Cycles in Mallows random permutations

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(joint with Jimmy He, Fiona Skerman and Teun Verstraaten)

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The Mallows model

We sample a permutation $\Pi_n \in S_n$ according to:

$$\mathbb{P}(\Pi_n = \pi) = \frac{q^{\mathsf{inv}(\pi)}}{\sum_{\sigma \in S_n} q^{\mathsf{inv}(\sigma)}}.$$

Here $q > 0$ is a parameter and

$$\mathsf{inv}(\pi) := |\{(i,j) : i < j \text{ and } \pi(i) > \pi(j)\}|,$$

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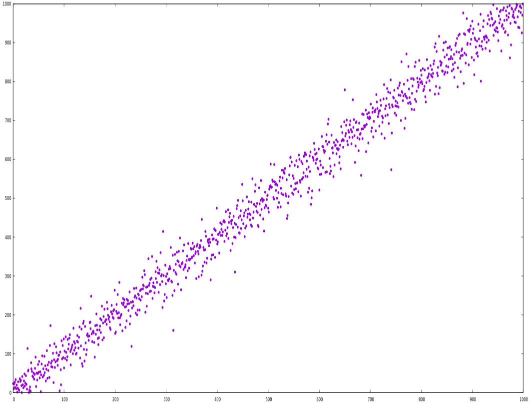
Intuition:

- when $0 < q < 1$ we stay "close to" the identity $i \mapsto i$,
- when $q > 1$ we stay "close to" the reverse map $i \mapsto n + 1 - i$.

A simulation



$(n = 1000, q = .95.)$

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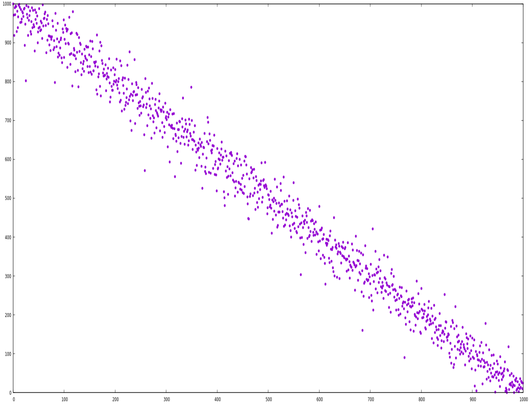

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Background

Introduced by C.L. Mallows in 1957 in the context of "statistical ranking theory".

> *"There is a fixed set of individuals being assessed by a population of judges, or by the same judge in repeated trials, on a particular attribute whose ranking is known a priori."*

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Also studied in connection with Markov chains, random colorings of the integers, stable matchings, random binary search trees, learning theory, exchangeability, point processes, statistical physics, genomics.

More background

Longest increasing subsequence:

$$\mathrm{LIS}(\pi) := \max\{k : \exists i_1 < \cdots < i_k \text{ s.t. } \pi(i_1) < \cdots < \pi(i_k)\}.$$

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When $q = 1$ then $\left(\mathrm{LIS}(\Pi_n) - 2\sqrt{n}\right)/n^{1/6}$ tends in distribution to the Tracy-Widom distribution [Baik+Deift+Johansson 1999].

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When $0 < q < 1$ (is fixed) then $(\mathrm{LIS}(\Pi_n) - \mu n) / (\sigma\sqrt{n})$ tends to a standard normal, for some $\mu = \mu(q), \sigma = \sigma(q)$. [Basu+Bhatnagar 2017].

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Other aspects that have been considered for the Mallows distribution include : longest common subsequences, “pattern avoidence”, the number of descents and the cycle structure (but only when $q = q(n) \to 1$ in this last case.)

A basic observation

If r_n denotes the map $i \mapsto n + 1 - i$ then

$$\mathrm{inv}(r_n \circ \pi) = \binom{n}{2} - \mathrm{inv}(\pi).$$

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That is $r_n \circ \Pi_n \overset{d}{=} \mathrm{Mallows}(n, 1/q)$.

Mallows' sampling algorithm for Π_n when $0 < q < 1$

Let $Z_1, \ldots, Z_n$ be independent with

$$\mathbb{P}(Z_i = k) = \frac{(1-q)q^{k-1}}{1-q^{n+1-i}} \quad (k = 1, \ldots, n+1-i).$$

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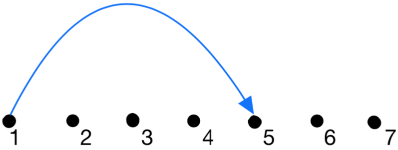
and for $1 < i \leq n$:

$\Pi_n(i) := Z_i$-th smallest element of $\{1, \ldots, n\} \backslash \{\Pi_n(1), \ldots, \Pi_n(i-1)\}$.

Why does this generate the right probability distribution? (1/2)

Note $Z_1 - 1$ is precisely the number of j with $\Pi_n(j) < \Pi_n(1)$.



(The four values of j such that $\Pi_n(j) \in \{1, \ldots, 4\}$ will be produce inversions of the form $(1, j)$.)

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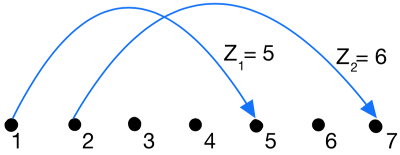

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Similarly, $Z_2 - 1$ is precisely the number of $j > 2$ with $\Pi_n(j) < \Pi_n(2)$.

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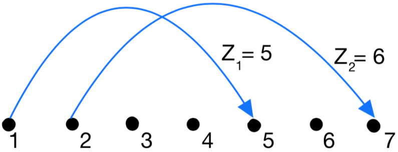

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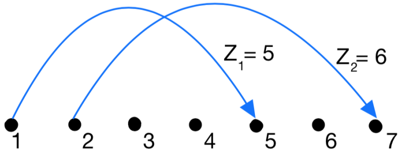

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Conclusion : $\mathrm{inv}(\Pi_n) = Z_1 + \cdots + Z_n - n$.

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For each $\pi \in S_n$ there are $k_1, \ldots, k_n$ such that

$$\mathbb{P}(\Pi_n = \pi) \quad = \quad \mathbb{P}(Z_1 = k_1, \ldots, Z_n = k_n)$$

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■

The infinite Mallows model (for $0 < q < 1$)

Let $Z_1, Z_2, \ldots$ be i.i.d. $\text{Geom}(1-q)$ and define the random bijection

$$\Pi : \mathbb{N} \to \mathbb{N},$$

by

$$\Pi(1) := Z_1,$$

and for $i > 1$:

$$\Pi(i) := Z_i\text{-th smallest element of } \mathbb{N} \setminus \{\Pi(1), \ldots, \Pi(i-1)\}.$$

Notation : $\text{Mallows}(\mathbb{N}, q)$.

Key property of Mallows($\mathbb{N}, q$)

There is a coupling of Π_n, Π such that, with probability $1 - o(1)$, $\Pi_n(i) = \Pi(i)$ for all $1 \le i \le n - \log^2 n$.

The bi-infinite Mallows model

For $0 < q < 1$, Gnedin+Olshanski 2012 introduce a random bijection

$$\Sigma : \mathbb{Z} \to \mathbb{Z},$$

with the property that Π_n is "locally approximated" by Σ.
(The precise definition of Σ is rather technical)

Notation : $\text{Mallows}(\mathbb{Z}, q)$.

Key properties of Mallows$(\mathbb{Z}, q)$

If $i_n, n - i_n \to \infty$ then

$$\Pi_n(i_n) - i_n \xrightarrow[n\to\infty]{d} \Sigma(0).$$

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More generally, for $k \in \mathbb{N}$ fixed

$$(\Pi_n(i_n - k) - i_n, \dots, \Pi_n(i_n + k) - i_n) \xrightarrow[n\to\infty]{d} (\Sigma(-k), \dots, \Sigma(k)).$$

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The probability mass function $z \mapsto \mathbb{P}(\Sigma(0) = z)$ has an explicit expression in terms of q-hypergeometric functions.

Cycles in permutations

A k-cycle in a permutation $\pi \in S_n$ is a set of indices $\{i_1, \ldots, i_k\}$ such that $\pi(i_1) = i_2, \ldots, \pi(i_{k-1}) = i_k, \pi(i_k) = i_1$.

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Notation : $C_k(\pi) := \#k\text{-cycles of } \pi$.

A first year undergrad exercise.

If d_n denotes the number of $\pi \in S_n$ with $C_1(\pi) = 0$ then

$$d_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor .$$

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In particular for $q = 1$:

$$\mathbb{P}(C_1(\Pi_n) = 0) \to 1/e.$$

Generalisation of the exercise

If $q = 1$ (uniform), a classical result of Gontcharoff [1941] and Kolchin [1976] says:

$$(C_1(\Pi_n), C_2(\Pi_n), \ldots, C_k(\Pi_n)) \xrightarrow{\text{d}} (\text{Po}(1), \text{Po}(1/2), \ldots, \text{Po}(1/k))\,,$$

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Curiously, the case when $q \neq 1$ is fixed has not previously been investigated, but the case when $q = q(n) \to 1$ has (by Gladkich+Peled 2018).

Cycles when $0 < q < 1$

Theorem. [He+M+Verstraaten 2023]
For $0 < q < 1$ there exist positive constants $m_1, m_2, \dots$ and an infinite matrix $P \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ such that for all $\ell \geq 1$ we have

$$\frac{1}{\sqrt{n}}\left(C_1(\Pi_n) - m_1 n, \dots, C_\ell(\Pi_n) - m_\ell n\right) \xrightarrow{\text{d}} \mathcal{N}_\ell(\underline{0}, P_\ell),$$

where $\mathcal{N}_\ell(\cdot,\cdot)$ denotes the ℓ–dimensional multivariate normal distribution and P_ℓ is the submatrix of P on the indices $[\ell] \times [\ell]$.

The constants $m_1, m_2, \ldots$

For $i = 1, 2, \ldots$ we have

$$m_i = (1/i) \cdot \mathbb{P}(\text{0 lies in an } i\text{-cycle of } \Sigma),$$

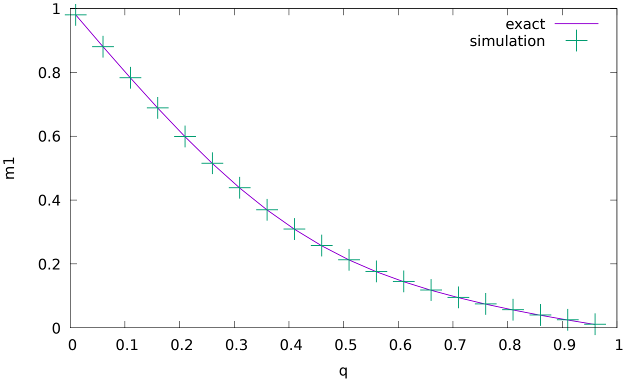
where $\Sigma \sim \text{Mallows}(\mathbb{Z}, q)$.

In particular

$$m_1 = \mathbb{P}(\Sigma(0) = 0) = {}_0\phi_1(-; q; q, q^3) \cdot (1-q) \cdot \prod_{i=1}^{\infty}(1-q^i),$$

where ${}_r\phi_s$ denotes the q-hypergeometric function.

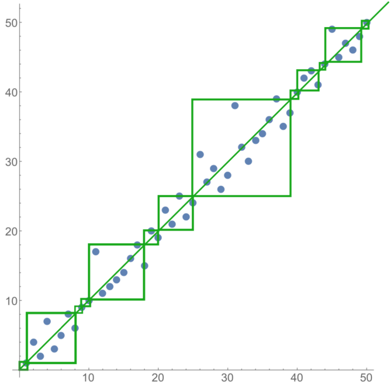
Plot of m_1



Curve : m_1 as a function of q.
Crosses : simulations with $n = 1000$, average of 10^5 tries is shown.

Sketch of the proof of the multivariate normal limit when $0 < q < 1$

When $0 < q < 1$ the Mallows models has a “renewal structure”:



(Plot taken from [Basu+Bhatnagar 2017].)

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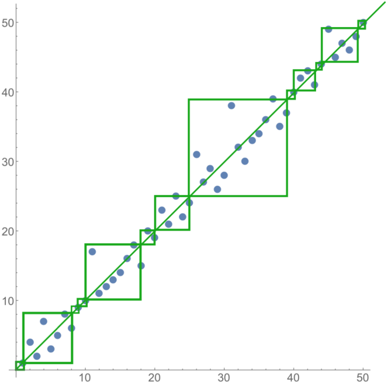

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The "blocks" are approximately i.i.d.

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$$T_1 := \inf\{t : \Pi[\{1,\ldots,t\}] = \{1,\ldots,t\}\},$$
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If $X_i = T_i - T_{i-1}$ and Y_i is the number of k-cycles in the i-th block then $(X_1, Y_1), (X_2, Y_2), \dots$ are i.i.d. and well behaved (all moments exist).

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Now $C_k(\Pi_n) \approx Y_1 + \dots + Y_N$ where N is the (random) value such that $T_N \leq n < T_{N+1}$.

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Now $C_k(\Pi_n) \approx Y_1 + \dots + Y_N$ where N is the (random) value such that $T_N \leq n < T_{N+1}$.
We apply a version of the CLT adapted to such "randomly stopped sums" (Gut+Janson 1983), and the Cramer-Wold device.

Even cycles when $q > 1$

Theorem. [He+M+Verstraaten 2023]
For $q > 1$ there exist positive constants $\mu_2, \mu_4, \ldots$ and an infinite matrix $Q \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ such that for all $\ell \geq 1$ we have

$$\frac{1}{\sqrt{n}}(C_2(\Pi_n) - \mu_2 n, \ldots, C_{2\ell}(\Pi_n) - \mu_{2\ell} n) \xrightarrow{\text{d}} \mathcal{N}_\ell(\underline{0}, Q_\ell),$$

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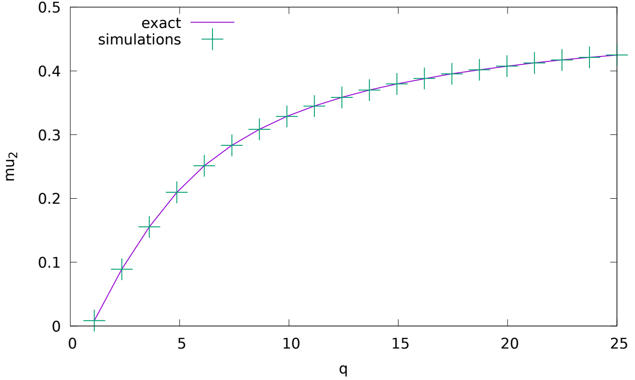
Note this is **only for even cycles**.

The constants $\mu_2, \mu_4, \dots$

Let $q > 1$ and $\Sigma, \Sigma' \sim \text{Mallows}(\mathbb{Z}, 1/q)$ be independent. For $i = 1, 2, \dots$ we have

$$\mu_{2i} = \frac{1}{2i} \cdot \mathbb{P}(0 \text{ is in an } i\text{-cycle of } \Sigma' \circ \Sigma)$$

Plot of μ_2



Odd cycles when $q > 1$.

Theorem. [He+M+Verstraaten 2023]
For $q > 1$, let $\Pi_n \sim \mathrm{Mallows}(n, q)$ and $\Sigma \sim \mathrm{Mallows}(\mathbb{Z}, 1/q)$ and let r, ρ denote the maps $i \mapsto -i$, respectively $i \mapsto 1 - i$.
We have

$$(C_1(\Pi_{2n+1}), C_3(\Pi_{2n+1}), \dots) \xrightarrow{\mathrm{d}} (C_1(r \circ \Sigma), C_3(r \circ \Sigma), \dots)$$

and

$$(C_1(\Pi_{2n}), C_3(\Pi_{2n}), \dots) \xrightarrow{\mathrm{d}} (C_1(\rho \circ \Sigma), C_3(\rho \circ \Sigma), \dots).$$

Moreover, the two limiting distributions above are distinct for all $q > 1$.

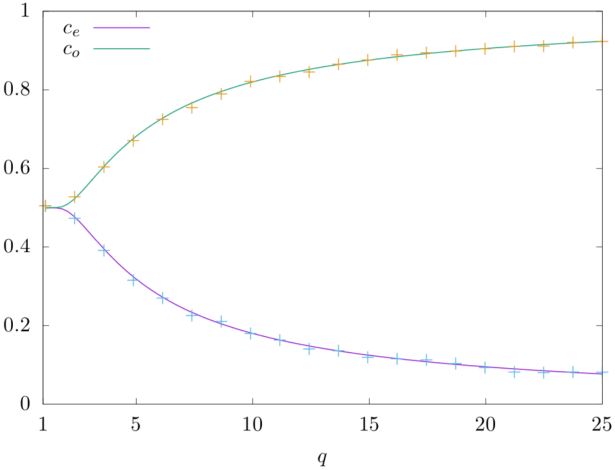
Expected number of fixed points when $q > 1$.

For $q > 1$, let $\Pi_n \sim \text{Mallows}(n, q)$ and $\Sigma \sim \text{Mallows}(\mathbb{Z}, 1/q)$. Then

$$\mathbb{E}C_1(\Pi_{2n+1}) \to \mathbb{P}(\Sigma(0) \text{ even}),$$

$$\mathbb{E}C_1(\Pi_{2n}) \to \mathbb{P}(\Sigma(0) \text{ odd}).$$

A plot of the expected number of fixed points when $q > 1$



Curves : expected no. of fixed points as a function of q.
Crosses : simulations with $n = 1000, 1001$. average no. fixed pts. in 10^5 tries is shown.

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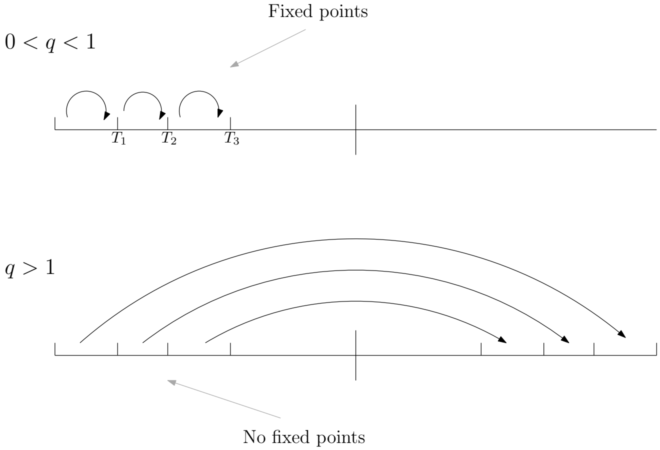

Even cycles for $q > 1$

Solution for even cycles: Define the "joint return times"

$$T_i := \inf \left\{ t > T_{i-1} : \begin{array}{l} \Pi_n[\{1,\dots,t\}] = r_n[\{1,\dots,t\}] \text{ and} \\ \Pi_n[r_n[\{1,\dots,t\}]] = \{1,\dots,t\} \end{array} \right\}$$

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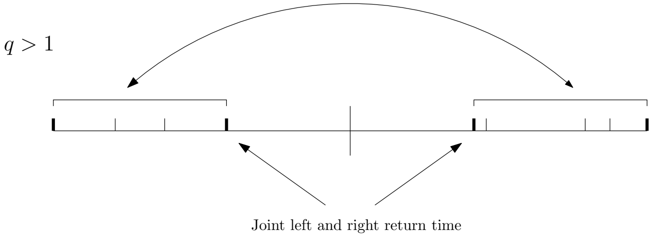

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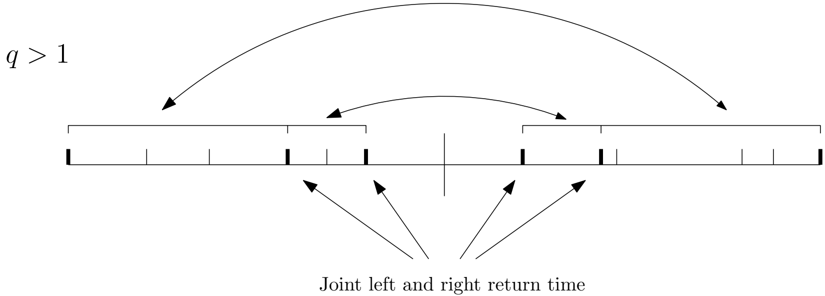

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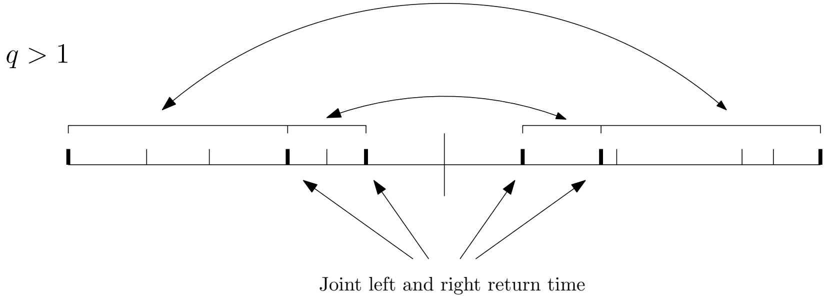

Odd cycles can only occur around the middle, and are sandwiched between the final joint return time.

Why the dependence on the parity of n?

Example $n = 4$:

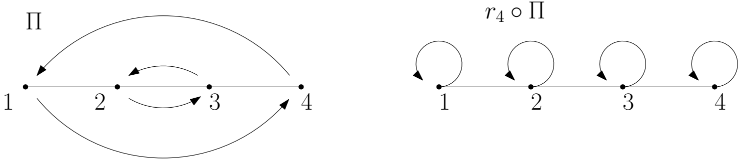

Figure: Candidates for images in $\Pi \in S_4$ that lead to fixed points in $r_4 \circ \Pi$

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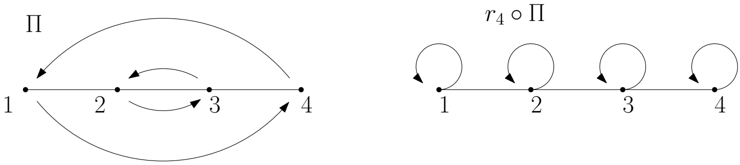

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Example $n = 5$:

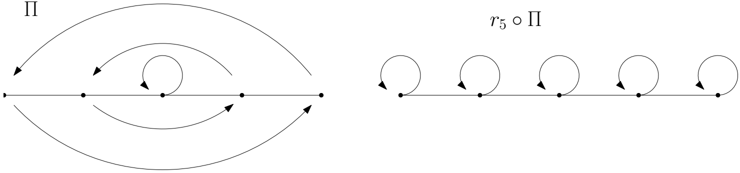

Figure: Candidates for images in $\Pi \in S_5$ that lead to fixed points in $r_5 \circ \Pi$

Open questions

- Recall $\lim_{n\to\infty} \mathbb{E} C_1(\Pi_{2n}), \lim_{n\to\infty} \mathbb{E} C_1(\Pi_{2n+1}) \to \frac{1}{2}$ as $q \downarrow 1$. Why? (At $q = 1$ this limit equals one.)

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- Apparently $\lim_{n\to\infty} \mathbb{E}C_1(\Pi_{2n}) < \frac{1}{2} < \lim_{n\to\infty} \mathbb{E}C_1(\Pi_{2n+1})$ for all $q > 1$.
  (We were not able to show it. Maybe someone better versed in q-hypergeometric functions can manage.)

Encore

How did we (=Fiona Skerman + TM + Teun Verstraaten) get interested?

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Limit laws for logic of random permutations.

Two logical languages of permutations

Two different (first order) languages of permutations:

theory of one bijection (TOOB)

and

theory of two total orders (TOTO)

names invented by Albert+Bouvel+Feray [JCTA, 2020]

TOOB

We are allowed to use the quantifiers $\forall, \exists$, variables $x, y, z, \dots$, the logical connectives $\wedge, \vee, \neg$, etc., brackets and the relation symbols $=, R$.

The variables range over $[n] := \{1, \dots, n\}$ and xRy just means that $\pi(x) = y$.

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- Non-example: the number of fixed points is even.
- Non-example: occurrence of the pattern 231.
(I.e. existence of $i_1 < i_2 < i_3$ with $\pi(i_3) < \pi(i_1) < \pi(i_2)$.)

TOTO

We are allowed to use the quantifiers $\forall, \exists$, variables $x, y, z, \dots$, the logical connectives $\wedge, \vee, \neg$, etc., brackets and the relation symbols $=, <_1, <_2$.

The variables range over $[n]$ and $x <_1 y$ just means that $x < y$ while $x <_2 y$ means that $\pi(x) < \pi(y)$.

TOTO examples

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- Non-example: existence of a fixed point ($\pi(i) = i$). [Albert-Bouvel-Feray]

Definition: zero-one/convergence law

For $(\Pi_n)_n$ a sequence of random permutations and $\mathcal{L} \in \{\mathsf{TOOB}, \mathsf{TOTO}\}$ we say Π_n satisfies the *zero-one law* if

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Results for TOOB

In 1989, Compton has already shown that when $q = 1$ (uniform distribution) the convergence law holds, but the zero-one law fails, for TOOB.

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(ii) If $q > 1$ then the **convergence law fails**.

Some words on the proof.

In TOOB we can only “see” the cycle structure.
We can exploit the results from earlier on the talk, combined with relatvely routine arguments from logic / random graphs.

Results for TOTO

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(The condition on q in **(ii)** can be improved.)

Some words on the proofs

Non-convergence when $q = 1$ was already shown by Foy+Woods 1990, using a variant of the "arithmetization" technique of Shelah+Spencer 1988.

For the case $q \to 1$ (but not very, very slowly), we "zoom" in on a small initial interval $\{1, \ldots, i\}$, so small that it behaves almost like the $q = 1$ case, and apply a construction similar to the Foy-Woods one there.

Thank you for your attention!