MINICOURSE 1, TUTORIAL

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TABLE 1. Problem Difficulty

No stars	Easiest
$\star$	Easy
$\star\star$	Moderate
$\star\star\star$	Look at Kesten's "Aspects"

We will assume the following: we have iid edge weights $\{\tau_e\}_{e \in E(\mathbb{Z}^d)}$ where $E(\mathbb{Z}^d)$ is the edge-set of first-passage percolation.

Let $\{e_1, \ldots, e_d\}$ be the unit directions in $\mathbb{Z}^d$. Let $E(\mathbb{Z}^d)$ be the set of nearest-neighbor edges on the lattice $\mathbb{Z}^d$. $\vec{0}$ is the origin in $\mathbb{Z}^d$ (and $\mathbb{R}^d$).

Let $T(x, y) = T([x], [y])$ be the usual extension of the passage time from $\mathbb{Z}^d \times \mathbb{Z}^d$ to $\mathbb{R}^d \times \mathbb{R}^d$ where $[x]$ is the unique lattice point in $x + [0, 1)^d$.

I will refer to the two manuscripts "50 years of first-passage percolation" by Auffinger, Damron and Hanson and "Aspects of first-passage percolation" by Kesten as "50 years" and "Aspects" respectively.

Problem 1. a) Let $\{a_n\}_{n=0}^{\infty}$ be a subadditive sequence; i.e.,

$$a_{n+m} \leq a_n + a_m$$

with $a_n \in [-\infty, \infty) \quad \forall n \geq 0$. Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$$

b) Show that $\mathbb{E}[T(\vec{0}, ne_1)]$ is a subadditive sequence.

Problem 2. Properties of the time-constant μ:

a) (Triangle inequality) For $x, y \in \mathbb{R}^d$, $\mu(x+y) \leq \mu(x) + \mu(y)$
b) (1-homogeneity) For $\lambda \in \mathbb{R}$, show $\mu(\lambda x) = |\lambda|\mu(x)$.
c) (Convexity) For $\lambda \in [0,1]$, $\mu(\lambda x + (1-\lambda)y) \leq \lambda\mu(x) + (1-\lambda)\mu(y)$
d) μ is invariant under the symmetries of $\mathbb{Z}^d$ that fix the origin $\vec{0}$. What are the relevant symmetries of $\mathbb{Z}^d$ here?

Problem 3. Let $f \colon \mathbb{R}^{\otimes E(\mathbb{Z}^d)} \to \mathbb{R}$ be the passage time $T(x,y)$ for some fixed $x, y \in \mathbb{Z}^d$ and fixed realization of the weights. Is f concave, convex or neither in any one of its coordinates, while keeping all the other coordinates fixed?

Problem 4. Prove the 1D projection of the shape theorem by hand, without using the Cox-Durrett theorem. Suppose

$$\mathcal{B}_1(n) = \{\lambda e_1 \colon T(0, \lambda e_1) \leq n, \lambda \in \mathbb{R}\}$$
$$\mathcal{B}_1 = \mathcal{B} \cap \{\lambda e_1 \colon \lambda \in \mathbb{R}\}$$

where $\mathcal{B} = \{x \in \mathbb{R}^d \colon \mu(x) \leq 1\}$ is the usual limit-shape.

If $\mathbb{E}[\min_{i=1,\ldots,2d} \tau_i] < \infty$, then for any $\epsilon > 0$, show that

$$\mathbb{P}\left((1-\epsilon)\mathcal{B}_1 \subset n^{-1}\mathcal{B}_1(n) \subset (1+\epsilon)\mathcal{B}_1 \text{ for all large enough } n\right) = 1$$

Problem 5. Show that (under the assumptions of the theorem) the Cox-Durrett shape theorem holding is equivalent to the following statement:

$$\overline{\lim_{|x|_1\to\infty}} \frac{T(0,x) - \mu(x)}{|x|_1} = 0$$

Problem 6 ($\star$)**.** Show that the results of Durrett-Liggett, Marchand, and Auffinger-Darmon on the flat spot imply if the iid edge-weights have distributions in $\mathcal{M}_p$, then the limit-shapes are are not polygonal. Recall that $\mathcal{M}_p$ consists of distributions F that satisfy

(1) $\text{supp}(F) \subseteq [1, \infty)$
(2) $F(1) \geq \vec{p_c}$

where $\vec{p_c}$ is the critical probability for oriented bond percolation on $\mathbb{Z}^d$ (Durrett, 1984, AoP), and supp is the support of the distribution F. To elaborate, if $\mathcal{B}$ is the usual limit shape, let $\mathcal{B}^1 = \mathcal{B} \cap \{x \in \mathbb{R}^2 \colon |x|_1 \leq 1\}$, and $\mathcal{B}^1_+ = \mathcal{B}^1 \cap [0, \infty)^2$. Let α_p be the asymptotic speed of directed percolation. Define

$$M_p = \left(\frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}} \right), N_p = \left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}} \right).$$

Note that M_p and N_p are points on $\{x \in [0, \infty)^2 \colon |x|_1 = 1, \}$ that are symmetric about $(1/2, 1/2)$. Let $[M_p, N_p] \subset \mathbb{R}^2$ be the line-segment connecting the two points M_p and N_p.

For $F \in \mathcal{M}_p$, the Durrett-Liggett theorem (Theorem 2.26 in "50 years") says that

(1) $\mathcal{B} \subseteq \mathcal{B}^1$
(2) If $p < \vec{p}_c$ then $\mathcal{B} \subsetneq \text{interior}(\mathcal{B}^1)$,
(3) if $p > \vec{p}_c$ then $\partial \mathcal{B}^1_+ = [M_p, N_p]$, and
(4) if $p = \vec{p}_c$ then $\partial \mathcal{B}^1_+ = \left(\frac{1}{2}, \frac{1}{2} \right)$.

Let $\beta_p = N_p \cdot e_1$, the projection of N_p on the x-axis. Suppose for the rest of the problem that $F(1) > \vec{p}_c$.

(1) Show that $\mu(e_1) \leq \frac{1}{\beta_p}$.
(2) Marchand showed that in fact this inequality is strict; i.e., $\mu(e_1) < \frac{1}{\beta_p}$. Auffinger-Damron showed that moreover, μ is differentiable at the points M_p and N_p. Using these two results, show that $\mathcal{B}$ is not a polygon.

Problem 7 ($\star$)**.** First, as a starter, show that

$$\mathbb{E}[\min_{i=1,\dots,2d} \tau_i] = \infty$$

implies that $\mu(e_1) = \infty$. *Hint:* Write $\mathbb{E}[\min_{i=1,\dots,2d} \tau_i]$ as a sum of probabilities, and note that $\tau_{ne_1, ne_1 \pm \theta} > Cn$ for all $\theta \in \{\pm e_1, \dots, \pm e_d\}$ implies that $T(0, ne_1) > Cn$.

We have seen that if $F(0) < p_c$ and

$$\mathbb{E}[\min_{i=1,\dots,2d} \tau_i^d] < \infty \tag{1}$$

then there is an $\epsilon > 0$ and a compact set $\mathcal{B}$ such that

$$\mathbb{P}\left((1-\epsilon)\mathcal{B} \subset n^{-1}\mathcal{B}(n) \subset (1+\epsilon)\mathcal{B} \text{ for all large enough } n\right) = 1$$

Show that if $F(0) \geq p_c$ then the above cannot hold. *Hint: Show that for any constant $C > 0$*

$$\overline{\lim_{|x|_1 \to \infty}} \frac{T(0,x)}{|x|_1} \geq C \quad \text{a.s}$$

Problem 8 ($\star$)**.** Recall $F(0) = p \in (0,1)$ and we call an edge open if $\tau_e = 0$ (in contrast to classical percolation). Let

$$W = \{x \in \mathbb{Z}^d \colon \exists \text{ path } \gamma, \text{ from } \vec{0} \text{ to } x \text{ only consisting of open edges } \}$$

Kesten defines

$$\begin{aligned} p_T &= \sup_p \{p \colon \mathbb{E}_p[\#W] < \infty]\} \\ p_H &= \inf_p \{p \colon \mathbb{P}_p(\#W = \infty) > 0]\} \\ p_C &= \sup_p \{p \colon \mathbb{P}_p(\#W = \infty) = 0]\} \end{aligned}$$

Show that $p_T \leq p_H$ and then $p_C = p_H$.

Problem 9 ($\star\star$)**.** Show that $\mu(e_1) > 0$ iff $F(0) < p_c$. To show that $F(0) < p_c \Rightarrow \mu(e_1) > 0$, you may assume the following statements:

If $F(0) < p_c$, then $\exists C, D, E > 0$ constants only depending on F and d such that

$$\mathbb{P}\left(\exists \text{ a self avoiding } \gamma, \gamma(0) = \vec{0}, |\gamma|_1 \geq n \text{ s.t } T(\gamma) \leq Cn\right) \leq De^{-En}$$

For the converse, show that if $\mu > 0$, then $F(0) \leq p_c$. From the next problem, we see that in fact $F(0) < p_c$. Let, recall the point-to-point and point-to-hyperplane passage times. Let

$$H_0 := \{v \in \mathbb{Z}^d : v \in \operatorname{span}(e_2, \dots, e_d)\}$$

be the hyperplane normal to e_1. Consider the point-to-point passage time

$$a_{0,n} = T(\vec{0}, ne_1)$$

and the point-to-hyperplane passage time

$$b_{0,n} = \inf_{v \in H_0} T(\vec{0}, ne_1 + v).$$

Kesten, Theorem 2.18 says that if $\mathbb{E}[\min_{i=1,\dots,2d} \tau_i] < \infty$, then $\exists\ \mu(e_1) < \infty$ s.t

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{b_n}{n} = \mu(e_1) \quad \text{a.s}$$

Kesten, Theorem 5.2 gives the lower-tail large deviations for $\theta_{0,n} = a_{0,n}$ or $b_{0,n}$. For any $\epsilon > 0 \quad \exists A(e), B(\epsilon) > 0$ such that

$$\mathbb{P}\left(\theta_{0,n} < n(\mu - \epsilon)\right) \leq Ae^{-Bn}$$

Problem 10. Recall the Cox-Kesten continuity theorem. Suppose $F_n \overset{d}{\Rightarrow} F$ (covergence in distribution) as $n \to \infty$. Then the associated time-constants satisfy:

$$\lim_{n\to\infty} \mu_{F_n}(e_1) = \mu_F(e_1)$$

Show that if $F(0) = p_c$, then $\mu_F(e_1) = 0$.