^1 T(p)\, w(p, \gamma_f) dp \quad \text{Where}$$

$$w(p, \gamma_f) = p\gamma_f\, e^{-p\gamma_f} \qquad \longrightarrow (1)$$

DISCRETISE USING 3- LAYERS

For the three layer problem:

$$R_f = e^{-\gamma f} + \int_0^{200} T_3 w(p, \gamma_f) dp + \int_{200}^{500} T_2 w(p, \gamma_f) dp + \int_{500}^{100} T_1 w(p, \gamma_f)\, dp$$

$\longrightarrow$ (2)

Rewrite as (after integrating)

$$\underbrace{\left[\int_0^{200} w\,(p,\gamma_f)\,dp\right]}_{a_{3f}} T_3 + \underbrace{\left[\int_{200}^{500} w\,(p,\gamma_f) dp\right]}_{a_{2f}} T_2 + \underbrace{\left[\int_{500}^{1000} w\,(p,\gamma_f)\,dp\right]}_{a_{1f}} T_1$$

$$= R_f - e^{-\gamma f} \longrightarrow (3)$$

Substituting for $w\,(p,\gamma_f)$ from (1) into (3) and integrating we get:

LINEAR MODEL

$$\boxed{a_{1f}T_1 + a_{2f}T_2 + a_{3f}T_3 = R_f - e^{-\gamma f}} \longrightarrow (4)$$

Generate date

i	f_i	γ_{fi}	$w\left(p,\gamma_{fi}\right)$	$e^{-\gamma_{fi}}$
1	0.9	$1/0\cdot 9$	$\frac{p}{0.9}\exp\left(\frac{-p}{0\cdot 9}\right)$	0.329
2	1.0	$1/0\cdot 7$	$\frac{p}{0.7}\exp\left(\frac{-p}{0\cdot 7}\right)$	0.240
3	1.1	$1/0\cdot 5$	$\frac{p}{0.5}\exp\left(\frac{-p}{0.5}\right)$	0.135
4	1.2	$1/0\cdot 3$	$\frac{p}{0.3}\exp\left(\frac{-p}{0.3}\right)$	0.036
5	1.3	$1/0\cdot 2$	$\frac{p}{0.2}\exp\left(\frac{-p}{0.2}\right)$	0.007

using the values in the Table and setting $T_1 = 0.9,\ T_2 = 0.85,\ T_3 = 0.875$

First compute $\left(R_{fi} - e^{-\gamma_{fi}}\right)$ for $1 \le i \le 5$ using the formula (4)

Define

$$z_i = \left(R_{fi} - e^{-\gamma_{fi}}\right) + v_i \longrightarrow (5)$$

$v_i \sim N\,(\,0,\ \sigma^2)$, the observation noise and generate $z = (z_1, z_2, z_3, z_4, z_5)$

Set up a liner least squares problem

$$\begin{bmatrix} a_1f_1 & a_2f_1 & a_3f_1 \\ a_1f_2 & a_2f_2 & a_3f_2 \\ a_1f_3 & a_2f_3 & a_3f_3 \\ a_1f_4 & a_2f_4 & a_3f_4 \\ a_1f_5 & a_2f_5 & a_3f_5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \longrightarrow (6)$$

And solve for $T^* = (T_1^*,\ T_2^*,\ T_3^*)$

$T(truth) = (0.9,\ 0.85,\ 0.875,)$ and

Find $\|T\ (truth) - T^*\|$

Plot $\|T\ (truth) - T^*\|$ Vs $\sigma^2 = 0.1,\ 0.5,\ 1.0,\ 1.5,\ 2.0$ and comment

10. <u>Non liner problem</u>

Let $T(p) = x_1(p - x_2)^2 + x_3$ for $0 \leq p \leq 1$

Let $z_i = \int_{a_i}^{b_i} T\ (p)\ dp.$ Define

$x = (x_1,\ x_2,\ x_3)^T, \quad z = (z_1, z_2, z_3, z_4)^T$

$h(x) = (h_1(x),\ h_2(x),\ h_3(x),\ h_4(x))^T.$
Set $z = h(x)$
Data:-

i	a_i	b_i	z_i
1	0.00	0.25	0.21
2	0.20	0.50	0.15
3	0.30	0.70	0.51
4	0.60	0.80	0.11

- Integrate: $z_i = \int_{a_i}^{b_i} T(p)dp = h_i(x) \quad 1 \leq i \leq 4$
- Set up: $f(x) = (z - h(x))^T(z - h(x))$
- Find $\nabla f(x),\ \nabla^2 f(x)$
- Update using the first – order approximation.

VI. MATRIX METTHODS: CHAPTER 9

1. Let $m = 10$ and $n = 8$. Generate a matrix $H \in \mathbb{R}^{m \times n}$ where $h_{ij} \in [-1, 1]$ uniformly randomly. Also generate $z \in \mathbb{R}^m$ where $z_i \in [-1, 1]$ uniformly randomly.
   1) Compute $(H^T H)$, $H^T z$
   2) Solve $(H^T H)x$, $H^T z$ using
      a) Cholesky decomposition
      b) QR decomposition
      c) S V D
   3) Compare the solution.

2. Let

$$H = \begin{bmatrix} 1.0 & 0.0 \\ 1.0 & 1.0 \\ 1.0 & 2.0 \\ 1.0 & 3.0 \end{bmatrix} \qquad z = \begin{bmatrix} 1.0 \\ 3.0 \\ 2.0 \\ 3.0 \end{bmatrix}$$

   a) Compute $z - H\,x,\ x = (x_1, x_2)^T$
   b) Compute $f(x) = (z - H\,x)^T\,(z - H\,x)$
   c) Plot the contours of $f(x)$ and identify graphically the minimum.

3. Following the exercises in chapter 9, LLD (2006), develop your own MATLAB program to solve
   a) a lower / upper triangular system
   b) LU - decomposition
   c) Cholesky decomposition
   d) QR decomposition

4. Compute number of operations needed by each of these algorithms.
5. Analyze the modified Gramm- Schmidt algorithm.

VII. OPTIMIZATION PROCEDURE: CHAPTER 10-12

1. Let $f(x) = \frac{1}{2}\, x^T A\, x - b^T x$ where $A \in \mathbb{R}^{n \times n}$ is SPD and $b \in \mathbb{R}^n$. Using gradient algorithm to minimize $f(x)$, show that

$$E\,(x_{k+1}) = \left[1 - \frac{(\Lambda_k^T \Lambda_k)^2}{(\Lambda_k^T A\, \Lambda_k)(\Lambda_k^T A^{-1} \Lambda_k)}\right] E(x_k)$$

Where $E\,(x) = f(x) - f(x^*)$ and $x^* = A^{-1}\, b$

2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$ and $f(x) = \frac{1}{2}\, x^T A\, x$

- using gradient algorithm, verify that $x_k = \left(\frac{\lambda-1}{\lambda+1}\right)^k \begin{pmatrix} \lambda \\ (-1)^k \end{pmatrix}$, $with\ x_0 = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$

- For $\lambda = 4$, plots the trajectory of the gradient algorithm.

3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $f(x) = \frac{1}{2}\, x^T A\, x$

- Verify that the trajectory $\{x_k\}$ of the gradient algorithm is given by
$$x_k = \left(\frac{1}{3}\right)^k \begin{bmatrix} 2 \\ (-1)^k \end{bmatrix}, x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
- Verify $f(x_{k+1}) = \frac{1}{9}\, f\,(x_k)$
- Plot $f(x_k)$ Vs k.

4. Verify the properties of conjugate direction and conjugate gradient methods as given is chapter 11, LLD (2006).

5. Let $m = 50$, $n = 40$, generate $H \in \mathbb{R}^{m \times n}$ and $z \in \mathbb{R}^m$ randomly.

- Compute $e(x) = z - Hx$ and $f(x) = \mathrm{e}^{\mathrm{T}}(x)\, \mathrm{e}\,(x) = (z - H\,x)^T\,(z - Hx)$

- Minimize $f(x)$ using
  (a) Gradient algorithm
  (b) Conjugate gradient algorithm.
  (c) Plot $f(x_k)$ Vs k for both the methods in the same plot and compare.

6. Prove that the Newton's algorithm for find the solution of $f(x) = 0$ converges quadratically.

VIII. Statistical Estimation: CHAPTER 13 -17

1. Define the following properties of an estimate
   a) Biased Vs unbiased
   b) Consistency
   c) Efficiency
   d) Linear Vs non linear

2. State and prove Gauss Markov theorem.

3. Following example 15.1.1, describe the maximum likelihood estimation.

4. Following example 16.2.1, analyze the Bayasian method for estimation.

5. Repeat problem (4) for the vector case described in Example 16.2.2.

6. Following the developments in chapter 17, derive an expression for the linear, minimum variance estimation.

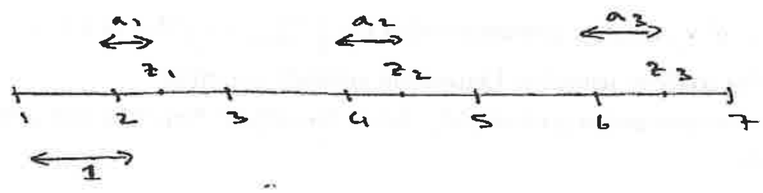
7. Referring to the Table 17.1.1, compare the Bayasian estimate is Example 16.2.2 and the linear minimum variance estimate in chapter 17. The equivalence between these two approaches rests on the matrix identity known as Sherman - Morrison – Woodbury formula.

8. Follow the derivation in section 17.2 to understand the underpinnings of the data assimilation phase in the Kalman filtering approach.

Note : All the reference to chapters and examples are from LLD (2006).

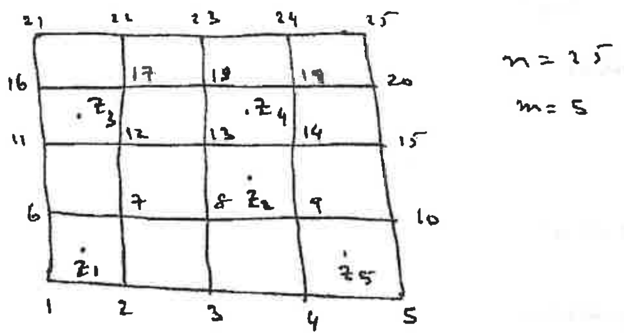
IX. BASIC DATA ASSIMILATION: CHAPTER 18-20

1. Consider 1-D uniform grid with n grid points and m observations : $n = 7,\ m = 3$



Compute the liner interpolation matrix $H \in R^{7\times3}$.

2. Consider a $2-D$ grid with n points and m- observations.



Compute $H \in R^{m\times n}$ bilinear interpolation matrix.

3. Compare Cressman's and Barne's scheme for spreading m observation on n - grid points $(m < n)$.

4. Derive the expression for Wieners optimal interpolation based on spatial covariance.

5. Following the developments in section 20.4, derive expression for the optimal estimates for the 3-D VAR problem using the second order approximation.

X. 4 –D VAR METHODS: CHAPTER 22-25

1. Following the development of fluvial dynamics in section 3.4, solve the problem 23.11 using Legrangian multiples method.

2. Let $x_{k+1} = a\, x_k$ with x_0 as I.C. and $z_k = hx_k + V_k$, $V_k \sim N\ (0, \sigma^2)$
   - Using $x_k = a^k x_0$, find the gradient of $Q(x) = \frac{1}{2} \sum_{k=1}^{N} \frac{(z_k - hx_k)^2}{\sigma^2}$ w.r.t. x_0
   - Compute the gradient using the Lagrangian multiplier method.
3. Following the development in section 23.5, derive the adjoint dynamics and develop the 4-D VAR framework.

4. Consider the non liner dynamics

   $\dot{x_1} = a\, x_1\, x_2$

   $\dot{x_2} = b\, x_1^2$ With $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

   With $a + b = 0$ (Burger's 2 mode)

   Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with

   $$\left.\begin{aligned} z_1 &= a\, x_1\ x_2 + v_1 \\ z_2 &= b x_1^2 + v_2 \end{aligned}\right\} v_i \sim N\ (0, \sigma^2)$$

   a) Discratize the model and express it as
   $$\left.\begin{aligned} x_{k+1} &= M(x_k) \\ z_k &= h(x_k) + v_k \end{aligned}\right\}$$
   by setting $a = \frac{1}{2}$, $b = \frac{-1}{2}$, and $\sigma^2 = 0.01$

   b) Let
   $$J(x_0) = \frac{1}{2} \sum_{k=1}^{10} \frac{(z_k - h(x_k))^2}{\sigma^2}$$
   - Draw the contours of $J(x_0)$ (Refer to Example 24.4.1)

   c) Minimize $J(x_0)$ using the 4-D VAR method.
5. Solve problem 24.17, 24.18

XI. KALMAN FILTERS: CHAPTERS 27-30

1. Reproduce the results of exercise 27.2.1, 27.2.2, 27.2.3, 27.2.4
2. Exercises : 27.3, 27.4, 27.5
3. Following section 28.2, explore the effect of cross correlation between model noise and observation noise on the structure of the Kalman Filter equation.
4. Following examples 28.7.1, describe the role of Polters algorithm in Kalman filtering.
5. Work through the details of the examples 29.1.1 and 29.1.2.
6. Derive the extended KF, and the second – order filters (See section 29.4)
7. Following the exercises 30.2 and 30.3 derive Ensemble Transform and Ensemble Adjustment Kalman Filters.