Percolation on nonamenable groups, old and new

§0: Percolation Basics

$G=(V,E)$ connected, locally finite graph
($\Rightarrow V, E$ countable) ← all degrees finite

Bernoulli p bond percolation on G: Delete or retain each edge of G independently at random with retention probability p. Denote law by $\mathbb{P}_p$, expectations by $\mathbb{E}_p$.

- Retained edges are called open
- Deleted edges are called closed
- Connected components are called clusters

Q: What do clusters look like? How does this depend on p?

The monotone coupling: Let $(U_e)_{e\in E}$ be i.i.d. uniform $[0,1]$ r.v.s. For each $p\in[0,1]$

$$\omega_p(e) := \mathbb{1}(U_e \leq p)$$

has the law of Bernoulli-p bond percolation and has

$$\omega_p \leq \omega_q \quad \text{when} \quad p\leq q.$$

Consequence: If $A \subseteq \{0,1\}^E$ is increasing then $\mathbb{P}_p(A)$ is increasing function of p.

$\uparrow$ $\omega \in A$ and $\omega' \geq \omega \Rightarrow \omega' \in A$

Harris-FKG inequality: If A, B are increasing then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

"Increasing events are positively correlated"

The critical probability:

must equal 1 by Kolmogorov's 0-1 law. $\downarrow$

$$p_c = p_c(G) = \inf\{p : \mathbb{P}_p(\exists \text{ an } \infty \text{ cluster}) > 0\}$$

Say that G has a non-trivial phase transition if $0 < p_c(G) < 1$.

Prop $p_c(G) \geq \dfrac{1}{\text{max degree} - 1}$

This is an equality on a regular tree.

Proof Let M = max degree. The number of simple paths of length n starting at some vertex v is at most $M(M-1)^{n-1}$. By Markov's inequality, the probability at least one such path is open is at most $M(M-1)^{n-1}p^n$. This goes to zero as $n \to \infty$ when $p < \frac{1}{M-1}$.

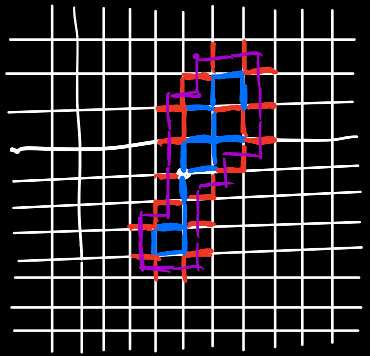
$\square$

Proving $p_c < 1$ is generally harder.

Prop If $d \geq 2$ then $p_c(\mathbb{Z}^d) \leq \frac{2}{3}$.

Proof Suffices by monotonicity to consider case $d=2$.

"Peierls argument"



If the cluster of the origin is finite, it must be surrounded by a dual circuit of closed edges.

There are at most $3^n \cdot n$ dual circuits of length n surrounding the origin

possible choices of path given x intercept.

choices of positive x intercept

> Similar argument for any one-ended finitely presented group (Babson & Benjamini)

So $p > 2/3 \Rightarrow$ Expected number of closed dual circuits surrounding the origin is finite.

Exercise $\Rightarrow$ No closed dual circuits surrounding the origin with positive probability. $\square$

Thm (Kesten) $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

⚠ We don't expect to be able to compute p_c in most examples.

G <u>nonamenable</u> if its Cheeger constant

$$h(G) = \inf \left\{ |\partial_E K| \Big/ \sum_{v \in K} \deg(v) \ : K \subseteq V \text{ finite} \right\}$$

Edges with one endpoint in K and the other outside K

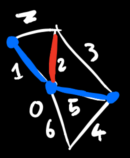
is positive, <u>amenable</u> otherwise

<u>Prop</u> (Benjamini-Schramm) If G is k-regular then $p_c \leq 1/1+hk$

Cheeger constant.

<u>Proof</u> Consider exploring the cluster of the origin one edge at a time as follows:

- Fix an enumeration $E = \{e_1, e_2 \dots\}$.
- At each step, let E_n be the minimal element of E that
  - (a) Has exactly one endpoint in the revealed part of the cluster of the origin.
  - (b) Has not already been revealed.



$\rightsquigarrow$ $\omega(E_1), \omega(E_2), \ldots, \omega(E_T)$

are distributed as iid coin flips

At the end of the procedure, we have

- An open spanning tree of the cluster of the origin
- All the closed edges in the boundary of the cluster
- Some additional closed edges with both endpoints in the cluster of the origin.

$\rightsquigarrow$ Revealed open $\leq \underbrace{|K|}_{}$ $\nwarrow$ # vertices

Revealed closed $\geq |\partial_E K| \geq h_K |K|$.

If $p > \frac{1}{1+h_K}$, positive probability that an ∞ sequence of iid coin flips $(B_i)_{i\geq 1}$ will satisfy $\frac{1}{N}\sum_{i=1}^{N} B_i > 1/_{1+h_K}$ $\forall N \geq 1$.

$\rightsquigarrow$ Positive probability not to have an ∞ cluster $\square$

Sharpness of the phase transition

Thm (Menshikov, Aizenman & Barsky '80s) If G is transitive and $p<p_c$ then $\mathbb{E}_p|K|<\infty$.

cluster of the origin.

= "susceptibility"

In fact, $\exists\, c_p>0$ such that

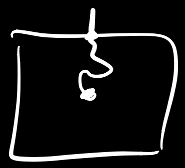
$$\mathbb{P}_p(|K|\geq n)\leq e^{-c_p n}\qquad \forall n\geq 1$$

This is a very important theorem!

Easy new proof due to Duminil-Copin & Tassion 2015

- For each $S\subseteq V$ finite and $p\in[0,1]$ define

$$\varphi_p(S)=p\sum_{e\in\vec{\partial}_E S}\mathbb{P}_p(0\xleftrightarrow{S}e^-)$$



Fixed "origin" vertex

oriented edges with $e^-\in S$, $e^+\notin S$.

- Define $\tilde{p}_c=\sup\{p:\exists\, \overset{\overset{0}{\cap}}{S}\subseteq V$ with $\varphi_p(S)<1\}$.

We will show $\tilde{p}_c=p_c$.

BK Inequality Let $A \subseteq \{0,1\}^E$, $\omega \in A$.

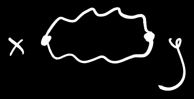
$W \subseteq E$ is a *witness* for A if knowing $\omega|_W$ guarantees A.

E.g. $A = \{x \leftrightarrow y\}$, W an open path from x to y.

$A \circ B = \{\exists$ disjoint witnesses for A and $B\}$

"disjoint occurence"

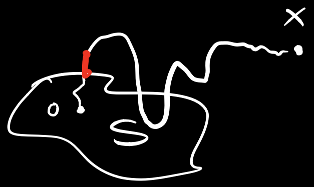
BK Inequality: If A, B increasing then



$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\,\mathbb{P}_p(B).$$

Step 1: If $p < \tilde{p}_c$ then $\exists\, S^{\overset{0}{\in}}$ with $\varphi_p(S) < 1$.

$x \notin S$



$$\{0 \leftrightarrow x\} \subseteq \bigcup_{e \in \partial_E^{\rightarrow} S} \{0 \overset{S}{\longleftrightarrow} e^-\} \circ \{e \text{ open}\} \circ \{e^+ \leftrightarrow x\}$$

BK Inequality $\Rightarrow$

$$\mathbb{E}_p|K| = \sum_x \mathbb{P}_p(0 \leftrightarrow x) \leq \sum_{x \in S} \mathbb{P}_p(0 \leftrightarrow x)$$

$$+ p \sum_{e \in \partial_E^{\rightarrow} S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} e^-) \sum_{x \notin S} \mathbb{P}_p(e^+ \leftrightarrow x)$$

Work inside $\Delta \subseteq V$ finite

look at supremal expected cluster size inside Δ

$$\leq |S| + \varphi_p(S)\,\mathbb{E}_p|K|$$

$$\Rightarrow \mathbb{E}_p|K| \le \frac{|S|}{1-\varphi_p(S)}$$

To do this properly, work in finite volume and take a limit.

So $p < \hat{p}_c \Rightarrow \mathbb{E}_p|K| < \infty$

$\Rightarrow \hat{p}_c \le p_c$.

Russo's formula: If A is increasing, depends on finitely many edges, then

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{e\in E} \mathbb{P}_p(e \text{ is pivotal for } A)$$

Given $\omega|_{E\setminus\{e\}}$, turning e on causes A to occur, turning e off causes A not to occur.

$$\frac{d}{dp}\mathbb{P}_p(A) = \frac{1}{1-p}\sum_{e\in E}\mathbb{P}_p\left(\begin{array}{l} e \text{ is a closed pivotal} \\ \text{for } A \end{array}\right)$$

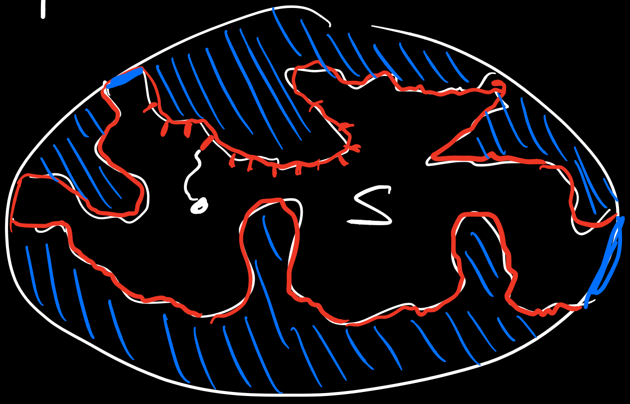
Claim

ball of radius r.

$$\frac{d}{dp}\mathbb{P}_p(0 \leftrightarrow B(0,r)^c)$$

$$\geq \frac{1-\mathbb{P}_p(0\leftrightarrow B(0,r)^c)}{p(1-p)} \inf_{S \text{ finite}} \varphi_p(S)$$

Why? Condition on set of vertices connected to $B(0,r)^c$, let S be the complement of this set



Conditional expected of number of closed pivotals is $\frac{1}{p}\varphi_p(S)$.

So, if $p > \tilde{p}_c$

$$\frac{d}{dp}\mathbb{P}_p(0 \leftrightarrow B(0,r)^c)$$

$$\geq \frac{1}{p(1-p)}\left(1-\mathbb{P}_p(0 \leftrightarrow B(0,r)^c)\right)$$

Integrate this differential inequality!

$$\mathbb{P}_p(0 \leftrightarrow B(0,r)^c) \geq \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}$$

$\forall p > \hat{p}_c$ and $r \geq 1$.

RHS doesn't decay as $r \to \infty$

$\Rightarrow \tilde{p}_c \geq p_c$.

§1: Uniqueness and non-uniqueness.

Thm (Newman & Schulman) If G is transitive then the number of ∞ clusters is either $0, 1$, or ∞ a.s.

Proof # of ∞ clusters is not random by ergodicity. Suppose that it is a.s. equal to $1 < k < \infty$.

$\rightsquigarrow \exists R$ st. $B(0,R)$ intersects all k infinite clusters with probability at least $1/2$.

With probability at least $1/2$, there is no infinite cluster that does not intersect $B(0,R)$

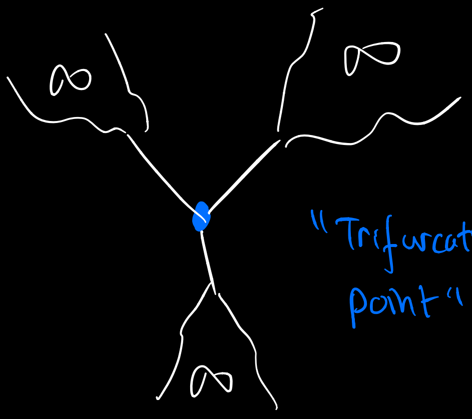
Independent of the edges inside $B(0,R)$!

Since with positive probability every edge in $B(0,R)$ is open, there is a unique infinite cluster with positive probability #

$\square$

Thm (Aizenman-Kesten-Newman / Burton-Keane)

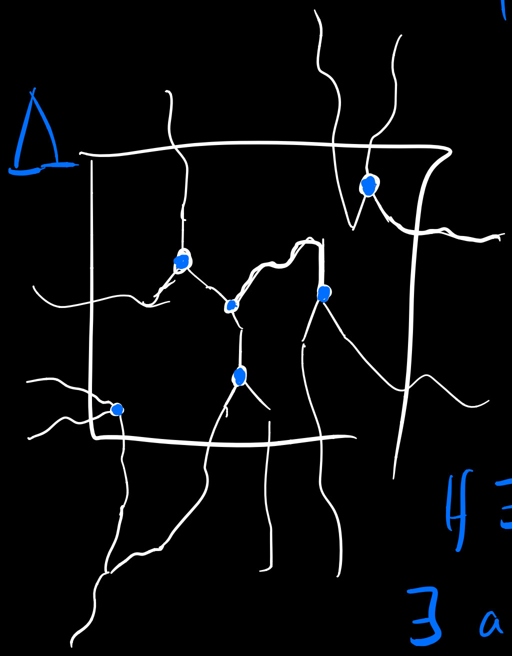
If G is transitive and amenable then there is at most 1 infinite cluster almost surely.



"Trifurcation point"

Transitivity ⇒ Every vertex has same prob to be trifurcation

$$|\partial_E \Delta| \ll |\Delta|$$



By linearity of expectation,

$E\#$ Trifurcations $= c\,|\Delta|$

Combinatorial fact:

If $\exists\, k$ trifurcations inside Δ then $\exists$ a collection of k edge disjoint open paths connecting Δ to ∞

$\#$ Trifurcations $\leq |\partial_E \Delta|$

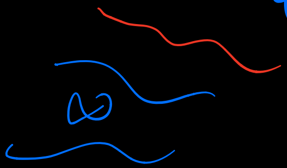
$\#$

Thm (Uniqueness monotonicity: Haggstrom, Peres, & Schonmann) If G is transitive and

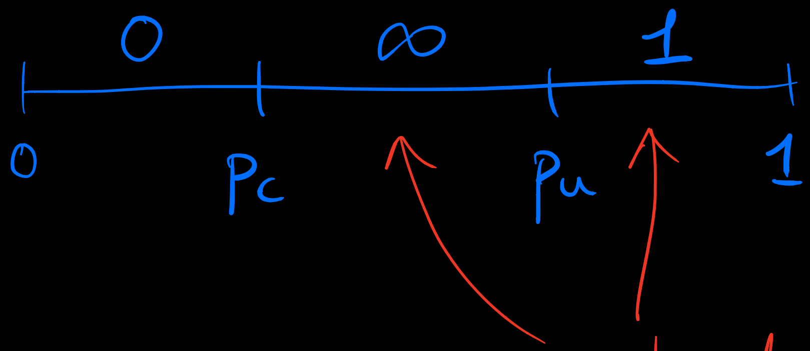
$$\mathbb{P}_p(\exists \text{ a unique } \infty \text{ cluster}) = 1 \quad \text{for some } p$$

then

$$\mathbb{P}_q(\underline{\exists \text{ a unique } \infty \text{ cluster}}) = 1 \quad \forall q \geq p$$



↑ This event is not increasing!



might be degenerate!

Understanding (non)uniqueness at p_u is a difficult problem.

E.g. tesselation of the hyperbolic plane then

$0 < p_c < p_u < 1$, no ∞ clusters at p_c,
unique ∞ cluster at p_u

Benjamini & Schramm 2001.

Proof (Via the Lyons-Schramm Sledgehammer)

Thm (Lyons-Schramm Indistinguishability)
If G is transitive and unimodular and $\mathcal{A} \subseteq \{0,1\}^E$ is a measurable, automorphism-invariant set of subgraphs then either

$$\mathbb{P}_p(\text{All } \infty \text{ clusters belong to } \mathcal{A}) = 1$$

or $$\mathbb{P}_p(\text{All } \infty \text{ clusters belong to } \mathcal{A}^c) = 1.$$

Unimodularity: $|\mathrm{Stab}_u v| = |\mathrm{Stab}_v u| \quad \forall u,v \in V.$
Equivalently, a transitive graph G is unimodular if the <u>mass-transport principle</u>

$$\sum_{v \in V} F(0,v) = \sum F(v,0)$$

holds for every $F : V^2 \to [0,\infty]$ such that

$$F(\gamma u, \gamma v) = F(u,v) \quad \forall \gamma \in \mathrm{Aut}(G),\ u,v \in V.$$

Suppose for contradiction that there is a unique ∞ cluster at p but ∞ many ∞ clusters at $q > p$

$\rightsquigarrow$ We can distinguish the ∞-q clusters by the invariantly defined property "performing p/q percolation gives at least one ∞ cluster almost surely". #

Similar argument carried out in the nonunimodular case by Tang or Häggström, Peres, Schonmann

<u>In fact</u>: It has been proven using indistinguishability that for G transitive

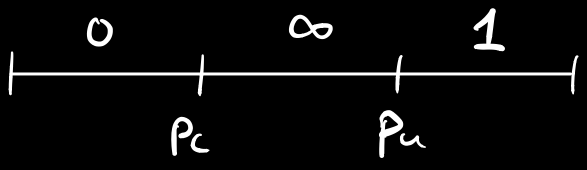
$$\begin{array}{c}\text{nonuniqueness} \\ \text{at } p\end{array} \iff \inf_{x,y} \mathbb{P}_p(x \leftrightarrow y) = 0.$$

Since the RHS condition is clearly monotone, this gives another proof of uniqueness monotonicity.

This was proven by Lyons & Schramm in the unimodular case and by Tang in the nonunimodular case.

Lecture 2

Last time:



- Uniqueness monotonicity
- $p_c = p_u$ for amenable transitive graphs
- Sharpness of the phase transition:

$$\mathbb{E}_p |K| < \infty \quad \text{for } p < p_c .$$

- $p_c \le 1/(1+\underset{\text{Cheeger constant}}{h}\,\underset{\text{degree}}{k}) < 1$ for nonamenable graphs.

$$\inf \left\{ \frac{|\partial_E K|}{\sum_{v \in K} \deg(v)} : K \text{ finite} \right\}$$

Big Conjecture (Benjamini & Schramm 1996)

If G is transitive and nonamenable then $p_c(G) < p_u(G)$.

- Grimmett and Newman:
  $0 < p_c < p_u < 1$ on $T_k \times \mathbb{Z}^d$ with k large.
- Perturbative results (Schonmann, Nagnibeda & Pak, ...)

  Every nonamenable group has a Cayley graph where the conjecture holds.
- Planar graphs (Lalley Benjamini & Schramm)
- "Cost > 1" Lyons
- Graphs with a nonunimodular transitive subgroup (e.g. products with trees) H. 2017
- Gromov hyperbolic graphs. H 2018

Another conjecture: If G is transitive and one-ended then $p_u(G) < 1$

Could be false...?

The operator theoretic approach: A first look

$G=(V,E)$ countable, $M \in [0,\infty]^{V^2}$

$\{f : \sum f(v)^2 < \infty\}$

$$\|M\|_{2\to 2} = \sup\left\{ \frac{\|Mf\|_2}{\|f\|_2} : f \in L^2(V) \setminus \{0\} \text{ finitely supported} \right\}$$

Similarly define $\|M\|_{q\to q}$ operator norm on $L^q(V)$.

Operator norm on $L^2(V)$.

$T_p(u,v) = \mathbb{P}_p(u \leftrightarrow v)$ two-point matrix

$$p_{2\to 2}(G) = \sup\{p : \|T_p\|_{2\to 2} < \infty\}$$

Conjecture (H. 2018) If G is transitive and nonamenable then $p_c(G) < p_{2\to 2}(G)$.

Known in all cases $p_c < p_u$ is known except the cost >1 case.

Note for contrast that $\|T_p\|_{1\to 1} = \|T_p\|_{\infty\to\infty} = \mathbb{E}_p|K|$ so that $p_c = p_{1\to 1} = p_{\infty\to\infty}$ by sharpness of the phase transition.

Observation: $p_{2\to 2} \leq p_u$.

Why? $T_p(u,v) \geq \mathbb{P}_p(u \leftrightarrow \infty)^2 > 0$ $\forall u,v \in V$ when $p > p_u$.
$\Rightarrow T_p$ unbounded.

Recall that if P is the simple random walk transition matrix then $\rho(G) = \|P\|_{2\to 2}$ is called the spectral radius of G.

Thm (Kesten) G nonamenable $\Leftrightarrow \rho < 1$. $\Leftrightarrow$ return probs decay exponentially.

Note that if G is k-regular and A is the adjacency matrix on G then $A = kP$.

Observation: $T_p \preccurlyeq \sum_{i=0}^{\infty} (pA)^i$

$\uparrow$ entrywise inequality

Why? $A^i(u,v)$ is an upper bound on the number of length i self-avoiding paths from u to v. Claim follows by Markov's inequality. $\square$

Note: If M,N are non-negative matrices with $M \preccurlyeq N$ then $\|M\| \leq \|N\|$

Corollary $\|T_p\|_{2\to 2} \leq 1/(1 - p\|A\|_{2\to 2})$.

In particular, $p_{2\to 2} \geq 1/\|A\|_{2\to 2} = 1/k\rho$

for k-regular G.

Thm (cf. Nagnibeda & Pak) If G is regular and $\rho \le 1/2$ then $p_c < p_{2\to 2}$.

Proof We have $p_c \le \frac{1}{1+kh}$ and $p_{2\to 2} \ge \frac{1}{k\rho}$

$\uparrow$ Cheeger constant

$$\frac{|\partial_E K|}{k|K|} = 1 - \frac{1}{|K|}\sum_{\substack{u\in K\\ v\in K}} P(u,v) = 1 - \frac{1}{|K|}\langle 1_K, P1_K\rangle$$

$$\ge 1-\rho$$

$\rightsquigarrow \boxed{h \ge 1-\rho}$ Easy part of "Cheeger's Inequality".

So $p_c \le \frac{1}{1+(1-\rho)k} < \frac{1}{(1-\rho)k}$ and $p_{2\to 2} \ge \frac{1}{\rho k}$

Clearly implies the claim □

Corollary Every nonamenable group has a Cayley graph with $p_c < p_{2\to 2} \le p_u$.

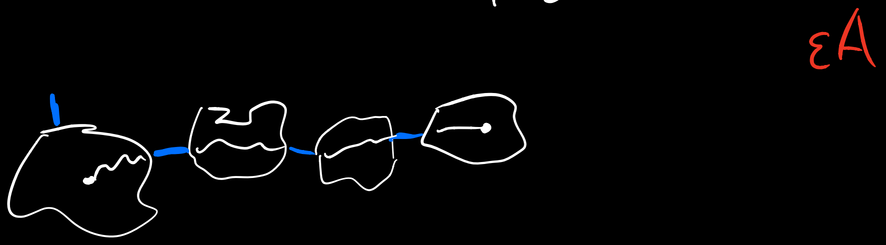
Proof Easy if we allow multigraphs (from generating multisets) since $\rho(S^k) = \rho(S)^k$.

$\llcorner$ multiset.

Also true without allowing multisets, but less obvious. (Thom 2013) □

Later we'll see that we can get a lot more out of the operator-theoretic approach!

Lemma $T_{p+\varepsilon} \preccurlyeq \sum_{i=0}^{\infty} (\varepsilon T_p A)^i T_p$



$$\{x \xleftrightarrow{p+\varepsilon} y\} \leq \bigcup_{i \geq 0} \bigcup_{e_1,\dots,e_i} \{x \xleftrightarrow{p} e_1^-\} \circ \{e_1 \text{ } (p+\varepsilon)\text{-open}, \underset{p \text{ closed}}{}\} \circ \{e_1^+ \xleftrightarrow{p} e_2^-\} \circ \dots$$

Apply union bound and ~~BK~~ Reimer inequality.

□

Corollary. $\|T_{p+\varepsilon}\|_{q\to q} \le \dfrac{\|T_p\|_{q\to q}}{1-\varepsilon\|A\|_{q\to q}\|T_p\|_{q\to q}}$

when $\varepsilon < 1/\|A\|_{q\to q}\|T_p\|_{q\to q}$

$\|M\|_{1\to 1}$ = maximum ℓ_1 norm of a column

- $\|T_{p_c}\|_{1\to 1} = \mathbb{E}_{p_c}|K| = \infty$

$= \|A\|_{1\to 1}$.

$\mathbb{E}_{p_c-\varepsilon}|K| \ge 1/\varepsilon \times (\text{max degree})$

"Mean-field lower bound".

- Similarly,

$\|T_{p_{2\to 2}}\|_{2\to 2} = \infty$, $\boxed{\|T_{p_{2\to 2}-\varepsilon}\| \ge \dfrac{1}{\varepsilon\|A\|_{2\to 2}}}$.

This inequality plays an important role in our analysis of percolation on hyperbolic groups.

§3: Critical Behaviour

- Are there infinite clusters at p_c?
- What does the distribution of finite critical clusters look like?

Believed that there exist critical exponents

e.g.

$$\mathbb{P}_{p_c}(|K| \geq n) \approx n^{-1/\delta}$$

$$\mathbb{P}_{p_c+\varepsilon}(|K| = \infty) \approx \varepsilon^{\beta}$$

$$\mathbb{E}_{p_c-\varepsilon}|K| \approx \varepsilon^{-\gamma}.$$

On the tree:

$$\mathbb{P}_{p_c}(|K| \geq n) \asymp n^{-1/2}$$

$$\mathbb{P}_{p_c+\varepsilon}(|K| = \infty) \asymp \varepsilon$$

$$\mathbb{E}_{p_c-\varepsilon}|K| \asymp \varepsilon^{-1}.$$

$\asymp$, $\lesssim$, $\gtrsim$

= Equalities and inequalities to within positive multiplicative constants

$f \lesssim g \iff f = O(g).$

What about $\mathbb{Z}^d$?

- Believed that same exponents as tree hold when $d > 6$ "Mean field critical behaviour"

- Proven for d large by Hara & Slade ('90s), $d \geq 11$ Fitzner & van der Hofstadt

- For $d < 6$, exponents should be different.

Only understood for $d=2$, site percolation on the triangular lattice

$$\mathbb{P}_{p_c+\varepsilon}(|K| = \infty) \approx \varepsilon^{5/36}$$

$$\mathbb{P}_{p_c}(|K| \geq n) \approx n^{-5/91}$$

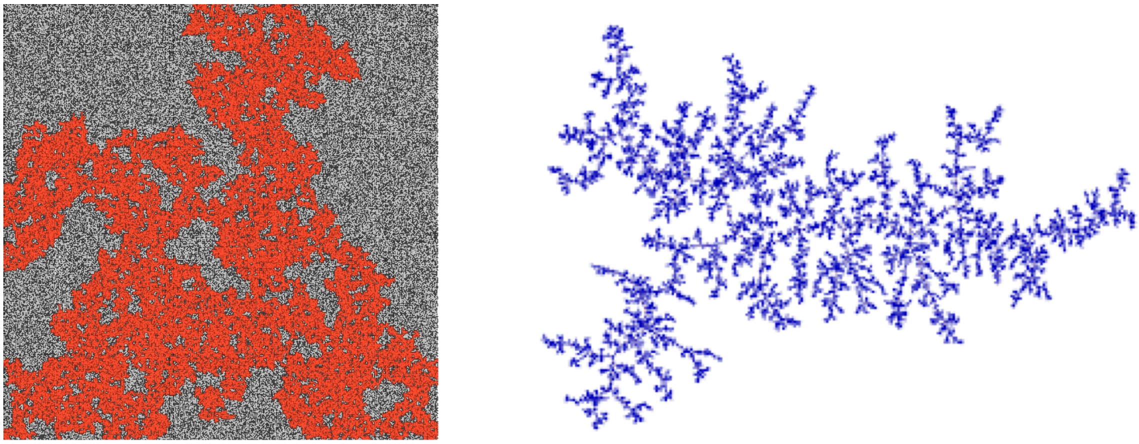
$$\mathbb{E}_{p_c}|K| \approx \varepsilon^{-43/18}$$

Predicted in the 70s by Nienhuis.

Kesten, Smirnov, Lawler, Schramm, Werner...

⚠ For $d = 3, 4, 5$, no reason for exponents to be rational (or algebraic).

- Intuitively, mean field critical behaviour ⟷ Critical percolation behaves similarly to critical branching random walk.



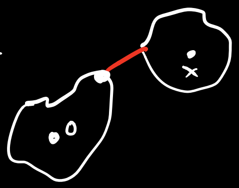
In low dimensions macroscopic cycles play a significant role; in high dimensions they do not.

<u>Conjecture</u>: Every transitive nonamenable graph has mean-field critical behaviour.

Seven dimensional volume growth should be sufficient!

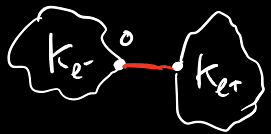
$\chi_p = \mathbb{E}_p|K|$ "susceptibility"

By Russo's formula,

$\frac{d}{dp}\chi_p = \frac{1}{1-p}\sum_e \sum_x \mathbb{P}_p\Big($  $\Big)$

If G is unimodular then

$$\frac{d}{dp}\chi_p = \frac{1}{1-p}\sum_{e^-=0}\mathbb{E}_p\Big[|K_{e^-}|\times|K_{e^+}|\,\mathbb{1}(e^-\not\leftrightarrow e^+)\Big]$$



This easily gives $\frac{d}{dp}\chi_p \leq C\chi_p^2$

We already saw this via a different method!

$$\underset{\text{Calculus}}{\rightsquigarrow} \quad \chi_{p_c-\varepsilon} \geq \frac{c}{\varepsilon}$$

To establish mean-field behaviour, key step is usually to prove complementary diff ineq

$$\frac{d}{dp}\chi_p \geq c\chi_p^2 \qquad p < p_c$$

Idea: conditioning two clusters not to intersect should have a mild effect in high dimensions

Why does this suffice?

$$\frac{d}{dp}\frac{1}{\chi_p} = -\frac{\frac{d}{dp}\chi_p}{\chi_p^2} \leq -c$$

$$\underset{=0}{\cancel{\frac{1}{\chi_{p_c}}}} - \frac{1}{\chi_{p_c-\varepsilon}} \leq -c\varepsilon$$

$$\Longrightarrow \chi_{p_c-\varepsilon} \leq 1/c\varepsilon.$$

The triangle condition

$$\nabla_p := \sum_{y,z} \mathbb{P}_p(0 \leftrightarrow y)\, \mathbb{P}_p(y \leftrightarrow z)\, \mathbb{P}_p(z \leftrightarrow 0)$$

$$= T_p^3(0,0)$$ Certainly finite if $p < p_{2\to 2}$!

Thm (Aizenman & Newman) If $\nabla_{p_c} < \infty$ then $\mathbb{E}_{p_c - \varepsilon} |K| \leq C \varepsilon^{-1}$ $\forall \varepsilon > 0$.

$\frac{d}{dp} \chi_p \geq c \chi_p^2$

$\uparrow$ depends on ∇_{p_c}, p_c, and the degree.

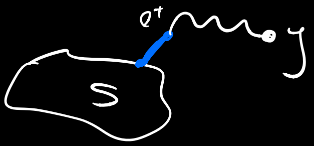
Now known that $\nabla_{p_c} < \infty$ implies many aspects of mean-field critical behavior.

So $p_c < p_{2\to 2} \Rightarrow$ Mean-field critical behavior.

A new derivation of mean-field critical behaviour

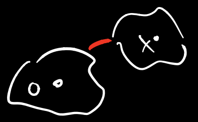
Let's see a simple derivation of $\frac{d}{dp}\chi_p \geq c\chi_p^2$ assuming $\boxed{p_c < p_{2\to 2}}$. Similar proof works assuming only $\nabla_{p_c} < \infty$ with a bit more work.

Define $\quad \Psi_p(S) := \sum_{e\in\vec{\partial}_E S} \sum_{y\notin S} \mathbb{P}_p(e^+ \leftrightarrow y \text{ off } S)$



By Russo's formula,

$$\frac{d}{dp}\chi_p = \frac{1}{1-p}\mathbb{E}_p\Big[\#\{x :$$



$$\}\Big]$$

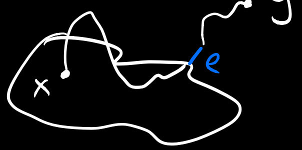
$$= \frac{1}{1-p}\mathbb{E}_p[\Psi_p(K)]$$

Suffices to prove that

$$\Psi_p(S) \geq c\chi_p|S| \qquad \forall S\subseteq V \text{ finite}, \; p<p_c.$$

⚠ UNDER THE TRIANGLE CONDITION (BUT NOT $p_c < p_{2\to 2}$), THIS INEQUALITY MAY NOT HOLD POINTWISE.

Fix $S \subseteq V$.



$x \in S,\ y \notin S$

$$\{x \leftrightarrow y\} \subseteq \bigcup_{e \in \partial_E^{\to} S} \{x \leftrightarrow e^-\} \circ \{e \text{ open}\} \circ \{e^+ \leftrightarrow y \text{ off } S\}$$

Bk + Union bound $\Rightarrow$

$$\sum_{x \in S} \sum_{y \notin S} \mathbb{P}_p(x \leftrightarrow y) \le p \sum_{e \in \partial_E^{\to} S} \sum_{x \in S} \mathbb{P}_p(x \leftrightarrow e^-) \sum_{y \notin S} \mathbb{P}_p(e^+ \leftrightarrow y \text{ off } S)$$

Write $f(v) = \mathbb{1}(v \in S) \sum_{\substack{e \in \partial_E^{\to} S \\ e^- = v}} \sum_{y \notin S} \mathbb{P}_p(e^+ \leftrightarrow y \text{ off } S)$

$\le \deg(v)\, \chi_p$

$$\sum_{x \in S} \sum_{y \notin S} \mathbb{P}_p(x \leftrightarrow y) \le p \sum_{v \in S} \sum_{x \in S} \mathbb{P}_p(x \leftrightarrow v) f(v)$$

$$= p \langle T_p \mathbb{1}_S, f \rangle$$

$$\langle T_p \mathbb{1}_S, f \rangle \le \| T_p \mathbb{1}_S \|_2 \| f \|_2$$

$$\le \| T_p \mathbb{1}_S \| \, \| f \|_1^{1/2} \, \| f \|_\infty^{1/2}$$

$$\sum_{x \in S} \sum_{y \notin S} \mathbb{P}_p(x \leftrightarrow y) = |S| \chi_p - \langle T_p \mathbb{1}_S, \mathbb{1}_S \rangle$$

$$\Psi_p(S) = \|f\|_1 \geq \frac{(\chi_p|S| - \langle T_p \mathbb{1}_S, \mathbb{1}_S\rangle)^2}{p^2 \|T_p \mathbb{1}_S\|_2^2 \, \|f\|_\infty}$$

$$\|f\|_\infty \leq \underset{=M}{(\text{max degree})} \times \chi_p \qquad \|\mathbb{1}_S\|_2^2 = |S|$$

$$\Psi_p(S) \geq \frac{(\chi_p - \|T_p\|_{2\to2})^2 \, |S|^{\cancel{2}}}{p^2 \|T_p\|_{2\to2}^2 \, \cancel{|S|} \, \chi_p}$$

Suppose that $\|T_{p_c}\|_{2\to2} < \infty$.

Since $\chi_p \uparrow \infty$ as $p \uparrow p_c$, there exists $\varepsilon > 0$ st. $\chi_p \geq 2\|T_{p_c}\|_{2\to2}$ for $p > p_c - \varepsilon$, so that

$$\Psi_p(S) \geq c\,\chi_p |S|$$

for every $p_c - \varepsilon \leq p < p_c$ and $S \subseteq V$ finite. □

§4: Continuity of the phase transition and power law bounds

Conjecture (Benjamini & Schramm '96) If G is transitive and $p_c < 1$ then there are no infinite clusters at p_c

BIG open problem for $\mathbb{Z}^d$, $d = 3, 4, 5$.

Theorem (Benjamini, Lyons, Peres, Schramm '99)
True for G unimodular and nonamenable.

Theorem (H. 2016) True for G of exponential growth.

(Input in nonunimodular case from Timár)

↑ Weaker than nonamenability, e.g. lamplighters.

Theorem (H. 2018) If G has exponential volume growth then $\exists$ constants $c, C > 0$ such that

$$\mathbb{P}_{p_c}(|K| \geq n) \leq C n^{-c} \quad \forall n \geq 1.$$

transitive and unimodular.

__Thm__ If G ← transitive has exponential growth, then G does not have a __unique__ infinite cluster at p_c.

__Proof__ Consider the quantity

$$\kappa_p(n) = \min\{\mathbb{P}_p(x \leftrightarrow y) : d(x,y) \le n\}$$

__Claim__ $\kappa_p(n+m) \ge \kappa_p(n)\kappa_p(m)$ "Supermultiplicativity"

Why? For each z with $d(x,z) \le n+m$ $\exists\, y$ with $d(x,y) \le n$, $d(y,z) \le m$.

$\mathbb{P}_p(x \leftrightarrow z) \ge \mathbb{P}_p(x \leftrightarrow y)\,\mathbb{P}_p(y \leftrightarrow z) \ge \kappa_p(n)\,\kappa_p(m)$

Claim follows by infimizing over z.

__Fekete's Lemma__: If $a(n)$ is a positive supermultiplicative sequence then $\sup_{n \ge 1} a(n)^{1/n} = \lim_{n\to\infty} a(n)^{1/n}$.

Let $p < p_c$. Then

numerator $\le \chi_p < \infty$ by sharpness! ↙

$$\sup_{n \ge 1} \kappa_p(n)^{1/n} = \lim_{n\to\infty} \kappa_p(n)^{1/n} \le \liminf_{n\to\infty} \left(\frac{\sum_{x \in B(0,n)} \mathbb{P}_p(0 \leftrightarrow x)}{|B(0,n)|} \right)^{1/n}$$

$gr(G) := \limsup_{n\to\infty} |B(0,n)|^{1/n}$

$$= 1/gr(G).$$

So $\boxed{\kappa_p(n) \leq gr(G)^{-n}}$ for every $p < p_c$ and $n \geq 1$.

For each x, y

$$\mathbb{P}_p(x \leftrightarrow y) = \sup_{r \geq 1} \mathbb{P}_p(x \leftrightarrow y \text{ inside } B(x,r))$$

supremum of increasing continuous functions

$\Rightarrow \mathbb{P}_p(x \leftrightarrow y)$ is left continuous in p for each $x, y \in V$.

$\Rightarrow \boxed{\kappa_{p_c}(n) \leq gr(G)^{-n} \quad \forall n \geq 1 \text{ also.}}$

This implies the claim. $\square$

Remark: Same proof gives that

$$\kappa_{p_{2\to 2}}(n) \leq gr(G)^{-n/2}$$

$\forall n \geq 1$, so that there is never uniqueness at $p_{2\to 2}$.

Open problem: When is $p_{2\to 2} = p_u$?

Lecture 3

Last time:
- Perturbative criteria for $p_c < p_{2\to 2}$
- Mean-field critical behaviour and the triangle condition.

$$\kappa_{p_c}(n) = \min \{ \mathbb{P}_{p_c}(x \leftrightarrow y) : d(x,y) \leq n \}$$
$$\leq gr(G)^{-n}$$

where $gr(G) = \limsup_{n\to\infty} |B(0,n)|^{1/n}$

exponential rate of growth.

Polynomial bounds via the Aizenman-Kesten-Newman method.

Thm (H. 2018) If G is a unimodular transitive graph with degree k and growth $gr(G) \geq g > 1$ then $\exists$ $C = C(k,g)$ and $c = c(k,g) > 0$ st.

$$\mathbb{P}_{p_c}(|K| \geq n) \leq C n^{-c} \quad \forall n \geq 1.$$

Aside:

Schramm's Locality Conjecture If G_n is a sequence of transitive graphs converging locally to a transitive graph G and $p_c(G_n) < 1$ $\forall n \geq 1$ then

$$p_c(G_n) \xrightarrow[n \to \infty]{} p_c(G)$$

"Given $p_c < 1$, the value of p_c is local".

Corollary The locality conjecture is true if $\exists g > 1$ st.

$gr(G_n) \geq g \quad \forall n \geq 1.$

Why? Prop (Pete) $p_c(G) \leq \liminf_{n\to\infty} p_c(G_n)$

"p_c is lower semicontinuous".

Pf We saw in the proof of sharpness that

$$\mathbb{P}_{p_c+\varepsilon}(|K|=\infty) \geq \frac{\varepsilon}{(1-p_c)(p_c+\varepsilon)} \geq \varepsilon$$

for every transitive graph. If R_n is the maximal radius st. G_n and G coincide up to radius R_n then

$$\mathbb{P}^{G}_{p_c(G_n)+\varepsilon}(|K|\geq R_n) \geq \mathbb{P}^{G_n}_{p_c(G_n)+\varepsilon}(|K|=\infty)$$

$$\geq \varepsilon \qquad \forall n, \varepsilon>0$$

and it follows that if $p \geq \liminf_{n\to\infty} p_c(G_n)+\varepsilon$ then

$$\mathbb{P}^{G}_{p}(|K|=\infty) \geq \limsup_{n\to\infty} \mathbb{P}^{G}_{p}(|K|\geq R_n) \geq \varepsilon.$$

This implies the claim. $\square$

To complete the proof that exponential growth $\Rightarrow$ locality, suffices to prove that

$$p_c(G_m) \leq p_c(G) + C R_m^{-c} \quad \forall m \geq 1.$$

By the theorem, $\exists C$ and $c>0$ st.

$$\mathbb{P}_p^{G_m}(|K| \geq n) \leq C n^{-c} \quad \forall n, m \geq 1.$$

$p \leq p_c(G_m)$

If G_m and G coincide up to radius R but $p_c(G_m) \geq p_c(G) + \varepsilon$ then

$$C R^{-c} \geq \mathbb{P}^{G_m}_{p_c(G)+\varepsilon}(|K| \geq R) \geq \mathbb{P}^{G}_{p_c(G)+\varepsilon}(|K| = \infty) \geq \varepsilon$$

which gives a contradiction when $\varepsilon > C R^{-c}$.

So $p_c(G_m) < p_c(G) + C R^{-c}$ as claimed.

$\square$

General principle: Uniform control of critical or near critical percolation for some class of graphs $\Rightarrow$ Locality for that class.

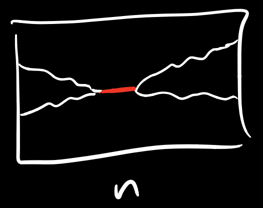
Problem: Let G be a transitive graph with degrees bounded by k. Then $\exists\, C = C(k)$ such that

$$\mathbb{E}_{p_c-\varepsilon}|K| \leq \underbrace{C^{C^{C^{1/\varepsilon}}}}_{\text{Or whatever function you like!}} \qquad \forall\, \varepsilon > 0.$$

This would establish locality!

The Aizenman–Kesten–Newman method

Thm (**AKN** '86) Percolation on a box in $\mathbb{Z}^d$.



$$\mathbb{P}_p\Big(\qquad \Big) \le \frac{C_p \log n}{\sqrt{n}}$$

- Bound only implicit in their work.
- Gained more popularity following the work of Cerf (2013).

Thm ("Two-ghost Inequality" H. 2018)

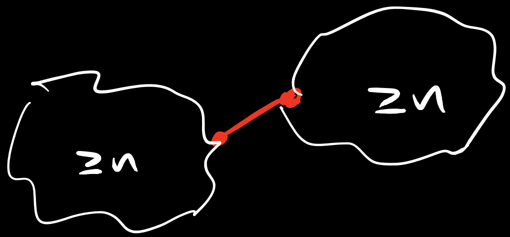
G transitive unimodular. Then

$$\sum_{e^-=0} \mathbb{P}_p(\mathcal{S}_{e,n}) \le 66 \deg(o) \sqrt{\frac{1-p}{pn}} \qquad \forall n \ge 1$$

$\lesssim 1/\sqrt{n}$

$\uparrow$

$\{e^-, e^+$ belong to distinct clusters, both touch at least n edges, at least one is finite$\}$



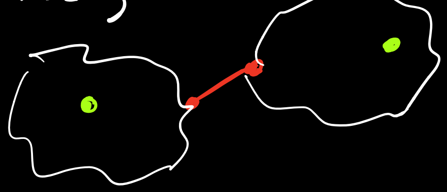
Let $h > 0$ and let $\mathcal{G}$ be an independent ghost field of intensity h on E, i.e., a random subset of E in which vertices are included independently at random with inclusion probability. Joint law $\mathbb{P}_{p,h}$.
$1-e^{-h} \approx h$

Ghost fields let us convert questions about the distribution of the volume of percolation clusters into connectivity-type events. At criticality we expect intuitively that $\{|K| \geq 1/h\}$ and $\{0 \leftrightarrow \mathcal{G}\}$ are "roughly the same".

Thm ("Two-ghost Inequality" H. 2018)
G transitive unimodular. Then

$$\sum_{e^-=0} \mathbb{P}_{p,h}(\mathscr{T}_e) \leq 33 \deg(0) \sqrt{\frac{1-p}{p} h} \quad \forall n \geq 1$$

$\{e^-, e^+$ belong to distinct clusters, both touch a ghost edge, and at least one is finite$\}$



To deduce Thm from Thm', use that

$$\mathbb{P}_{p,h}(T_e) \geq (1-e^{-hn})^2\,\mathbb{P}_p(S_{e,n})$$

and optimize over h.

For each finite subgraph H of G define the *fluctuation*

$$h_p(H) = p|\underbrace{\partial H}_{\text{edges that touch but do not belong to } H}| - (1-p)|\underbrace{E_o(H)}_{\text{edges belonging to } H.}|$$

If we explore the cluster of the origin one edge at a time then

$$p\#(\text{Revealed closed edges at time } n)$$
$$-(1-p)\#(\text{Revealed open edges at time } n)$$

is a random walk on $\mathbb{R}$ with iid, mean zero increments, stopped when we explore the whole cluster.

$h_p(K)$ = Final value of this martingale.

Aside: If F is a function on subgraphs of G then under mild assumptions we have

$$\frac{d}{dp}\mathbb{E}_p F(K) = \frac{d}{dp}\sum p^{\#\text{open}}(1-p)^{\#\text{closed}} F(C)$$
$$= \frac{1}{p(1-p)}\mathbb{E}_p\left[h_p(K)F(K)\right]$$

This makes the fluctuation $h_p(K)$ an important quantity in many contexts.

Key Lemma

$$\sum_{e^-=0} \mathbb{P}_{p,h}(\tilde{T}_e)$$
$$\le \frac{2\deg(0)}{p}\mathbb{E}_{p,h}\left[\frac{|h_p(K_0)|}{|E(K_0)|}\underbrace{\mathbb{1}\big(|K_0|<\infty \text{ and } \overset{\substack{\text{edges touching } K_0\\ \downarrow}}{E(K_0)}\cap \mathcal{G}\neq\emptyset\big)}_{\mathbb{1}(|K_0|<\infty)(1-e^{-h|E(K_0)|})}\right]$$

Mass-transport for edge-functions:

$$\sum_{v\in V} F(o,v) = \sum_{v\in V} F(v,o)$$

$F: V^2 \to [0,\infty]$

$F(\gamma u,\gamma v) = F(u,v)$

(oriented) Edge version:

$$\sum_{e_1^-=o}\ \sum_{e_2\in E^{\to}} F(e_1,e_2) = \sum_{e_1^-=o}\ \sum_{e_2\in E^{\to}} F(e_2,e_1) \quad (*)$$

$F: (E^{\to})^2 \to [0,\infty]$

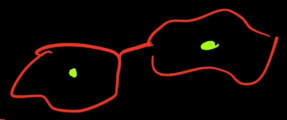
$F(\gamma e_1,\gamma e_2) = F(e_1,e_2)$

Similarly, if $F:(E^{\to})^2 \to \mathbb{R}$ is diagonally invariant and satisfies

$$\sum_{e_1^-=o}\ \sum_{e_2\in E^{\to}} |F(e_1,e_2)| < \infty$$

then $(*)$ holds. (Why? Apply $(*)$ to positive and negative parts of F.)

Proof of Key Lemma

$\mathcal{T}_e = \{$  $\}$

$\mathcal{F}_e :=$ {Every cluster touching e is finite}

$\mathcal{G}_e :=$ {$\exists$ a finite cluster touching e and $\mathcal{G}$}

Observe

$$\mathbb{1}(\mathcal{T}_e \cap \mathcal{F}_e) = \mathbb{1}(e \text{ closed}) \# \{\text{finite clusters touching } e \text{ and } g\}$$

$$- \mathbb{1}(\{e \text{ closed}\} \cap \mathcal{G}_e)$$

Taking expectations:

$$\mathbb{P}_{p,h}(\mathcal{T}_e \cap \mathcal{F}_e) = \mathbb{E}\big[\mathbb{1}(e \text{ closed}) \# \{\text{finite clusters touching } e \text{ and } g\}\big]$$

$$- \mathbb{P}_p(\{e \text{ closed}\} \cap \mathcal{G}_e \cap \mathcal{F}_e)$$

$$- \mathbb{P}_p(\{e \text{ closed}\} \cap \mathcal{G}_e \setminus \mathcal{F}_e)$$

$\mathcal{T}_e \cap \mathcal{F}_e$ and $\{e \text{ closed}\} \cap \mathcal{G}_e \setminus \mathcal{F}_e$,

have union $\mathcal{T}_e$ modulo a null set (in which there is an ∞ cluster not touching g)

So

$$\mathbb{P}_{p,h}(\mathcal{T}_e) = \mathbb{E}\big[\mathbb{1}(e \text{ closed}) \# \{\text{finite clusters touching } e \text{ and } g\}\big]$$

$$- \mathbb{P}_p(\{e \text{ closed}\} \cap \mathcal{G}_e \cap \mathcal{F}_e)$$

$$= \mathbb{E}\big[\mathbb{1}(e \text{ closed}) \# \{\text{finite clusters touching } e \text{ and } g\}\big]$$

$$- \frac{1-p}{p} \mathbb{P}_p(\{e \text{ open}\} \cap \mathcal{G}_e \cancel{\cap \mathcal{F}_e})$$

$\rightsquigarrow$

$$\mathbb{P}_{p,h}(\tau_e) = \mathbb{E}\left[\left(\mathbb{1}(e \text{ closed}) - \frac{1-p}{p}\mathbb{1}(e \text{ open})\right) \cdot \#\{\text{finite clusters touching } e \text{ and } g\}\right].$$

So far we have not used anything about the graph!

Define $F : \vec{E} \times \vec{E} \to \mathbb{R}$ by

Write $\sum\{a_i : i \in I\} = \sum_{i \in I} a_i$

$$F(e_1, e_2) =$$

$$\mathbb{E}_{p,h} \sum \left\{ \frac{\mathbb{1}(e_1 \text{ closed}) - \frac{1-p}{p}\mathbb{1}(e_1 \text{ open})}{2|E(K)|} : \begin{matrix} K \text{ a finite cluster} \\ \text{touching } e_1, e_2 \text{ and} \\ \text{the ghost field} \end{matrix} \right\}$$

Multiset being summed over has size ≤ 2

$\rightsquigarrow$ F satisfies integrability required for signed MTP.

$$\sum_{e^- = 0} \mathbb{P}_{p,h}(\tau_e) = \sum_{e_1^- = 0} \sum_{e_2 \in \vec{E}} F(e_1, e_2)$$

$$= \sum_{e_1^- = 0} \sum_{e_2 \in \vec{E}} F(e_2, e_1)$$

$$= \frac{1}{p} \sum_{e_i^-=0} \mathbb{E}_{p,h} \sum \left\{ \frac{h_p(K)}{|E(K)|} : K \text{ a finite cluster touching } e_i \text{ and ghost} \right\}$$

$$\leq \frac{2}{p} \deg(o)\, \mathbb{E}_{p,h} \frac{|h_p(K_o)|}{|E(K_o)|} \mathbb{1}\left(|K_o| < \infty,\ E(K_o) \cap \mathcal{G} \neq \emptyset\right)$$

as required $\square$

Now, as we discussed before, exploring the cluster of the origin one edge at a time

$$Z_n := \begin{aligned} &p\,\#(\text{Revealed closed edges at time } n) \\ &- (1-p)\#(\text{Revealed open edges at time } n) \end{aligned}$$

$T = |E(K_o)|$ stopping time

$Z_T = h_p(K_o)$

$$\sum_{e^-=0} \mathbb{P}_{p,h}(\mathcal{T}_e) \leq \frac{2}{p} \deg(o)\, \mathbb{E}_p \left[\frac{|Z_T|}{T} (1 - e^{-hT}) \mathbb{1}(T < \infty) \right]$$

WTP $\leq C\sqrt{h}$

So it suffices to prove

$$\mathbb{E}_p\left[\frac{|Z_T|}{T}(1-e^{-hT})\mathbb{1}(T<\infty)\right] \le C\sqrt{p(1-p)h}$$

We have

$$\frac{1-e^{-hx}}{x}$$

$$\mathbb{E}_p\left[\frac{|Z_T|}{T}(1-e^{-hT})\mathbb{1}(T<\infty)\right]$$

$$\le \sum_{k\ge 0}\frac{1-e^{-h2^k}}{2^k}\,\mathbb{E}_p\left[\max_{2^k\le n\le 2^{k+1}}|Z_n|\mathbb{1}(2^k\le T\le 2^{k+1})\right]$$

Z_n is a martingale with

$$\mathbb{E}_p Z_n^2 = \sum_{i=1}^{n}\mathbb{E}_p(Z_i - Z_{i-1})^2 \le p(1-p)n$$

↑ orthogonality of martingale increments.

Doob's L^2 maximal inequality $\Rightarrow$

$$\mathbb{E}_p\left[\max_{2^k\le n\le 2^{k+1}}|Z_n|^2\right] \le 4\,\mathbb{E}_p|Z_{2^{k+1}}|^2 \le 8p(1-p)2^k$$

$$
\begin{aligned}
\mathbb{E}_p\left[\frac{|Z_T|}{T}\left(1-e^{-hT}\right)\mathbb{1}(T<\infty)\right]
&\le \sum_{k\ge 0} \frac{1-e^{-h2^k}}{2^k}\sqrt{8p(1-p)2^k} \\
&= \sqrt{8p(1-p)}\;\underbrace{\sum_{k\ge 0} \frac{1-e^{-h2^k}}{2^{k/2}}}_{\le C\sqrt{h}}
\end{aligned}
$$

use $1-e^{-h2^k} \le h2^k$ for $k \le \log_2(1/h)$,

$1-e^{-h2^k} \le 1$ for $k > \log_2(1/h)$

$$\sum_{k=0}^{\log_2(1/h)} h2^{k/2} + \sum_{k>\log_2(1/h)} 2^{-k/2} \le C\sqrt{h}.$$

This completes the proof!

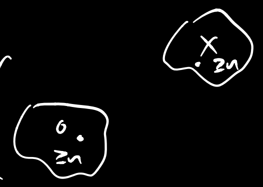
Deducing power law upper bounds from the two-ghost inequality

Fix $p < p_c$.

$\mathbb{P}_p\Big($  $\Big) \le \dfrac{p^{-d(0,x)}}{1-p} \underbrace{\sup_e \mathbb{P}_p(S_{e,n})}_{\le C n^{-1/2}}$

0,x in distinct clusters of size at least n

Why? Force edges in a geodesic 0→x to be open one at a time. On the event in question, there must be a point during this procedure where we have an event $S_{e,n}$ for some e. Doing the bookkeeping to see how this surgery affects probabilities gives the claim.

$\mathbb{P}_p(|K_0|, |K_x| \ge n) \le \mathbb{P}_p\Big($  $\Big)$

$+ \mathbb{P}_p(0 \leftrightarrow x)$

$$\rotatebox{90}{$\le$}$$

$$\mathbb{P}_p(|K_0| \ge n)^2$$

$$\mathbb{P}_p(|K_o| \geq n)^2 \leq \frac{p^{-r}}{1-p} \sup_e \mathbb{P}_p(S_{e,n}) + \kappa_p(r)$$

$$\leq C p^{-r} n^{-1/2} + gr(G)^{-r}$$

Take $r = c \log n$ for an appropriately small constant $c > 0$. $\square$

More applications of this method:

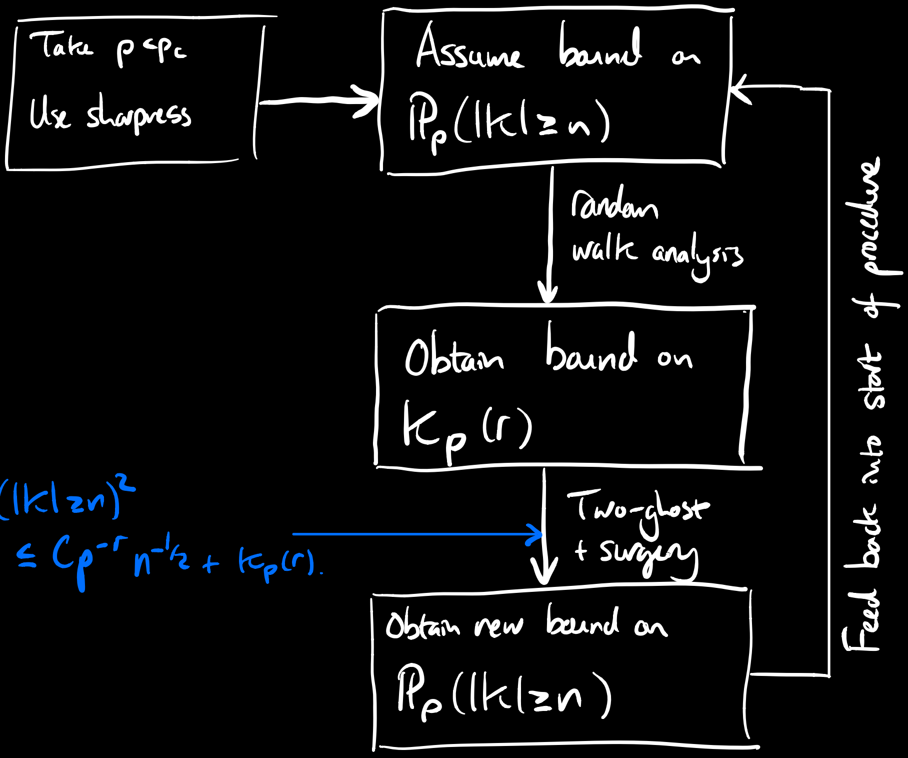
- Hermon & H. 2018: No percolation at p_c for certain groups of intermediate growth
- H. 2020: Continuity of the phase transition for the Ising model on nonamenable groups
- H. 2020: Power law bounds for critical long-range percolation on $\mathbb{Z}^d$.

Thm (Herman & H.) Suppose G is a unimodular transitive graph st. $p_n(0,0) \le Ce^{-cn^{\gamma}}$ some $C,c>0$, $\gamma > 1/2$. Then there is no percolation at criticality on G.

Applies to some groups of intermediate growth like Erschler's piecewise automatic groups

↖ Yield groups of intermediate growth with arbitrarily fast subexponential heat kernel decay.

Bootstrapping method:



$$\mathbb{P}_p(|K|\ge n)^2 \le Cp^{-r} n^{-1/2} + \kappa_p(r).$$

$p < p_c$, X indep random walk

$$\mathbb{P}(X_0 \leftrightarrow X_k) = \mathbb{E} \sum_{x \in K_0} P^k(0,x)$$

$\wedge|$

$K_p(K)$

$$= \mathbb{E} \sum_{y \in V} \sum_{x \in K_0} \frac{\mathbb{1}(y \in K_0)}{|K_0|} P^k(0,x)$$

$$\overset{MTP}{=} \mathbb{E} \sum_{y \in V} \sum_{x \in K_y} \frac{\mathbb{1}(0 \in K_y)}{|K_y|} P^k(y,x)$$

$$= \mathbb{E} \frac{1}{|K_0|} \sum_{x,y \in K_0} P^k(y,x) = \mathbb{E} \frac{1}{|K_0|} \langle P^k \mathbb{1}_{K_0}, \mathbb{1}_{K_0} \rangle$$

__Note__ If G __nonamenable__ then $\langle P^k \mathbb{1}_{K_0}, \mathbb{1}_{K_0} \rangle \leq \rho^k |K_0|$

$$\mathbb{P}_p(X_0 \leftrightarrow X_k) \leq \rho^k \quad \forall k \geq 1, \; p < p_c$$

"Schramm's Lemma"

This inequality holds for __any__ automorphism-invariant percolation model with no infinite components.

__Idea__: In amenable setting, $\frac{1}{|A|} \langle P^k \mathbb{1}_A, \mathbb{1}_A \rangle$ cannot be bounded by something small uniformly in A.

__But__: If return probabilities decay very fast, dependence on A is mild.

Prop If G satisfies $p_n(0,0) \le Ce^{-cn^\gamma}$ then

$$\underbrace{\frac{1}{|A|}\langle P^k 1_A, 1_A\rangle}_{\mathbb{P}(X_k \in A \mid X_0 \text{ uniform on } A)} \le C' e^{-c' \min\left\{\frac{k}{\log^{\frac{1-\gamma}{\gamma}}|A|},\, k\gamma\right\}}$$

$\forall A \subseteq V$ finite and $k \ge 1$.

$$\mathbb{P}_p(X_0 \leftrightarrow X_k) \le C' \mathbb{E}_p e^{-c' \min\left\{\frac{k}{\log^{\frac{1-\gamma}{\gamma}}|K_0|},\, k\gamma\right\}}$$

$$\overset{\text{Calculus}}{\le} C'' \mathbb{E}_p\left[e^{\log^\beta |K_0|}\right] e^{-c(\beta) k^{\underset{\overset{\shortparallel}{\alpha}}{\frac{\beta\gamma}{1-(1-\beta)\gamma}}}}$$

$\forall \beta \in (0,1]$

$\gamma > 1/2 \rightsquigarrow \exists \beta \in (0,1]$ with $\underset{\alpha =}{\beta\gamma / 1-(1-\beta)\gamma} > \beta$

$$\mathbb{P}_p(|K| \ge n)^2 \le Cp^{-k} n^{-1/2} + C'' \mathbb{E}_p\left[e^{\log^\beta |K_0|}\right] e^{-c(\beta) k^\alpha}$$

$k = c \log n$

$$\mathbb{P}_p(|K| \ge n) \le C\left(1 + \mathbb{E}_p\left[e^{\log^\beta |K_0|}\right]\right)^{1/2} e^{-c(\log n)^\alpha}$$

$\rightsquigarrow$

$$\mathbb{E}_p e^{\log^\beta |K_0|} \le C\sqrt{1 + \mathbb{E}_p e^{\log^\beta |K_0|}}$$

Rearrange $\rightsquigarrow$ $\mathbb{E}_p e^{\log^{\beta}|K_0|} \leq C' \quad \forall p < p_c$

$\uparrow$ doesn't depend on p!

$\rightsquigarrow$ Same bound holds at p_c! $\square$

Lecture 4: Percolation on hyperbolic graphs

Some things to recall:

$T_p(u,v) := \mathbb{P}_p(u \leftrightarrow v)$ two-point matrix

$$p_{2\to2}(G) := \sup\{p : \|T_p\|_{2\to2} < \infty\}$$
$\leq p_u.$

Conjecture (H. 2018) If G is transitive nonamenable then $p_c(G) < p_{2\to2}(G)$

Sprinkling lemma: $T_{p+\varepsilon} \preccurlyeq \sum_{i=0}^{\infty} (\varepsilon T_p A)^i T_p$

entrywise inequality.

$\rightsquigarrow \|T_p\|_{2\to2} \geq \frac{c}{p_{2\to2}-p}$ as $p \uparrow p_{2\to2}$.

To prove the conjecture, it therefore suffices to prove

$$\|T_{p_c-\varepsilon}\|_{2\to2} = o\left(\tfrac{1}{\varepsilon}\right) \text{ as } \varepsilon \downarrow 0.$$

Proof strategy:

Step a) Prove $\|T_{p_c-\varepsilon}\|_{1\to1} = \chi_p = \mathbb{E}_p|K|$ satisfies

$$\|T_{p_c-\varepsilon}\|_{1\to1} = O(1/\varepsilon).$$

STP

$$\frac{d}{dp}\chi_p \geq c\chi_p^2$$

This is predicted to hold in the nonamenable case since it is part of mean-field critical behaviour. But still open in general!

Step b) Prove that

$$\left\| \frac{T_{p_c-\varepsilon}}{\|T_{p_c-\varepsilon}\|_{1\to1}} \right\|_{2\to2} = o(1) \quad \text{as } \varepsilon \downarrow 0$$

This is a symmetric, bistochastic matrix! Can therefore be interpreted as the transition matrix of a random walk.

At each step, this walk samples a size-biased cluster then jumps to a uniform point in that cluster.

Step b is plausible :

- As $p \uparrow p_c$, the matrix $\frac{T_p}{\|T_p\|_{1\to 1}}$ becomes highly "spread out".

  $T_p(u,v) \leq 1/\chi_p$ small $\forall u,v \in V$.

- "Generic" spread out random walk matrices on a nonamenable transitive graph have small norm.

  E.g. $\|P^k\|_{2\to 2} = \|P\|_{2\to 2}^k \to 0$ as $k \to \infty$.

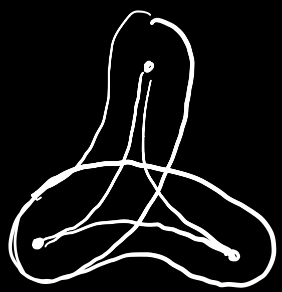
- However, not every spread out matrix on a nonamenable group has small norm.

  Consider e.g. a random walk on $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ that takes big jumps on one copy of $\mathbb{Z}$.

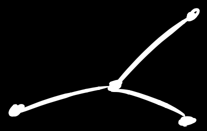
We will rule out these obstructions using hyperbolic geometry!

Thm (H. 2018) If G is a transitive, nonamenable Gromov hyperbolic graph then $p_c(G) < p_{2\to 2}(G)$.

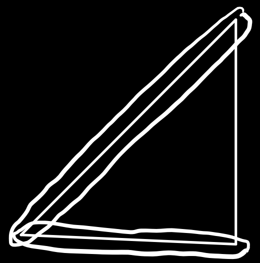
<u>Gromov hyperbolicity</u>: $\exists \delta \geq 0$ (don't think small) s.t. for each geodesic triangle, each side of the triangle is contained in the δ-neighbourhood of the other two sides.



"triangles look like tripods"



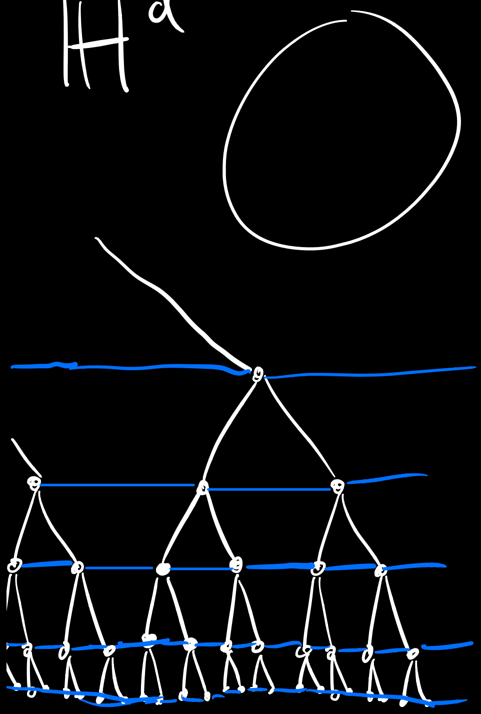
Trees are Gromov-hyperbolic with $\delta = 0$.



$\mathbb{R}^d$ is not Gromov-hyperbolic when $d \geq 2$.

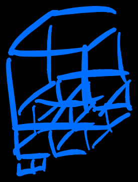
⚠ $\mathbb{R}^1 \cong \mathbb{H}^1$ is hyperbolic but amenable

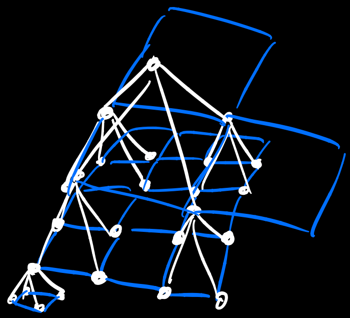
$\mathbb{H}^d$



- d-dimensional open ball
- small line segment ε away from the boundary has length $\simeq$ (Euclidean length)$/\varepsilon$.

$\mathbb{H}^2$ rough isometric to "Binary tree plus horizontal edges"





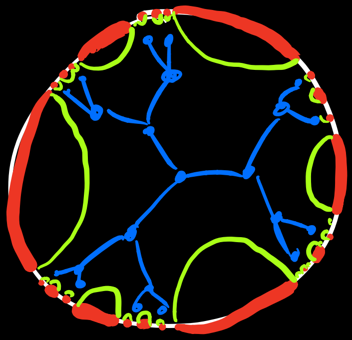
Similar dyadic graphical models for higher dimensional hyperbolic spaces.

<u>Thm</u> (Bonk & Schramm) If G is a bounded degree Gromov hyperbolic graph then $\exists$ $d \geq 1$, a convex set $\Gamma \subseteq \mathbb{H}^d$, a function $\varphi : V \to \Gamma$ and constants λ and C st.

- $| \lambda d(\varphi(u), \varphi(v)) - d(u,v) | \leq C \quad \forall u,v \in V$

- $\forall x \in \Gamma \; \exists u \in V$ with $d(\varphi(u), x) \leq C$.

For infinite transitive graphs, we may take Γ to be the convex hull of its ideal boundary points.



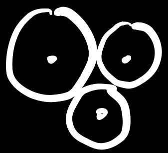
E.g. convex hull of a $2/3$ Cantor set

$\simeq$ regular tree

Geodesics in $\mathbb{H}^d$ $\cong$ Arcs of circles orthogonal to the boundary.

Upshot: For many purposes, we can pretend that we're in standard hyperbolic space.

The magic lemma



Thm (Benjamini Schramm 2001)

"From the perspective of a typical point, any finite set of vertices in $\mathbb{R}^d$ ~~looks like~~ it accumulates to at most two points of $\mathbb{R}^d \cup \{\infty\}$, one of which is ∞!"

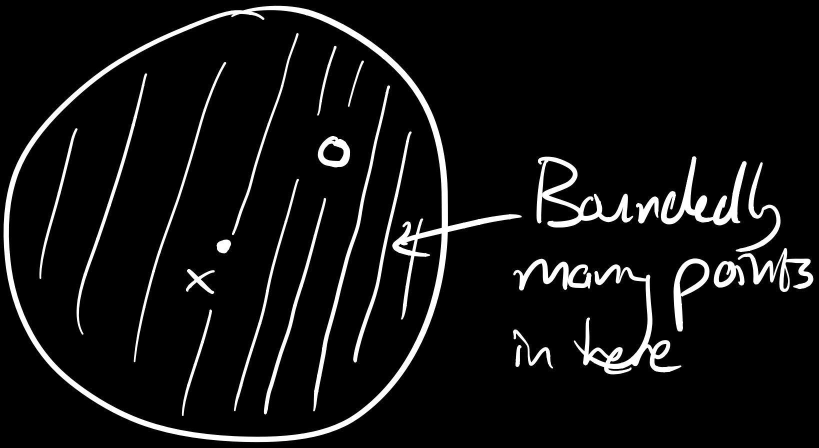
Let $A \subseteq \mathbb{R}^d$ be finite. For each $x \in A$ define the isolation radius

$$\rho(x) = \min\{|x-y| : y \in A \setminus \{x\}\}.$$

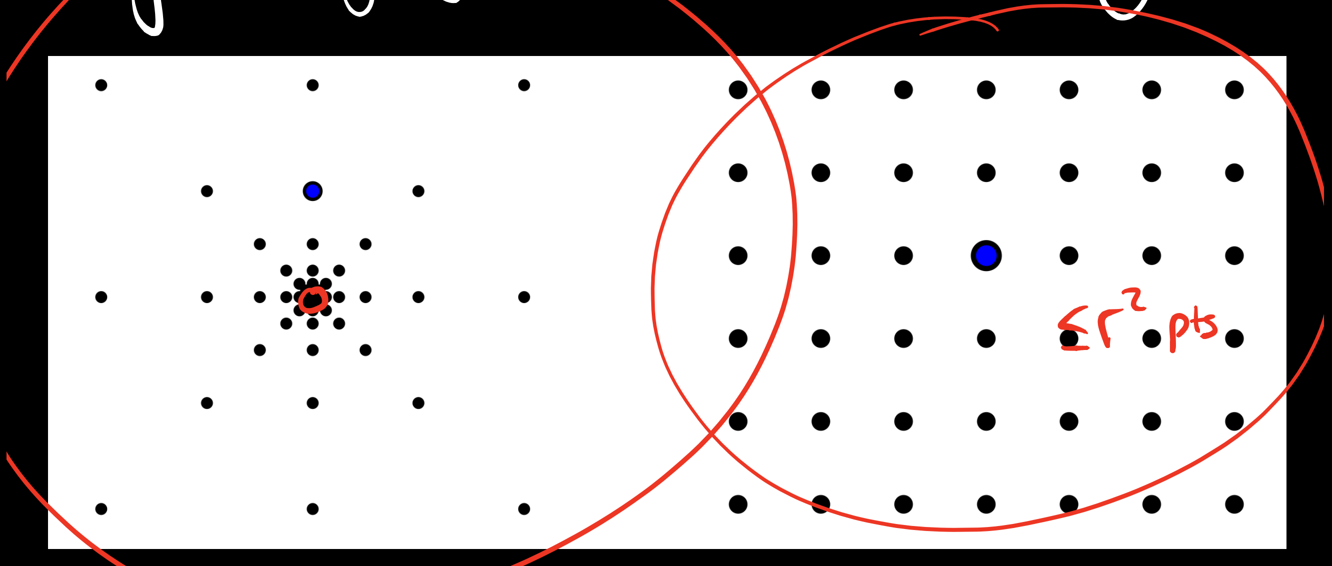
For each $\varepsilon > 0$ and $r < \infty$ $\exists\, C(\varepsilon, r, d) < \infty$ and a set $A' \subseteq A$ with $|A'| \geq (1-\varepsilon)|A|$ st.

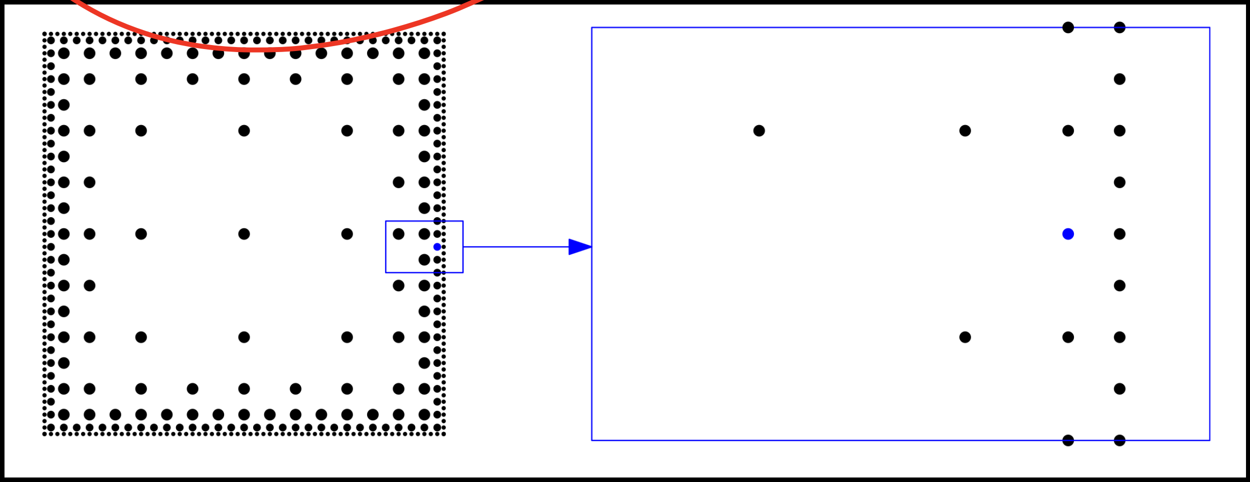
$\forall x \in A'$ $\exists y \in A$ st.

$$|A \cap B(x, r\rho(x)) \setminus B(y, \tfrac{\rho(x)}{r})| \leq C$$



In fact, for fixed ε, d, $C \approx r^d \log r$.

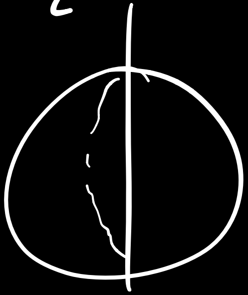




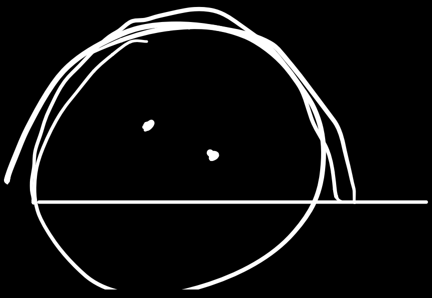
Hyperbolic Magic Lemma (H. 2018): Call a set of points A δ-separated if $d(x,y) \geq \delta$ $\forall$ distinct $x,y \in A$ /

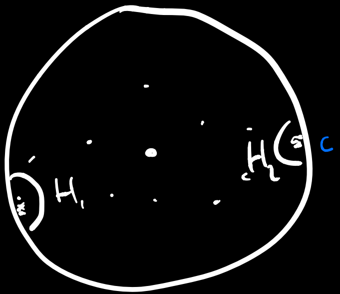
"From the perspective of a typical point, any finite δ-separated set of points in $\mathbb{H}^d$ looks like it accumulates to at most two points of the ideal boundary $\partial \mathbb{H}^d$!"

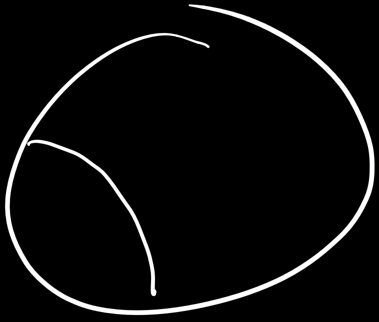
Let $\delta > 0$ and let $A \subseteq \mathbb{H}^d$ be a finite δ-separated subset of $\mathbb{H}^d$. For every $\varepsilon > 0$ and $r < \infty$ $\exists$ $C = C(d,\delta,\varepsilon,r) < \infty$ and $A' \subseteq A$ with $|A'| \geq (1-\varepsilon) A$ such that for every $x \in A'$ there exist half-spaces H_1, H_2 with $d(x,H_i) \geq r$ such that

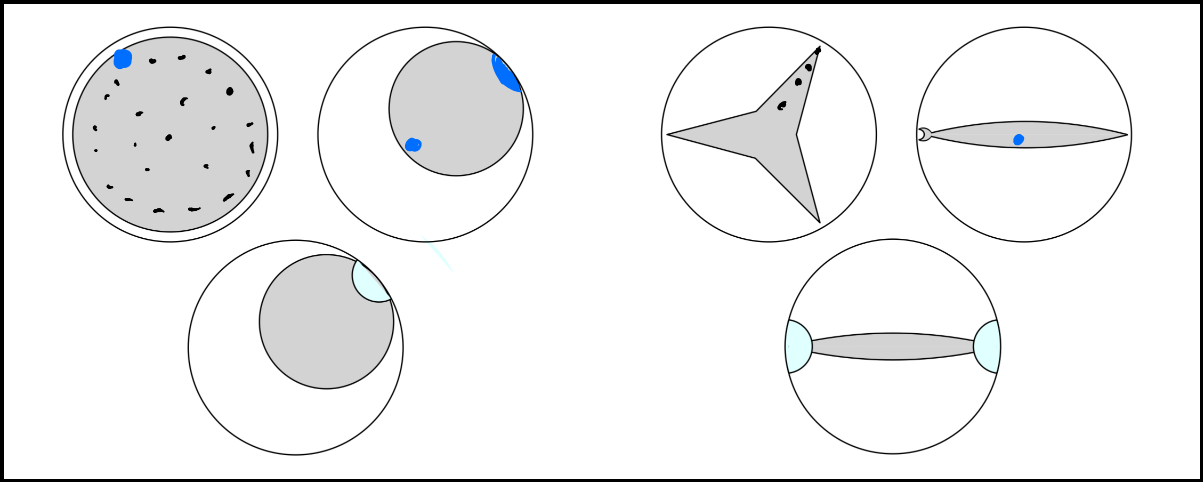


$$|A \setminus (H_1 \cup H_2)| \leq C$$









The hyperbolic magic lemma follows by applying the Euclidean magic lemma to the upper half-space model.

Why is this relevant to us?

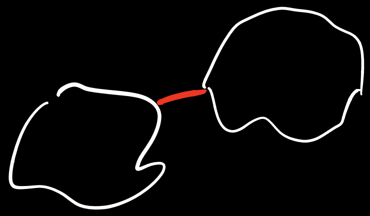
The magic lemma is used in both steps of our strategy to prove $p_c < p_{2\to 2}$ for hyperbolic graphs.

Step a) $\chi_p = \|T_p\|_{1\to 1} \leq \frac{C}{p_c - p}$ $p<p_c$.

As we discussed earlier, it suffices

to prove the complementary differential inequality

$$\frac{d}{dp}\chi_p \geq c\chi_p^2$$



$$)($$

$$\mathbb{E}_p |K_{e^-}| \cdot |K_{e^+}| \mathbb{1}(e^- \leftrightarrow e^+)$$

$)($ a surgery argument.

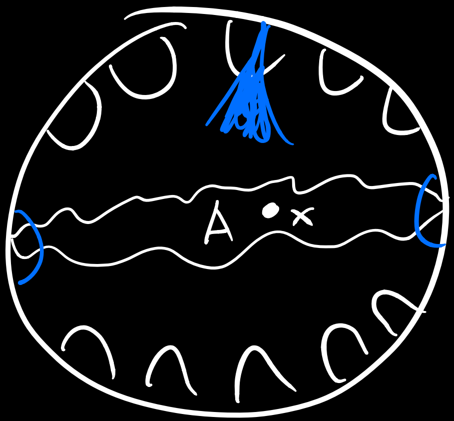
$$\mathbb{E}_p |K_x| \cdot |K_y| \mathbb{1}(x \leftrightarrow y)$$

with $d(x,y)$ bounded.

Consequence of the magic lemma:

For each $\delta > 0$ and $d \geq 2$ there exists R such that if $A \subseteq \mathbb{H}^d$ is finite and δ-separated then $\exists A' \subseteq A$ with $|A'| \geq \frac{1}{2}|A|$

such that if $x \in A'$ $\exists$ a half-space H with $d(x, H) \leq R$ and $A \cap H = \emptyset$



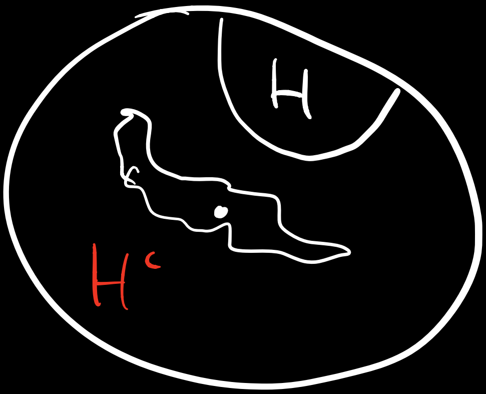
Moreover, when $\varphi : G \to \Gamma$ is the embedding of G into $\mathbb{H}^d$ via Bonk-Schramm, we may take H such that Γ accumulates to the part of the boundary in the interior of H.

Interpretation: Most points in any finite, δ-separated set $A \subseteq \mathbb{H}^d$ are near the boundary of the convex hull of A.

We aren't really using the full power of the magic lemma here.

When G is unimodular and $p < p_c$, "the origin is uniform on its cluster"

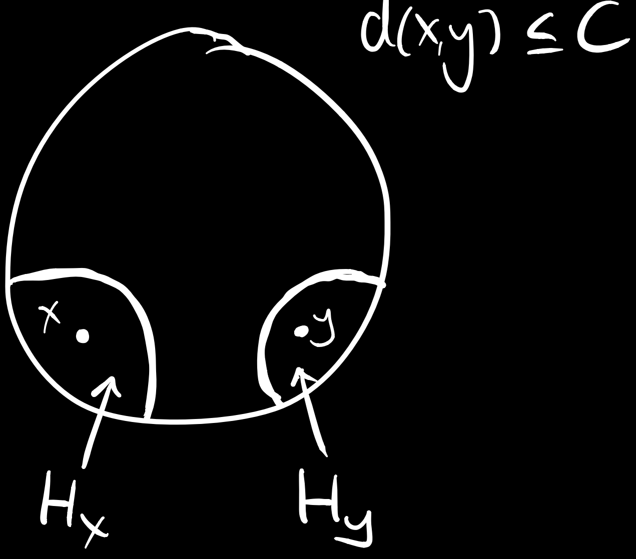
$\rightsquigarrow$ Given $|K|$, $\exists$ such a half-space within distance R of 0 with prob $\geq 1/2$.



A compactness argument $\rightsquigarrow$ $\exists$ H_p^c with $d(0, \partial H^c) \leq R$ st.

$$\mathbb{E}_p |K| \mathbb{1}(K \subseteq H_p^c) \geq c\, \mathbb{E}_p |K|.$$

H_p can be taken to satisfy same non-degeneracy assumptions as before.



$$\mathbb{E}|K_x|\mathbb{1}(K_x \subseteq H_x) \geq c\chi_p$$

$$\mathbb{E}|K_y|\mathbb{1}(K_y \subseteq H_y) \geq c\chi_p$$

$$\Rightarrow \mathbb{E}|K_x|\cdot|K_y|\mathbb{1}(x \not\leftrightarrow y) \geq c^2\chi_p^2$$

$$\rightsquigarrow \frac{d}{dp}\chi_p \geq c'\chi_p^2$$

$$\rightsquigarrow \chi_{p_c-\varepsilon} = O(1/\varepsilon)$$

Completes step (a)!

Step (b) Need to prove

$$\left\| \frac{T_p}{\|T_p\|_{1\to 1}} \right\|_{2\to 2} \to 0 \text{ as } p \uparrow p_c.$$

Theorem (Cheeger '70, Buser '82, Dodziuk '84, Mohar '88, ...)

If $P \in [0,\infty]^{V^2}$ is a symmetric stochastic matrix and we define its Cheeger constant

$$h(P) = \inf\left\{ \frac{\sum_{u \in W, v \notin W} P(u,v)}{|W|} : W \subset V \text{ finite} \right\}$$

then

$$1 - h(P) \leq \|P\|_{2\to 2} \leq \sqrt{1 - h(P)^2}$$

By Cheeger's inequality, it suffices to prove that

$$1 - h\left(\frac{T_p}{\|T_p\|_{1\to 1}}\right) = \sup\left\{ \frac{\sum_{u,v \in W} T_p(u,v)}{|W| \cdot \|T_p\|_{1\to 1}} : W \subset V \text{ finite} \right\}$$

is small when p is close to p_c.

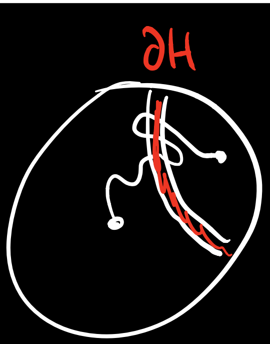
Idea: $\frac{1}{|W|} \sum_{u,v \in W} T_p(u,v)$ has an interpretation in terms of "looking at W from a typical point".

By magic lemma, suffices to prove that

$$\sum_{v \in H} T_p(u,v) = \mathbb{E}_p|K_u \cap H|$$
$$\le \varepsilon(R)\,\chi_p$$

when $d(u,H) \ge R$, where $\varepsilon(R) \to 0$ as $R \to \infty$.

How can we show this???



Suitably interpreted!

By BK,

$$\mathbb{E}_p|K_u \cap H| \le \mathbb{E}_p|K_u \cap \partial H| \cdot \chi_p$$

Idea 1: Suppose G a Cayley graph of a group Γ.

If $A \subseteq \Gamma$ generates a free semigroup

$$\langle A \rangle \cong \mathbb{F}_A$$

then $\sum_{a \in A} T_p(\mathrm{id}, a) \leq 1 \quad \forall p \leq p_c$.

Why? By FKG

$$\sum_{x \in \langle A \rangle} T_p(\mathrm{id}, x) = \sum_{n \geq 0} \sum_{a_1, \dots, a_n \in A} T_p(\mathrm{id}, a_1 \cdots a_n)$$

$$\geq \sum_{n \geq 0} \Big(\sum_{a \in A} T_p(\mathrm{id}, a) \Big)^n$$

When $p < p_c$

$$\sum_{x \in \langle A \rangle} T_p(\mathrm{id}, x) \leq \chi_p < \infty$$

and the claim follows.

Idea 2: Boundaries of half-spaces are essentially free.

More precisely, there is a set A st $\partial H \subseteq$ (C neighbourhood of A) and $\langle A \rangle \cong \mathbb{F}_A$.

$$\rightsquigarrow \mathbb{E}_p |K_n \cap \partial H| \leq C$$

not good enough!
need a small constant.

Stack K hyperplanes — still has

"approximately free" property!

So at least one of these hyperplanes has $\mathbb{E}_p |K_0 \cap \partial H| \leq \frac{C}{K}$

$\rightsquigarrow$ We get the bound we want on the inner half-space! $\square$