Problems for Course 3

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1. Suppose the passage time distribution t_e is compactly supported away from 0. That is, there are $0 < a < b < \infty$ such that $a \le t_e \le b$. Let

$$G := \#Geo(\underline{0}, n\mathbf{e}_1),$$

be the volume of the intersection of all geodesics from $\underline{0} = (0, \dots, 0)$ and $n\mathbf{e}_1 := (n, 0, \dots, 0)$. Show that

$$G \le \frac{b}{a} n$$

almost surely.

2. (All-pairs linear estimate for the passage time) Suppose the edge weights have finite second moment. Denote by

$$T(x, y) = \inf_{\gamma : x \leftrightarrow y} \sum_{e \in \gamma} t_e$$

the passage time for x to y. Let $S \subset \mathbb{Z}^d$ be a finite set.

(a) Suppose $x, y \in S$. Show that there is a constant C_0 independent of S such that we can build $2d$ disjoint paths from x to y with length at most $C_0|x - y|_1 = C_0 \sum_{i=1}^d |x_i - y_i|$.

(b) Show that for any $x, y \in S$,

$$\mathbb{P}(T(x, y) \ge \lambda) \le \left(\frac{C_0 \mathrm{diam}(S) \mathrm{Var}(t_e)}{(\lambda - C_0 \mathrm{diam}(S) \mathbb{E}[t_e])^2} \right)^{2d}.$$

(c) Conclude that there is a constant C_1 such that for all finite subsets S:

$$\mathbb{E}[(\max_{x,y \in S} T(x, y))^2] \le C_1 (\mathrm{diam}(S))^2. \tag{1}$$

3. (Geodesics are one-dimensional) Recall Kesten's lemma: assuming $\mathbb{P}(t_e = 0) < p_c(d)$, there are constants $a, C_2 > 0$ such that for all n

$$\mathbb{P}\Big(\exists \gamma \text{ self avoiding with } \#\gamma \ge n, T(\gamma) < an\Big) \le \exp(-C_2 n)$$

Let $\mathcal{G}$ be the set of all finite self-avoiding geodesics. Use Kesten's lemma and (1) to show the existence of a constant C_3 such that any finite set $E \subset \mathbb{Z}^d$,

$$\mathbb{E}[\max_{\gamma \in \mathcal{G}} \#(E \cap \gamma)] \le C_3 \mathrm{diam}(E),$$

where

$$\mathrm{diam}(E) = \max_{x,y \in E} |x - y|.$$

4. (Averaging trick) Assume $\mathbb{E}[t_e^2] < \infty$ and $\mathbb{P}(t_e = 0) < p_c(d)$. Let $B_m = \{x \in \mathbb{Z}^d : \max |x_i| \le m\}$. Define, for $m > 0$:

$$F_m = \frac{1}{\# B_m} \sum_{z \in B_m} T(z, z + n\mathbf{e}_1).$$

Select $m = \lfloor n^{1/4} \rfloor$. Use the results of the previous problems to show the existence of a constant C_4 independent of n such that

$$|\mathrm{Var}(T(0, n\mathbf{e}_1)) - \mathrm{Var}(F_m)| \le C_4 n^{1/4}.$$

5. (Benaim and Rossignol's inequality)

   (a) For a random variable $X \ge 0$, denote the *entropy*:

   $$\mathrm{Ent}(X) := \mathbb{E}\Big[X \log \frac{X}{\mathbb{E}[X]}\Big].$$

   Show that

   $$\mathbb{E}\Big[X^2 \log \frac{\mathbb{E}[X^2]}{(\mathbb{E}[X])^2}\Big] \le \mathrm{Ent}(X^2).$$

   This inequality is due to Falik and Samorodnitsky.

   (b) Let F be a function of N independent random variables $X_1, \ldots, X_N$. Define $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ and define

   $$\Delta_i F := \mathbf{E}[F \mid \mathcal{F}_i] - \mathbb{E}[F \mid \mathcal{F}_{i-1}].$$

   Show that

   $$\sum_{i=1}^N \mathbb{E}\Big[|\Delta_i F|^2 \log \frac{\mathrm{Var}(F)}{\sum_{i=1}^N (\mathbb{E}[|\Delta_i F|])^2}\Big] \le \sum_{i=1}^N \mathrm{Ent}(|\Delta_i F|^2).$$

6. (Logarithmic Sobolev inequality for Bernoulli) Let $f : \{\pm 1\} \to \mathbb{R}$ and X be uniformly distributed on $\{\pm 1\}$.

   (a) Show that

   $$\mathrm{Ent}(f(X)) = \mathbb{E}[f(X) \log \frac{f(X)}{\mathbb{E}[f(X)]}] \le \frac{1}{2} |f(-1) - f(1)|^2.$$

(b) Show that if $f : \{\pm 1\}^N \to \mathbb{R}^N$, then

$$\mathrm{Ent}(f(X_1, \dots, X_N)) \le \sum_{i=1}^{N} \mathbb{E}[\mathrm{Ent}_i(f(X_1, \dots, X_N))],$$

where

$$\mathrm{Ent}_i f(X_1, \dots, X_N) = \int f(X_1, \dots, X_N) \log \frac{f(X_1, \dots, X_N)}{\mathbb{E}[f(X_1, \dots, X_N)]} \, \mathbb{P}(\mathrm{d}X_i).$$

Conclude that

$$\mathrm{Ent} f(X_1, \dots, X_N) \le 2 \sum_{i=1}^{N} \mathbb{E}[(f(X_1, \dots, X_i, \dots, X_N) - f(X_1, \dots, -X_i, \dots, X_N)^2].$$