Talk 1: Coulomb branches & vortices through the lens of 3d TQFT

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Recall: 3d $\mathcal{N}=4$ theories

Labeled by $G_c \circlearrowright \mathbb{H}^n = T^*V$ here or $\mathcal{Y}$ smooth H.K. space or ...

compact linear quaternionic rep

$G := (G_c)_{\mathbb{C}}$

reductive

$\mathcal{T}_{G,V}$ $\mathcal{T}_{\mathcal{Y}}$

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* Spin or Spin_c TQFT's
* dg (cohomological, $\mathbb{Z}$ graded)
* may not be finite on closed M^3 $\rightsquigarrow$ noncompact TQFT
* $\mathcal{Z}^{A,B}_{\mathcal{Y}}$ similar, but may have framing anomaly or be only $\mathbb{Z}/2$ graded

<u>Sidebar</u>: 2d A & B models are very similar, (2)-1-0 dg TQFT's

E.g. 2d A-model w/ target $X = \mathbb{C}^*$

$$Z(S^1) = SH^\bullet(X) \simeq \mathbb{C}[x, x^{-1}]$$

$$Z(T^2) = \mathrm{Tr}_{\mathbb{C}[x,x^{-1}]} 1 = \infty$$

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More so, just as in 2d, $\exists$ <u>3d mirror symmetry</u>

$$\mathcal{T}_{(G,V)} \leftrightarrow \mathcal{T}_{(G',V')} \quad \text{or} \quad \mathcal{T}_{(G,V)} \leftrightarrow \mathcal{T}_{\mathcal{Y}} \quad \text{or} \quad ?$$

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$$\Rightarrow \quad Z^A_{(G,V)} \simeq Z^B_{(G',V')}$$

$$Z^B_{(G,V)} \simeq Z^A_{(G',V')}$$

(etc.)

$\rightsquigarrow$ rich dualities in geometry & rep theory

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Practically,

Z:

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	$\longmapsto$	$\mathrm{Hom}_{\mathbb{B}dy}(-,-)$
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For $Z^A_{(G,V)}$, $Z^B_{(G,V)}$, Z^A_y, defining the extended structure has been an active area of study (& progress!) in the last 5-ish years

Z^B_y : prototype by Kapustin-Rozansky-Saulina '08

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σ : section of $Ad(E)$

ϕ : section of $Ad(E)_{\mathbb{C}}$ homological degree $|\phi| = 2$

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Note: the hyperkahler moment maps for $G_c \circlearrowright T^*V$ $\quad \mu^i : \Sigma \to \mathfrak{g}^*$

are components of a G_c-valued <u>1-form</u> 'μ' $\quad i = 1,2,3$

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Classical EOM include:

$$\boxed{\begin{aligned} \not{\partial}_A \mathbb{X} &= 0 \\ * F_A &= \mu(\mathbb{X}) + d_A \sigma \end{aligned}}$$

$$\phi \cdot \mathbb{X} = \sigma \cdot \mathbb{X} = 0$$

$$d_A \phi = 0$$

$$[\phi, \bar{\phi}] = [\sigma, \phi] = [\sigma, \bar{\phi}] = 0$$

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$$* F_{z\bar{z}} = \mu_{\mathbb{R}}(X, Y)$$

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$\boxed{\text{symplectic vortex equations}}$

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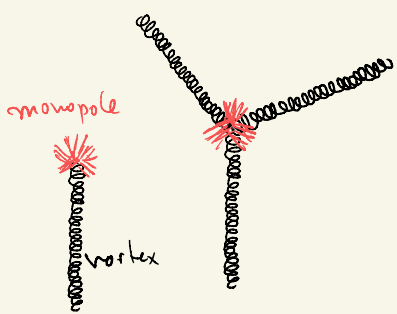
Picture:



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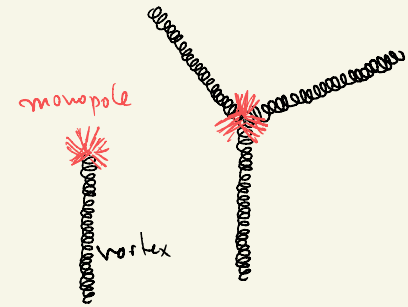

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Examples today: (dg) vector spaces

- $Z(\Sigma)$, Σ a smooth surface
- $Z(S^2) \rightsquigarrow$ Coulomb branch algebra, BFN
- $Z(S^2) \circlearrowleft Z(\mathbb{C})$, $Z(D^2)$ $\rightsquigarrow$ Vortices $\simeq$ Verma, actions on quantum/symplectic cohomology

Σ a smooth, cpt Riemann surface

cf. [Bullimore-Ferrari-Kim '21]

Expect $Z^A_{(G,V)}(\Sigma) \simeq H^\bullet(\mathcal{M}(\Sigma; G, V))$

$\mathcal{M}(\Sigma; G, V)$ = time-independent solutions to A-twist EOM on $\Sigma \times (0, \varepsilon)$

i.e. symplectic vortex moduli space on Σ

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$$\underset{\text{alg geom}}{\simeq} \Gamma\left(\Sigma, (T^*V \otimes K_\Sigma^{1/2}) /\!\!/ G\right)$$

algebraic symplectic quotient, $\mu_{\mathbb{C}} = 0$

twisted maps to the stack $T^*V /\!\!/ G$

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Make this nicer by introducing a stability condition

$\rightsquigarrow$ quasimaps to $T^*V^{stab} /\!\!/ G$

Has interesting wall-crossing phenomena, cf. A. Ferrari's talk

$\underline{\mathcal{E} = \mathbb{C}}$ cf [Bullimore-TD-Gaiotto-Hilburn-Kim '17]

generalizing [Braverman-Feigin-Rybnikov-Finkelberg '10]

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$$\simeq H^\bullet(\text{quasimaps } f : \mathbb{P}^1 \to T^*V^{stab}/\!\!/G \text{ st. } f(\infty) = v)$$

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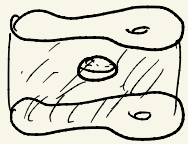
<u>So what?</u>

These spaces naturally get actions of $Z(S^2)$!

Connect "vortices" w/ rep theory.

$Z\Big($  $\Big) : Z(S^2) \otimes Z(\Sigma) \to Z(\Sigma)$

$(\Sigma \times I) \setminus B^3$

$$\Sigma = D \cup_{D^*} D \quad (\text{ravido}) \quad \approx S^2$$

for algebraic convenience

$D = \operatorname{Spec} \mathbb{C}[[z]]$

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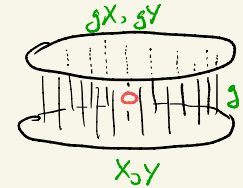
$D \cup_{D^*} D =$



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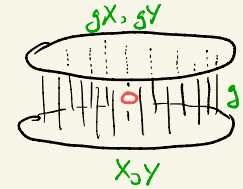

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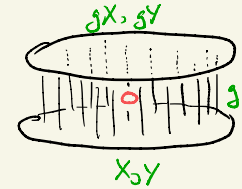

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space of Hecke modifications of symplectic vortices

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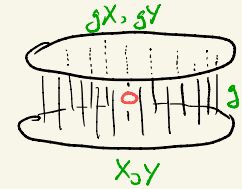

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$$\underset{\substack{\text{BFN} \\ \text{"conj"}}}{=} H^{BM}_\bullet\left(G[\![z]\!] \backslash \{(X,g) \in V[\![z]\!] \times G(\!(z)\!) \mid gX \in V[\![z]\!]\} / G[\![z]\!] \right)$$

$R_{G,V}$

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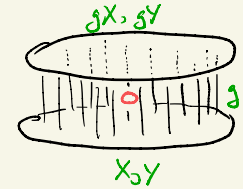

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▸ Product from convolution

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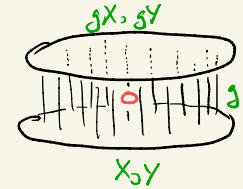

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$$\underset{\substack{\text{BFN}\\ \text{"conj"}}}{=} H^{BM}_{\bullet}\left({}_{G[\![z]\!]}\backslash \{ (X,g) \in V[\![z]\!] \times G(\!(z)\!) \mid gX \in V[\![z]\!] \} / {}_{G[\![z]\!]} \right)$$

▷ Product from convolution

$Z\Big($ $\Big) : Z(D \cup_{D^*} D) \otimes Z(D \cup_{D^*} D) \to Z(D \cup_{D^*} D)$

▷ Loop rotation $U(1)_{\mathrm{loop}}$ acts on the moduli space

loop equivariance (Ω background) $\rightsquigarrow$ non-commutative product

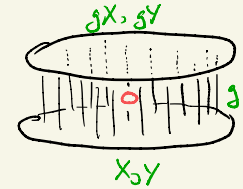
$$\mathcal{E} = D \cup_{D^*} D \ \text{(ravido)} \approx S^2$$

for algebraic convenience

$D = \mathrm{Spec}\,\mathbb{C}[[z]]$

$D^* = \mathrm{Spec}\,\mathbb{C}((z))$

$D \cup_{D^*} D =$



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BFN "conj"

- Product from convolution

$Z\Big($ $\Big) : Z(D \cup_{D^*} D) \otimes Z(D \cup_{D^*} D) \to Z(D \cup_{D^*} D)$

- Loop rotation $U(1)_{loop}$ acts on the moduli space

loop equivariance (Ω background) $\rightsquigarrow$ non-commutative product

Examples:

$G = \mathbb{C}^*$, $V = 0$

$Z(D \cup_{D^*} D) \simeq \mathbb{C}[\mathbb{C} \times \mathbb{C}^*]$

$G = \mathbb{C}^*$, $V = \mathbb{C}$

$Z(D \cup_{D^*} D) \simeq \mathbb{C}[\mathbb{C} \times \mathbb{C}]$

G reductive, $V = 0$

$Z(\ ) = \mathbb{C}[N \backslash\!\!\backslash T^*G /\!\!/ N]$

Back to actions

$$Z^A_{(G,N)}(S^2) \circlearrowleft Z^A_{(G,N)}(\mathbb{C}, v)$$

$\uparrow$

base pointed maps

$(\mathbb{P}^1, \infty) \to (T^*V/\!/G, v)$

Back to actions

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[BDGHK'17] [Zijun Zhou '21]

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[BDGHK'17] [Zijun Zhou'21]

$Z^A_{(G,V)}(S^2) \circlearrowright Z^A_{(G,V)}(\Sigma)$ is relatively unstudied.

Back to actions

$\mathcal{Z}^A_{(G,V)}(S^2) \circlearrowright \mathcal{Z}^A_{(G,V)}(\mathbb{C}, v)$ produces modules supported on Lagrangians in $\mathcal{M}_{Coul} = \mathrm{Spec}\, \mathcal{Z}^A_{(G,V)}(S^2)$

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[BDGHK'17] [Zijun Zhou'21]

$\mathcal{Z}^A_{(G,V)}(S^2) \circlearrowright \mathcal{Z}^A_{(G,V)}(\Sigma)$ is relatively unstudied.

$\circlearrowleft$

commutes with $MCG(\Sigma)$

<u>Conjecture</u>: $\mathcal{Z}^A_{(G,V)}(\Sigma)$ is finitely generated as a $\mathcal{Z}^A_{(G,V)}(S^2)$ module.

Suffices to understand MCG action on generators.

Connection w/ 2d A-models

Basic idea: the 3d A twist is compatible w/ a 2d A-twist on its boundary.

see C. Teleman's talks

[Teleman '14/ICM]

[Bullimore-TD-Garotta-Hilburn-(Kim) '16,'17]

[Teleman '18] [González-Mak-Pomerleano '22]

Connection w/ 2d A-models

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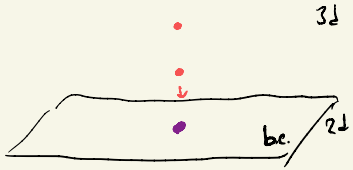

or

$Z\Big($ $\Big) : Z(S^2) \otimes Z(D^2, bc) \to Z(D^2, bc)$

D^2 w/ b.c. on $\partial(D^2)$

Connection w/ 2d A-models

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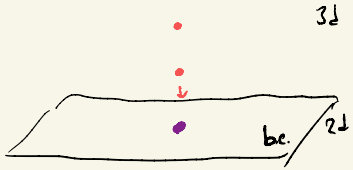

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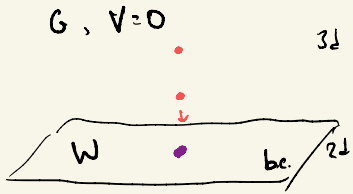
Example: G, $V=0$



W: vec space w/ G-action

Connection w/ 2d A-models

see C. Teleman's talks
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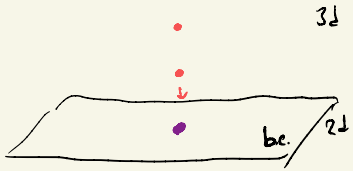

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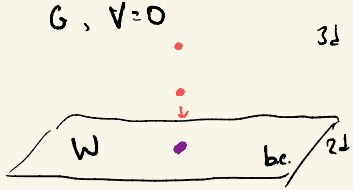

W: vec space w/ G-action

$$Z(S^2) = \mathbb{C}[\, N \,\backslash\!\!\backslash\, T^*G \,/\!\!/\, N \,]$$

$$\circlearrowright$$

$$Z(D^2, W) = H^\bullet\big(W((z)) / G[\![z]\!] \big)$$

$$\simeq QH^\bullet_G(W)$$

Module w/ Lagrangian support determined by the "Hori-Vafa" mirror superpotential.