A probabilistic approach to study discrete sgps of Lie gps.

$G =$ ss lie gp

$\Gamma \leq G \qquad\qquad M = \Gamma\backslash G/K$

$\mathrm{Sub}(G) = \{\text{closed sgps}\}$

$\cup|$

$\mathrm{Sub}_d(G) = \{\text{discrete sgps}\}$

$\mathrm{prob}(\mathrm{Sub}_d(G))$

Most hyperbolic manifold are non-arithmetic

A. Levit 2014 $\quad \underline{\underline{n \geq 4}}$

$\rho_n(v) =$ # of n-hyp mflds of vol $\leq v$
up to commensurability

$$\begin{tikzcd}
 & M_3 \arrow[dl] \arrow[dr] & \text{finite} \\
M_1 \arrow[rr, phantom, "\sim"] & & M_2
\end{tikzcd}$$

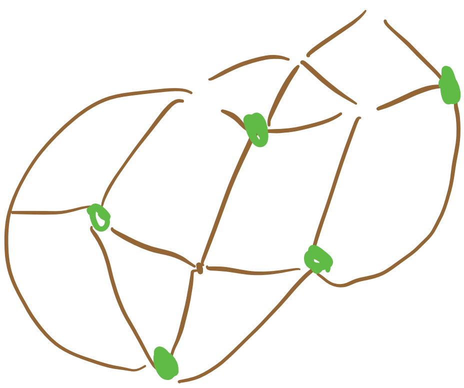
Thm $\quad \rho_n(v) \approx v^v$

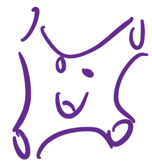
$a v \log v \le \log \rho_n(v) \le b v \log v$

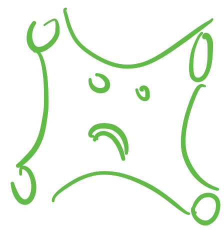
Levit, G
2014

Burger, G, Lubotzky, Mozes 2002

# of arithmetic $\le v^{B(\log v)^{\varepsilon}}$







IRS - invariant random subgroups

$G \curvearrowright \mathrm{Sub}(G)$ by conjugation

μ is an IRS if it is G-inv.

$\mathrm{IRS}(G) = \mathrm{prob}(\mathrm{Sub}(G))^{\mathrm{inv}}$

Thm (Kazhdan - Margulis)

$\exists\ 1 \in U \subseteq G$ open s.t. $\forall\ \Gamma \le G$ lattice
$\exists\ g \in G$ $\ g\Gamma g^{-1} \cap U = \{1\}$.

$$G/\Gamma \to \mathrm{Sub}(G)$$
$$g\Gamma \mapsto g\Gamma g^{-1}$$

$\mathcal{F} \subseteq \mathrm{IRS}$ is <u>w.u.d</u> if $\exists\ 1 \in U \subseteq G$ open s.t. $\forall\ \mu \in \mathcal{F}$

$$\mu(\Gamma : \Gamma \cap U = \{1\}) \geq \tfrac{1}{2}.$$

<u>Thm</u> $\mathrm{IRS}_d(G)$ is w.u.d.

<u>Thm</u> ('7s) (ABBGNRS)

$\mathrm{rk}\, G \geq 2$, G has (T).

$M_n = \Gamma_n \backslash G/K$ $\quad \Gamma_n \leq G$ irreducible

$\mathrm{vol}(M_n) \to \infty$

$\forall\ R$ $\quad \dfrac{\mathrm{vol}((M_n)_{\geq R})}{\mathrm{vol}(M_n)} \to 1$

SRS - stationary random subgroups

$\mu \in \mathrm{Prob}(G)$

$\nu \in \mathrm{Sub}(G)$ is μ-stationary if $\mu * \nu = \nu$.

$\Gamma \leq \mathrm{Sub}_d(G)$

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \mu^{(i)} * \delta_\Gamma \longrightarrow \nu_\infty \quad \mu\text{-stationary}$$

Fraczyk, G

Def

$\Delta \subseteq G$ is <u>confined</u> if $\exists\, C \subseteq G$ compact s.t. $C \cap \Delta^g \setminus \{1\} \neq \emptyset \quad \forall g \in G$.

$\Updownarrow$ for Δ discrete

$M = \Delta \backslash G/K$ has bounded inj-rad.

$\Delta \subseteq G$ is <u>not confined</u> iff $\exists\, g_n \in G$

$g_n \Delta g_n^{-1} \longrightarrow \langle 1 \rangle$.

Thm (Fraczyk, G) If G is semisimple and ($\mathrm{rk}\, G \geq 2$) has property (T) then $\Delta \leq G$ is confined $\Leftrightarrow$ Δ contains a lattice in a factor.

Example If $\Gamma \leq G$ is a uniform lattice $(1) \neq \Delta \triangleleft \Gamma$ then Δ is confined in G.

A. Levit U. Bader

G higher rank $G = SL_3(\mathbb{R})$ –
$\boxed{G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})}$ –

$\Gamma \leq G$ an irreducible lattice

$(1) \neq N \triangleleft \Gamma \Rightarrow |\Gamma/N| < \infty.$

$\Gamma = SL_3(\mathbb{Z})$ (T)

$\Gamma = SL_2(\mathbb{Z}[\sqrt{2}])$ $\times$

Thm (B,G,L) Let G be a higher rank ss Lie gp.
$\Gamma \leq G$ an irreducible lattice.
$\Delta \leq \Gamma$ is confined $\iff$ $|\Gamma : \Delta| < \infty$.

$1 \neq N \triangleleft \Gamma \implies$ N is confined in Γ

$a \in N \setminus \{1\}$ $\quad a \in N^{\sigma} = N \quad \forall \sigma \in \Gamma$.

N is not necessarily confined in G.

$\Gamma = F_2 \underset{12}{\leq} SL_2(\mathbb{Z}) \leq SL_2(\mathbb{R})$

$N = \Gamma' \triangleleft \Gamma$ $\qquad$ N is not confined in $SL_2(\mathbb{R})$.

A URS is a minimal G-inv set in $\mathrm{Sub}(G)$,

A URS is reducible $(G = G_1 \times G_2)$ if it consists of product groups $\Gamma_1 \times \Gamma_2$ contained in proper factors.

$\Delta \leq G$ is irreducibly confined if any URS $\subseteq \overline{\Gamma^G}$ is irreducible.

Thm (BGL) In higher rank

i.c. $\iff$ i.l.