Tutorial (exercises on lec. 1)

(1) For $x \in \mathbb{Z}^d$, let
$$\rho(x) = \lim_{n\to\infty} T(x, \partial B(n)),$$
where $B(n) = [-n, n]^d$. Show that
$$\rho(x) = \inf\{T(\gamma) : \gamma \text{ is an infinite, edge self-avoiding path starting at } x\}.$$

(2) If $(t_e)$ is i.i.d. and Bernoulli with $\mathbb{P}(t_e = 0) = p = 1 - \mathbb{P}(t_e = 1)$, then show that $\rho(0) < \infty$ a.s. if and only if $\mathbb{P}(\text{there is an infinite connected set of edges } e \text{ with } t_e = 0) = 1$.

(3) Suppose that $(t_e)$ is i.i.d. Show that a.s., there exists a unique geodesic between any $x, y \in \mathbb{Z}^d$ if and only if the common distribution of $t_e$ is continuous.

(4) Without using the shape theorem, show that if $(t_e)$ is i.i.d. with $\mathbb{P}(t_e = 0) = 0$, then there exists $c > 0$ such that
$$\mathbb{P}\left(\liminf_{\substack{x\in\mathbb{Z}^d \\ |x|\to\infty}} \frac{T(0,x)}{|x|} \geq c\right) = 1.$$

(5) Find an infinite connected graph and an i.i.d. distribution for weights $(t_e)$ on the edges of this graph such that a.s., there are at least two vertices between which there is no geodesic.

(6) Same as the last question, but $t_e$ should satisfy $\mathbb{P}(t_e = 0) > 0$.

(7) Show that if an infinite connected graph has bounded degree, and the distribution of $(t_e)$ is i.i.d., then for each $M > 0$ $\mathbb{P}(\text{there exist vertices } x, y \text{ with } |x - y| \geq M \text{ between which there is a geodesic}) = 1$.

(8) Show that if $\gamma_1, \gamma_2$ are infinite geodesics that coalesce, then their Busemann functions are equal; that is, $B_{\gamma_1}(x, y) = B_{\gamma_2}(x, y)$ for all $x, y \in \mathbb{Z}^d$.

(9) For $K \geq 1$, consider the infinite tube $[-K, K]^{d-1} \times \mathbb{Z}$ with edges between nearest neighbors. Suppose that $(t_e)$ is i.i.d. exponential with parameter 1. Let $N$ be as in the lecture:
$$N = N(\omega) = \sup\{k \geq 0 : \exists k \text{ edge-disjoint infinite geodesics}\}.$$
Prove that $\mathbb{P}(N = 2) = 1$.

(10) Show that Hoffman's $f$ is measurable relative to the cylinder sigma-algebra. Recall that $f$ is defined as follows. We assume that $N = 1$ a.s. on $\mathbb{Z}^d$, when $(t_e)$ is i.i.d. with continuous common distribution. For an outcome $\omega$, we pick any arbitrary infinite geodesic $\gamma$ (if there is one) and set $f(x, y) = B_\gamma(x, y)$, the Busemann function for $\gamma$.