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Circle swimmers, gravitaxis, and the harmonic oscillator

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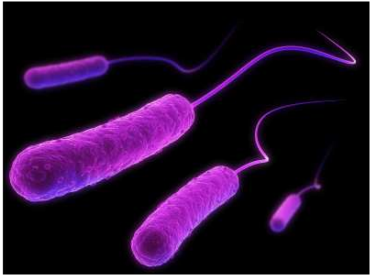
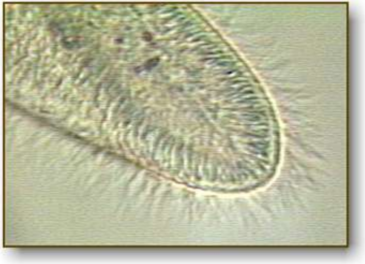
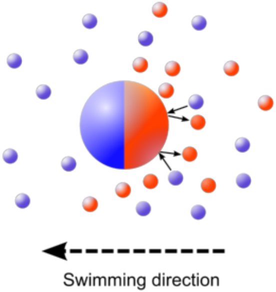
Self-propelled agents



Bacteria using flagella to swim



Paramecium uses cilia

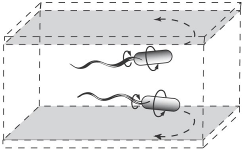


self-propelled Janus particle

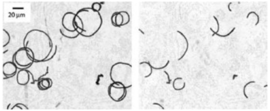
- Swimming mechanism by shape deformations or induced gradients in the fluid
- intrinsically far from equilibrium
- recent experimental progress to build artificial self-propelled particles
- plethora of collective phenomena (flocking, swarms, phase separation, trapping,...)
- mostly simulational studies
- lacking: complete characterization of single particle motion

Circle swimmers

Escherichia coli



DiLuzio

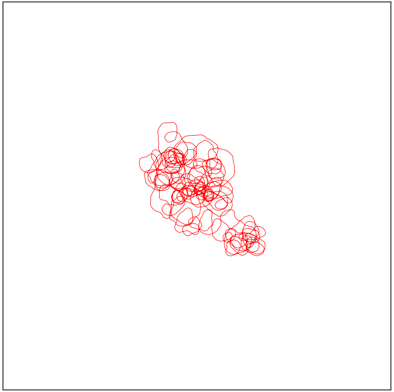


Lauga

- Chirality of flagellar motion
- Hydrodynamic coupling close to boundaries
  → circular motion
- angular drift velocity ω

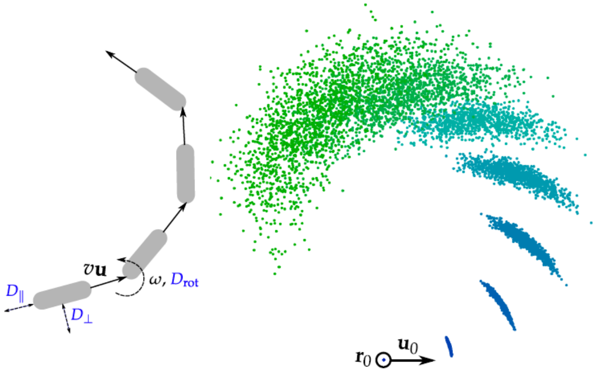
$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t) = \omega + \zeta(t)$$

$$\langle \zeta(t)\zeta(t')\rangle = 2D_{\mathrm{rot}}\delta(t-t')$$



Model set-up

Active Brownian Particle – circle swimmer



- Active propulsion with constant velocity v along the long axis $\mathbf{u}, |\mathbf{u}| = 1$
- Rotational diffusion D_{rot}
- Anisotropic translational diffusion $D_{\|}, D_{\perp}$
- possibly angular drift ω
- ignores microscopic origin of propulsion, effective description
- simplistic model encoding persistent random walk
  persistence length $\ell = v/D_{rot}$
  persistence time $\tau_{rot} = 1/D_{rot}$
  period $T = 2\pi/\omega$

Stochastic equations

Circle swimmer

angular drift

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t) = \omega + \zeta(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = v\,\mathbf{u}(t) + \eta(t) = v\begin{pmatrix}\cos\vartheta(t)\\ \sin\vartheta(t)\end{pmatrix} + \eta(t)$$

position   fixed velocity   orientation

$$\langle\zeta(t)\zeta(t')\rangle = 2D_{\mathrm{rot}}\delta(t-t'), \quad \langle\eta_i(t)\eta_j(t')\rangle = [2D_\perp\delta_{ij} + 2(D_\parallel - D_\perp)u_i(t)u_j(t)]\delta(t-t')$$

independent Gaussian white noise

- translational anisotropy $\Delta D = D_\parallel - D_\perp$, mean diffusion coefficient $\bar{D} = (D_\parallel + D_\perp)/2$
- characteristic length $a = \sqrt{3\bar{D}/D_{\mathrm{rot}}}/2$ replaces radius of the particle
- dimensionless parameters

anisotropy $\Delta D/\bar{D}$ Péclet number $\mathrm{Pe} = va/\bar{D}$ quality factor $M = \omega/2\pi D_{\mathrm{rot}}$

Circle swimmer – angular motion

angular diffusion with drift

- conditional probability $\mathbb{P}(\vartheta, t|\vartheta_0), t > 0$

$$\partial_t \mathbb{P} = D_{\text{rot}} \partial_\vartheta^2 \mathbb{P} - \omega \partial_\vartheta \mathbb{P}$$

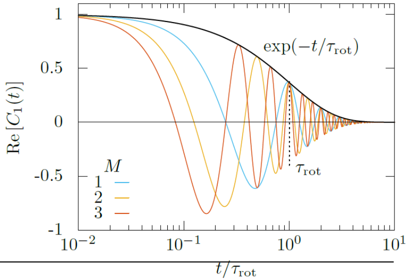
- 2π-periodic solution: **wrapped normal distribution with drift**

$$\mathbb{P}(\vartheta, t|\vartheta_0) = \frac{1}{\sqrt{4\pi D_{\text{rot}} t}} \sum_{n=-\infty}^{\infty} \exp\left\{ \frac{-(\vartheta - \vartheta_0 - \omega t + 2\pi n)^2}{4 D_{\text{rot}} t} \right\} = \sum_{\nu=-\infty}^{\infty} e^{-\nu^2 D_{\text{rot}} t} e^{i\nu(\vartheta - \vartheta_0 - \omega t)}$$

- angular correlation functions, $\nu, \mu \in \mathbb{Z}, t > 0$

$$\langle e^{i\nu\vartheta(t)} e^{-i\mu\vartheta(0)} \rangle = \int_0^{2\pi} \mathrm{d}\vartheta \int_0^{2\pi} \frac{\mathrm{d}\vartheta_0}{2\pi}\, e^{i\nu\vartheta} e^{-i\mu\vartheta_0} \mathbb{P}(\vartheta t|\vartheta_0)$$

$$\langle e^{i\nu\vartheta(t)} e^{-i\mu\vartheta(0)} \rangle = \delta_{\mu\nu} \exp(-\nu^2 D_{\text{rot}} t + i\nu\omega t)$$



Circle swimmer - mean-square displacement

pedestrian calculation

- increment

$$\Delta x(t) := x(t) - x(0) = \int_0^t [v \cos\vartheta(t') + \eta_x(t')]\mathrm{d}t' \qquad \Delta y(t) := y(t) - y(0) = \int_0^t [v \sin\vartheta(t') + \eta_y(t')]\mathrm{d}t'$$

- mean-square displacement

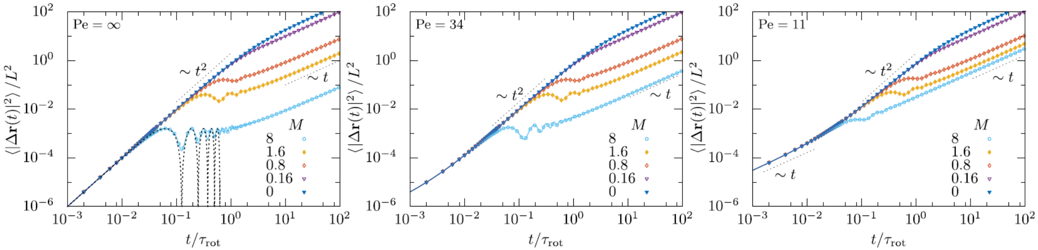
$$\langle \Delta x(t)^2\rangle + \langle \Delta y(t)^2\rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''[v^2 \underbrace{\langle \cos\vartheta(t')\cos\vartheta(t'') + \sin\vartheta(t')\sin\vartheta(t'')\rangle}_{\langle\cos[\vartheta(t'')-\vartheta(t')]\rangle = \exp(-D_\mathrm{rot}|t'-t''|)\cos[\omega(t'-t'')]} + \underbrace{\langle \eta_x(t')\eta_x(t'') + \eta_y(t')\eta_y(t'}_{4\bar{D}\delta(t'-t'')}$$

$$\langle|\Delta\mathbf{r}(t)|^2\rangle = 4\bar{D}t + \frac{2v^2}{(D_\mathrm{rot}^2 + \omega^2)}\Big[D_\mathrm{rot}^2(D_\mathrm{rot}t - 1) + \omega^2(D_\mathrm{rot}t + 1) + e^{-D_\mathrm{rot}t}[(D_\mathrm{rot}^2 - \omega^2)\cos(\omega t) - 2D_\mathrm{rot}\omega\sin(\omega t)]\Big] \xrightarrow{t\to\infty} 4D_\mathrm{eff}t$$

rotational time $\tau_\mathrm{rot} = 1/D_\mathrm{rot}$, **effective diffusion coefficient** $D_\mathrm{eff} = \bar{D} + v^2 D_\mathrm{rot}/[2(D_\mathrm{rot}^2 + \omega^2)]$

- higher moments $\langle|\Delta\mathbf{r}(t)|^n\rangle$ become tedious

Circle swimmer – mean-square displacment

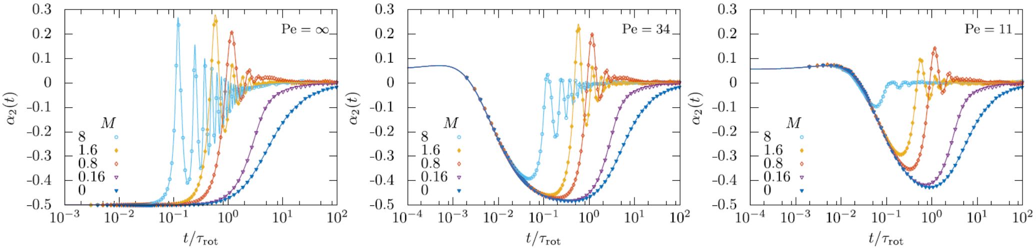


- oscillatory behavior for larger quality factor $M \gtrsim 1$
- without noise $D_{\mathrm{rot}} = \bar{D} = 0$ deterministic motion $|\Delta\mathbf{r}(t)| = 2R|\sin(\omega t/2)|$ with radius $R = v/\omega$
- crossover to long-time diffusion $D_{\mathrm{eff}} = \bar{D} + v^2 D_{\mathrm{rot}}/[2(D_{\mathrm{rot}}^2 + \omega^2)]$ ✓

$$D_{\mathrm{eff}}/\bar{D} = 1 + \frac{2}{3}\frac{\mathrm{Pe}^2}{1 + 4\pi^2 M^2}$$

→ **enhanced diffusion** but suppressed by **angular drift**

Circle swimmer – non-Gaussian parameter



- lengthy explicit expression for

$$\alpha_2(t) = \frac{1}{2}\frac{\langle|\Delta\mathbf{r}|^4\rangle}{\langle|\Delta\mathbf{r}(t)|^2\rangle^2} - 1$$

- $\alpha_2(t) \to 0$ for $t \to \infty$ by central limit theorem
- without noise $D_{\rm rot} = \bar{D} = 0$ deterministic motion $|\Delta\mathbf{r}(t)| = 2R|\sin(\omega t/2)|$ ➔ $\alpha_2(t) = -1/2$.
- initial value with noise $\alpha_2(t \to 0) = \Delta D^2/8\bar{D}^2$
- oscillatory behavior for larger quality factor $M \gtrsim 1$

Fokker-Planck equation

conditional probability density $\mathbb{P}(\mathbf{r},\vartheta,t|\vartheta_0)$ (Green function)

Perrin equation (Markov process)

$$\partial_t \mathbb{P} = D_{\mathrm{rot}}\partial_\vartheta^2 \mathbb{P} - \omega\partial_\vartheta\mathbb{P} - v\mathbf{u}\cdot(\partial_{\mathbf{r}}\mathbb{P}) + \partial_{\mathbf{r}}\cdot\left[D_{\|}(\partial_{\mathbf{r}}\mathbb{P}) - \Delta D(\mathbb{I}-\mathbf{u}\mathbf{u})\cdot(\partial_{\mathbf{r}}\mathbb{P})\right]$$

orientational diffusion angular drift active propulsion anisotropic diffusion $\Delta D = D_{\|} - D_{\perp}$

- Fokker-Planck equation for non-equilibrium dynamics
- coupling between orientational diffusion, angular drift, active propulsion, and translation
- spatial Fourier transform $\tilde{\mathbb{P}}(\mathbf{k},\vartheta,t|\vartheta_0) = \int \mathrm{d}^2 r \exp(-i\mathbf{k}\cdot\mathbf{r})\mathbb{P}(\mathbf{r},\vartheta,t|\vartheta_0)$

$$\partial_t\tilde{\mathbb{P}} = D_{\mathrm{rot}}\partial_\vartheta^2\tilde{\mathbb{P}} - \omega\partial_\vartheta\tilde{\mathbb{P}} - iv\mathbf{u}\cdot\mathbf{k}\tilde{\mathbb{P}} - [D_\perp k^2 + \Delta D(\mathbf{u}\cdot\mathbf{k})^2]\tilde{\mathbb{P}}$$

- solution for intermediate scattering function

$$F(\mathbf{k},t) = \langle \exp\left(-i\mathbf{k}\cdot\Delta\mathbf{r}(t)\right)\rangle = \int_0^{2\pi}\mathrm{d}\vartheta\int_0^{2\pi}\frac{\mathrm{d}\vartheta_0}{2\pi}\,\tilde{\mathbb{P}}(\mathbf{k},\vartheta,t|\vartheta_0)$$

Christina Kurzthaler *et al*, Soft Matter (2017)

Generalized Mathieu functions

Separation ansatz

- choose coordinates $\mathbf{k}$ in x-direction

$$\partial_t \tilde{\mathbb{P}} = \left[D_{\mathrm{rot}} \partial_\vartheta^2 - \omega \partial_\vartheta - ivk \cos\vartheta - \left(D_\perp k^2 + \Delta D\, k^2 \cos^2\vartheta \right) \right] \tilde{\mathbb{P}}$$

- separation ansatz $\exp(-\lambda t) z(\vartheta)$

eigenvalue problem

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} - 2\pi M \frac{\mathrm{d}}{\mathrm{d}\vartheta} - \frac{ivk}{D_{\mathrm{rot}}} \cos\vartheta - \frac{\Delta D\, k^2}{D_{\mathrm{rot}}} \cos^2\vartheta - \frac{D_\perp k^2}{D_{\mathrm{rot}}} + \frac{\lambda}{D_{\mathrm{rot}}} \right) z(\vartheta) = 0$$

- substitution $x = \vartheta/2$, **non-Hermitian** eigenvalue problem $Lz(x) = az(x)$

Sturm-Liouville operator

$$L = L(q, c, M) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2q\cos(2x) - c^2\cos^2(2x) - 4\pi M \frac{\mathrm{d}}{\mathrm{d}x}$$

deformation parameters $q = 2ivk/D_{\mathrm{rot}}$, $c^2 = 4\Delta D k^2/D_{\mathrm{rot}}$, $M = \omega/2\pi D_{\mathrm{rot}}$
eigenvalue $a = 4(\lambda - D_\perp k^2)/D_{\mathrm{rot}}$

Generalized Mathieu functions

Summary of properties

- scalar product for π-periodic functions

$$\langle \varphi | \psi \rangle = \frac{1}{\pi} \int_0^\pi \varphi(x)^* \psi(x) \mathrm{d}x$$

adjoint operator $$L^+ = L^+(q, c, M) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2q^* \cos(2x) - c^2 \cos^2(2x) + 4\pi M \frac{\mathrm{d}}{\mathrm{d}x}$$

- left and right eigenfunctions

$$L r_m = a_m r_m, \qquad L^+ l_m = a_m^* l_m, \qquad \langle l_m | r_n \rangle = \delta_{mn}$$

→ $l_m(q, c, M, x)^* = r_n(q, c, M, -x)$

Generalized Mathieu functions – cont'd

- Define *your own eigenfunctions* (call them generalized Mathieu functions) π-periodic for x deformation of complex exponentials functions $\exp(2i\pi n x)$

$$\textbf{Fourier expansion} \qquad \mathrm{ee}_{2n}(q,c,M,x) = \sum_{m=-\infty}^{\infty} A_{2m}^{2n} e^{2mix} \qquad n \in \mathbb{Z}$$

- **orthogonality and normalization**

$$\int_0^\pi \mathrm{ee}_{2m}(q,c,M,x)\mathrm{ee}_{2n}(q,c,M,-x)\mathrm{d}x = \pi\delta_{mn} \qquad n,m \in \mathbb{Z}$$

- **completeness**

$$\pi\text{-periodic} \qquad f(x) = \sum_{n=-\infty}^{\infty} \gamma_{2n}\, \mathrm{ee}_{2n}(q,c,M,x)$$

$$\text{Fourier coefficients} \qquad \gamma_{2n} = \frac{1}{\pi}\int_0^\pi f(x)\, \mathrm{ee}_{2n}(q,c,M,-x)\, \mathrm{d}x$$

Anisotropic self-propulsion

- Solution Fokker-Planck equation

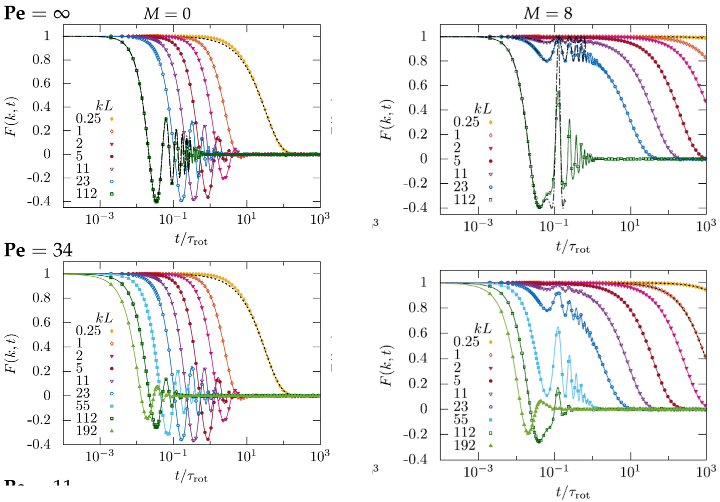
$$\tilde{\mathbb{P}}(k,\vartheta,t|\vartheta_0) = \frac{e^{-k^2 D_\perp t}}{2\pi} \sum_{n=-\infty}^{\infty} e^{-a_{2n}(q) D_{\mathrm{rot}} t/4} \mathrm{ee}_{2n}(q,c,M,\vartheta/2) \mathrm{ee}_{2n}(q,c,M,-\vartheta_0/2)$$

- **intermediate scattering** function (ISF) after averaging and marginalizing

$$F(k,t) = \frac{e^{-k^2 D_\perp t}}{4\pi^2} \sum_{n=-\infty}^{\infty} e^{-a_{2n}(q) D_{\mathrm{rot}} t/4} \left[\int_0^{2\pi} \mathrm{d}\vartheta \, \mathrm{ee}(q,c,M,\vartheta/2) \right]^2$$

C Kurzthaler *et al*, Soft Matter (2017)

Circle swimmer – intermediate scattering function



- ISF in 2D for isotropic motion

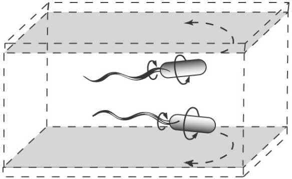
$$F(k,t) = \langle J_0(k|\Delta\mathbf{r}(t)|)\rangle$$

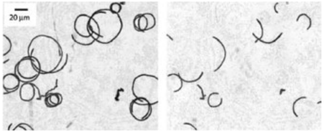
- chiral swimming pattern leads to **oscillations** around a plateau
- **ideal motion**
  $|\Delta\mathbf{r}(t)| = 2R|\sin(\omega t/2)|$

→ $F(k,t) = J_0(2kR|\sin(\omega t/2)|)$
→ oscillations between 1 and $J_0(2kR)$

Circle swimmers

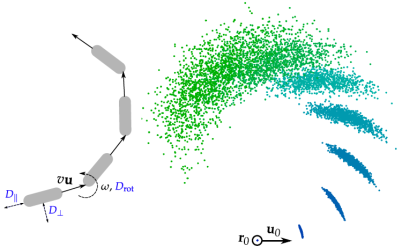
Escherichia coli





DiLuzio Lauga

- Chirality of flagellar motion
- Hydrodynamic coupling close to boundaries
  → circular motion
- angular drift velocity ω
- quality factor $M = \frac{\omega/2\pi}{D_{\rm rot}}$
- exact solution of intermediate scattering function in terms of generalizations of Mathieu functions



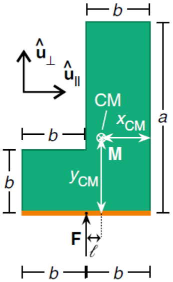
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = v\mathbf{u}(t)$$

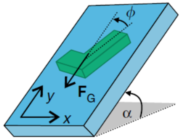
$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t) = \omega + \zeta(t)$$

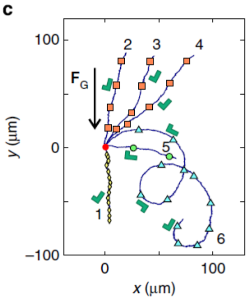
$$\langle \zeta(t)\zeta(t')\rangle = 2D_{\rm rot}\delta(t-t')$$

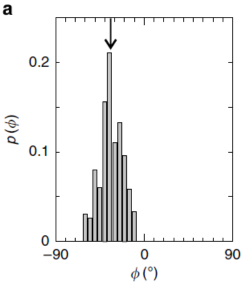
Kurzthaler *et al* Soft Matter (2017)

Gravitaxis









- active L-shaped particles are **circle swimmers**
- can move upwards against **gravity**
- transition from a **locked** to **periodic** phase

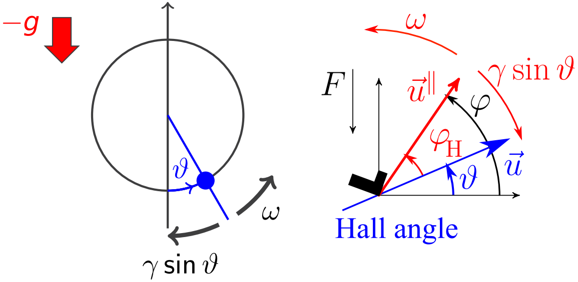
ten Hagen *et al*, Nature Communications (2014)

Minimal model for gravitaxis

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = v\mathbf{u}(t) := v\begin{pmatrix}\cos\vartheta(t)\\ \sin\vartheta(t)\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t) = \omega - \gamma\sin\vartheta(t) + \zeta(t)$$

$$\langle\zeta(t)\zeta(t')\rangle = 2D_{\mathrm{rot}}\delta(t-t')$$



driven pendulum

- simplified model of ten Hagen *et al*, Nature Communications (2014)
- maps to overdamped noisy driven pendulum / tilted washboard ... but geometry is rotated
- without noise $D_{\mathrm{rot}} = 0$ and drive $\omega = 0$ the active particle moves **horizontally**

Fold bifurcation

Classical motion (without noise)

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta = \omega - \gamma \sin\vartheta$$

- **locked phase**: stable fixed point, locked angle $\vartheta_* \in [0, \pi/2]$ for $\gamma \geq \omega$ (unstable fixed point at $\pi - \vartheta_*$)

$$\sin\vartheta_* = \omega/\gamma \qquad \text{constant } \textbf{drift velocity} \qquad \mathbf{vu} = v\begin{pmatrix}\cos\vartheta_* \\ \sin\vartheta_*\end{pmatrix} = v\begin{pmatrix}\sqrt{1-\omega^2/\gamma^2} \\ \omega/\gamma\end{pmatrix}$$

- **periodic phase**: no fixed point

$$\textbf{period} \qquad T = \int_0^{2\pi}\frac{\mathrm{d}\vartheta}{\omega - \gamma\sin\vartheta} = \frac{2\pi}{\sqrt{\omega^2-\gamma^2}} \qquad \text{diverges as} \qquad \gamma \uparrow \omega$$

$$\textbf{average drift} \qquad \frac{1}{T}\int_0^T v\begin{pmatrix}\cos\vartheta(t) \\ \sin\vartheta(t)\end{pmatrix}\mathrm{d}t = \frac{v}{T}\int_0^{2\pi}\begin{pmatrix}\cos\vartheta \\ \sin\vartheta\end{pmatrix}\frac{\mathrm{d}\vartheta}{\omega-\gamma\sin\vartheta} = v\begin{pmatrix}0 \\ \omega/\gamma - \sqrt{(\omega/\gamma)^2-1}\end{pmatrix}$$

- **bifurcation** at $\gamma_c = \omega$

Tilted-washboard potential

- motion in effective potential

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta = -\frac{D_{\text{rot}}}{k_B T}\partial_\vartheta U + \zeta(t)$$

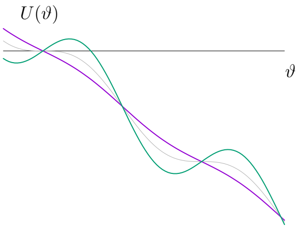
$$\frac{D_{\text{rot}}}{k_B T}U(\vartheta) = -\omega\vartheta - \gamma\cos\vartheta$$

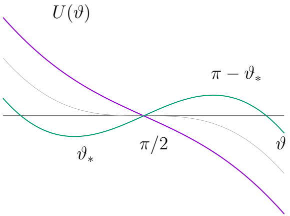
non-periodic due to angular drift → **nonequilibrium**
local **barrier** for $\gamma > \omega$ → crossing by **fluctuations**
no barrier for $0 < \gamma < \omega$

- harmonic approximation close in the locked phase

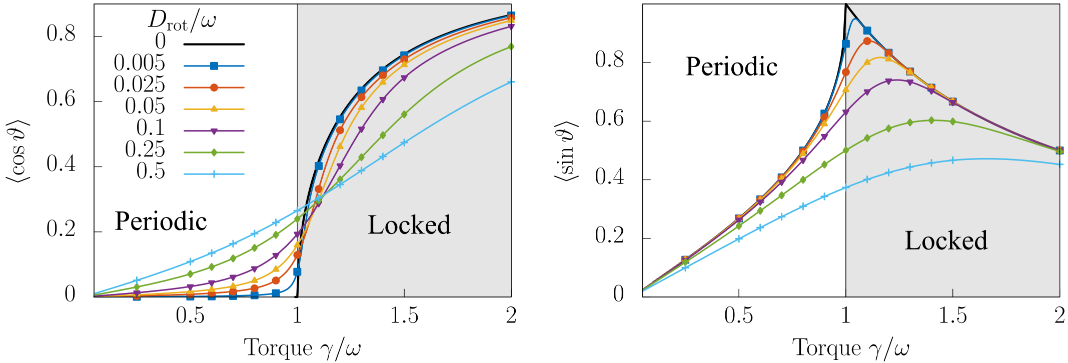
$$\frac{D_{\text{rot}}}{k_B T}U(\vartheta) = \frac{D_{\text{rot}}}{k_B T}U(\vartheta_*) + \frac{(\vartheta-\vartheta_*)^2}{2}\sqrt{\gamma^2-\omega^2} + O(\vartheta-\vartheta_*)^3$$

becomes **soft** upon approaching the bifurcation $\gamma \downarrow \omega$





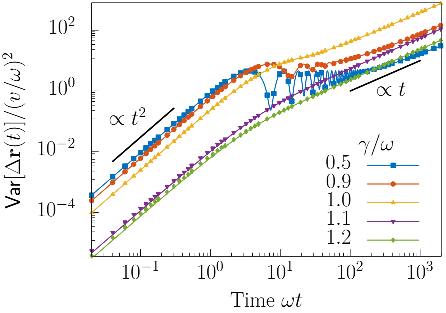
Mean drift



- analytic solution for stationary state $p^{st}(\vartheta)$ known
  → calculate numerically the mean drift and compare to simulation
- fluctuations smooths out the transition
- net horizontal motion in the periodic phase only due to fluctuations

Risken, *The Fokker-Planck equation*

Fluctuations

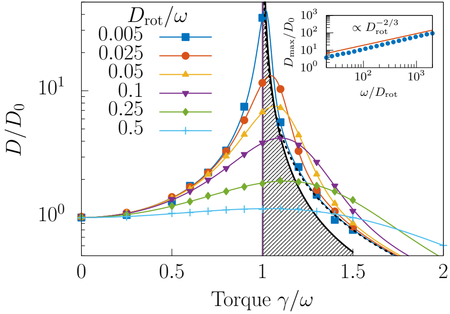


variance $\text{Var}[\Delta\mathbf{r}(t)] := \langle[\Delta\mathbf{r}(t) - \langle\Delta\mathbf{r}(t)\rangle]^2\rangle$

with increment $\Delta\mathbf{r}(t) := \mathbf{r}(t) - \mathbf{r}(0)$

- results for **stochastic simulations**
- initially $\text{Var}[\Delta\mathbf{r}(t)]^2\rangle$ grows $\propto t^2$
- prefactor depends drastically on the torque γ
- oscillations below the classical bifurcation
- long-time behavior diffusive $\propto t$

Diffusion coefficient



diffusion coefficient $D := \lim_{t\to\infty} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}[\Delta\mathbf{r}(t)]$

- normalize by free circle swimmer $D_0 = v^2 D_{\rm rot}/[2(D_{\rm rot}^2+\omega^2)] \propto D_{\rm rot}$ as $D_{\rm rot} \to 0$
- **resonance** emerging close to bifurcation as $D_{\rm rot} \downarrow 0$
- maximal relative diffusivity scales $\propto D_{\rm rot}^{-2/3}$ on parametric curve
- **goal:** rationalize the resonance and the scaling **analytically!**

Fokker-Planck equation

conditional probability density $\mathbb{P}(\mathbf{r}, \vartheta, t|\vartheta_0)$ (Green function)

Fokker-Planck equation (Markov process)

$$\partial_t \mathbb{P} = D_{\text{rot}} \partial_\vartheta^2 \mathbb{P} - \partial_\vartheta \left[(\omega - \gamma \sin\vartheta) \mathbb{P} \right] - v\mathbf{u} \cdot (\partial_\mathbf{r} \mathbb{P})$$

orientational diffusion angular drift active propulsion

- spatial Fourier transform $\tilde{\mathbb{P}}(\mathbf{k}, \vartheta, t|\vartheta_0) = \int \mathrm{d}^2 r \exp(-i\mathbf{k} \cdot \mathbf{r}) \mathbb{P}(\mathbf{r}, \vartheta, t|\vartheta_0)$

$$\partial_t \tilde{\mathbb{P}} = D_{\text{rot}} \partial_\vartheta^2 \tilde{\mathbb{P}} - \partial_\vartheta [\omega - \gamma \sin\vartheta] \tilde{\mathbb{P}} - iv\mathbf{u} \cdot \mathbf{k} \tilde{\mathbb{P}} =: (\mathcal{L} + \delta\mathcal{L}_\mathbf{k}) \tilde{\mathbb{P}}$$

- solution for intermediate scattering function

$$F(\mathbf{k}, t) = \langle \exp(-i\mathbf{k} \cdot \Delta\mathbf{r}(t)) \rangle = \int_0^{2\pi} \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\vartheta_0 \, \tilde{\mathbb{P}}(\mathbf{k}, \vartheta, t|\vartheta_0) p^{\text{st}}(\vartheta_0)$$

average with respect to **stationary distribution**

Eigenfunctions

Matrix representation

- Hilbert space of periodic square-integrable functions $f(\vartheta) \in L^2[0, 2\pi]$

scalar product $$\langle f|g\rangle = \int_0^{2\pi} \mathrm{d}\vartheta\, f(\vartheta)^* g(\vartheta)$$

- make isomorphism manifest by generalized eigenstates $f(\vartheta) = \langle \vartheta | f\rangle$
  $\{|n\rangle : n \in \mathbb{Z}\}$ ONB in $\mathcal{H}$ with real-space representation $\langle \vartheta | n\rangle = \exp(in\vartheta)/\sqrt{2\pi}$.
  non-Hermitian matrix representation

$$[\mathcal{L}]_{mn} = \langle m|\mathcal{L}n\rangle = \int_0^{2\pi} \frac{\mathrm{d}\vartheta}{2\pi} e^{-im\vartheta} \mathcal{L} e^{in\vartheta} = (-D_{\mathrm{rot}} m^2 - im\omega)\delta_{mn} + \frac{\gamma}{2} m(\delta_{m,n+1} - \delta_{m,n-1})$$

$$[\delta\mathcal{L}_{\mathbf{k}}]_{mn} = \langle m|\delta\mathcal{L}_{\mathbf{k}} n\rangle = \int_0^{2\pi} \frac{\mathrm{d}\vartheta}{2\pi} e^{-im\vartheta} \delta\mathcal{L}_{\mathbf{k}} e^{in\vartheta} = -\frac{ik_x v}{2}(\delta_{m,n+1} + \delta_{m,n-1}) - \frac{k_y v}{2}(\delta_{m,n+1} - \delta_{m,n-1})$$

→ **tridiagonal matrix** easy to diagonalize

- right and left **eigenstates**

$$\mathcal{L}|r_\lambda\rangle = \lambda|r_\lambda\rangle \qquad \mathcal{L}^\dagger|l_\lambda\rangle = \lambda|l_\lambda\rangle$$

Formal solution

- stationary state $p^{\mathrm{st}}(\vartheta) = \langle \vartheta | r_0 \rangle$ is eigenstate with eigenvalue 0, $\langle \vartheta | l_0 \rangle = 1$

$$\rightarrow \qquad F(\mathbf{k},t) = \int_0^{2\pi} \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\vartheta_0 \langle l_0 | \vartheta \rangle \langle \vartheta | e^{(\mathcal{L}+\delta\mathcal{L}_{\mathbf{k}})t} \vartheta_0 \rangle \langle \vartheta_0 | r_0 \rangle = \langle l_0 | e^{(\mathcal{L}+\delta\mathcal{L}_{\mathbf{k}})t} r_0 \rangle$$

- Expansion in powers of the wave vector yield moments

$$F(\mathbf{k},t) = \langle \exp[.i\mathbf{k}\cdot\Delta\mathbf{r}(t)] \rangle = 1 - i\mathbf{k}\cdot\langle \Delta\mathbf{r}(t) \rangle - \frac{1}{2}\langle [\mathbf{k}\cdot\Delta\mathbf{r}(t)]^2 \rangle + O(|\mathbf{k}|^3)$$

- time-dependent **perturbation theory** in $\delta\mathcal{L}_{\mathbf{k}}$

Dyson representation $\qquad e^{(\mathcal{L}+\delta\mathcal{L}_{\mathbf{k}})t} = e^{\mathcal{L}t} + \int_0^t \mathrm{d}s\, e^{\mathcal{L}(t-s)} \delta\mathcal{L}_{\mathbf{k}} e^{(\mathcal{L}+\delta\mathcal{L}_{\mathbf{k}})s}$

$$= e^{\mathcal{L}t} + \int_0^t \mathrm{d}s\, e^{\mathcal{L}(t-s)} \delta\mathcal{L}_{\mathbf{k}} e^{\mathcal{L}s} + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, e^{\mathcal{L}(t-s)} \delta\mathcal{L}_{\mathbf{k}} e^{\mathcal{L}(s-u)} \delta\mathcal{L}_{\mathbf{k}} e^{\mathcal{L}u} + O(\delta\mathcal{L}_{\mathbf{k}})^3$$

Born series

...collecting results

- with $e^{\mathcal{L}t}|r_0\rangle = |r_0\rangle$, $\langle e^{\mathcal{L}^\dagger t} l_0| = \langle l_0|$

$$F(\mathbf{k},t) = 1 + \int_0^t \mathrm{d}s\, \langle l_0|e^{\mathcal{L}(t-s)}\delta\mathcal{L}_\mathbf{k} e^{\mathcal{L}s} r_0\rangle + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \langle l_0|e^{\mathcal{L}(t-s)}\delta\mathcal{L}_\mathbf{k} e^{\mathcal{L}_\mathbf{k}(s-u)}\delta\mathcal{L}_\mathbf{k} e^{\mathcal{L}u} r_0\rangle + O(|\mathbf{k}|^3)$$

$$= 1 + t\langle l_0|\delta\mathcal{L}_\mathbf{k} r_0\rangle + \sum_\lambda \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, \langle l_0|\delta\mathcal{L}_\mathbf{k} e^{\mathcal{L}(s-u)} r_\lambda\rangle\langle l_\lambda|\delta\mathcal{L}_\mathbf{k} r_0\rangle + O(|\mathbf{k}|^3)$$

$$= 1 + t\langle l_0|\delta\mathcal{L}_\mathbf{k} r_0\rangle + \sum_\lambda \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, e^{-\lambda(s-u)} \langle l_0|\delta\mathcal{L}_\mathbf{k} r_\lambda\rangle\langle l_\lambda|\delta\mathcal{L}_\mathbf{k} r_0\rangle + O(|\mathbf{k}|^3)$$

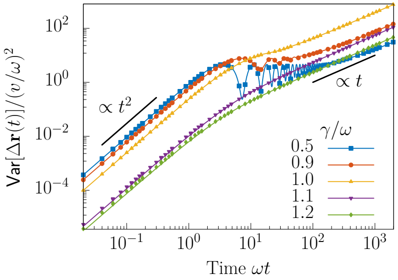
$$= 1 + t\langle l_0|\delta\mathcal{L}_\mathbf{k} r_0\rangle + \sum_\lambda \frac{e^{-\lambda t} + \lambda t - 1}{\lambda^2} \langle l_0|\delta\mathcal{L}_\mathbf{k} r_\lambda\rangle\langle l_\lambda|\delta\mathcal{L}_\mathbf{k} r_0\rangle + O(|\mathbf{k}|^3)$$

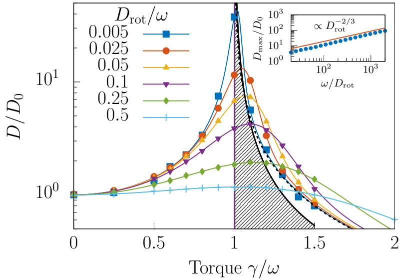
- mean drift and variance along $\mathbf{n} = \mathbf{k}/k$

$$\mathbf{n} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\langle \Delta\mathbf{r}(t)\rangle = \frac{i}{k}\langle l_0|\delta\mathcal{L}_\mathbf{k} r_0\rangle$$

$$\mathrm{Var}[\mathbf{n}\cdot\Delta\mathbf{r}(t)] = \frac{2}{k^2}\sum_{\lambda\neq 0} \frac{1 - \lambda t - e^{-\lambda t}}{\lambda^2}\langle l_0|\delta\mathcal{L}_\mathbf{k} r_\lambda\rangle\langle l_\lambda|\delta\mathcal{L}_\mathbf{k} r_0\rangle$$

Resonance





- **analytic theory** describes all data
- diffusion coefficient for motion along $\mathbf{n} = \mathbf{k}/k$

$$D_{\mathbf{n}} := \lim_{t\to\infty} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}[\mathbf{n}\cdot\Delta\mathbf{r}(t)] = \frac{-1}{k^2}\sum_{\lambda\neq 0}\frac{1}{\lambda}\langle l_0|\delta\mathcal{L}_{\mathbf{k}} r_\lambda\rangle\langle l_\lambda|\delta\mathcal{L}_{\mathbf{k}} r_0\rangle$$

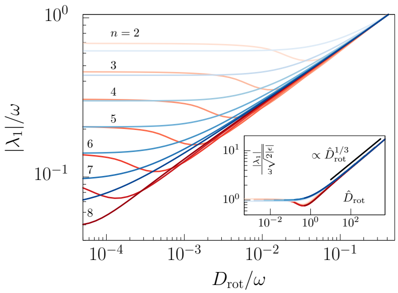
- discarded terms due to translational diffusion and noise can be readily **included**, only $\delta\mathcal{L}_{\mathbf{k}}$ changes
- How to explain the **resonance**?

Eigenvalues

- **all** **eigenvalues** go to **zero** as $D_{\mathrm{rot}} \to 0$ upon approaching the resonance $\varepsilon = (\gamma - \omega)/\omega \to 0$
- all curves have the same shape
  → **scaling behavior** with reduced rotational diffusion coefficient $\hat{D}_{\mathrm{rot}} := |\varepsilon|^{-3/2} D_{\mathrm{rot}}/\omega$

$$\lambda_n/\omega = \sqrt{2|\varepsilon|}\Lambda_{n,\pm}(\hat{D}_{\mathrm{rot}})$$

- close to resonance in locked phase
  $\lambda_n \propto n$ **harmonic potential**
  → $\Lambda_{+,n}(\hat{D}_{\mathrm{rot}}) \to n$ for $\hat{D}_{\mathrm{rot}} \to 0$



separation parameter $\varepsilon = (\gamma - \omega)/\omega = \pm 10^{-n/3}$

Harmonic approximation

- linearize around fixed point ϑ_* for small **separation parameter** $\varepsilon = (\gamma - \omega)/\omega \ll 1$

$$\dot{\vartheta}(t) = -\frac{1}{\tau}[\vartheta(t) - \vartheta_*] + \zeta(t), \qquad \langle \zeta(t)\zeta(t')\rangle = 2D_{\text{rot}}\delta(t-t')$$

relaxation rate $$\frac{1}{\tau} = \sqrt{\gamma^2 - \omega^2} \propto \omega\sqrt{2\varepsilon} \qquad \text{for } \varepsilon \to 0^+$$

- **eigenvalues** $\lambda_n = n/\tau, n \in \mathbb{N}_0$ approach zero
- **perturbing operator** upon linearization in harmonic approximation

$$\delta\mathcal{L}_{\mathbf{k}} = iv\mathbf{u}\cdot\mathbf{k} = -ik_x v\cos\vartheta_* - ik_y v\sin\vartheta_* + (ik_x v\sin\vartheta_* - ik_y v\cos\vartheta_*)(\vartheta - \vartheta_*) + O(..)$$

→ drift velocity approaches classical value

- relevant transition matrix matrix elements couple only to excited state

$$\langle l_0|\delta\mathcal{L}_{\mathbf{k}} r_\lambda\rangle = \langle l_\lambda|\delta\mathcal{L}_{\mathbf{k}} r_0\rangle = iv\sqrt{D_{\text{rot}}\tau}(k_x\sin\vartheta_* - k_y\cos\vartheta_*)\delta_{\lambda,1}.$$

→ only a single term contributes

Harmonic approximation

- ...within harmonic approximation

$$\mathrm{Var}[\mathbf{n}\cdot\Delta\mathbf{r}(t)] = 2\left(\frac{t}{\tau} - 1 + e^{-t/\tau}\right)(v\tau)^2 D_{\mathrm{rot}}\tau(n_x \sin\vartheta_* - n_y\cos\vartheta_*)^2$$

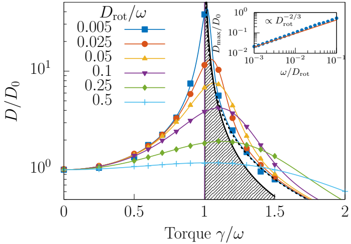
$$D_{\mathbf{n}} = (v\tau)^2 D_{\mathrm{rot}}(n_x\sin\vartheta_* - n_y\cos\vartheta_*)^2$$

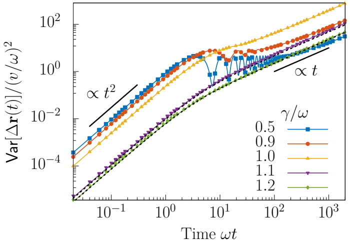
- picture should hold if relaxation rate $1/\tau$ large to **Kramer's escape rate** $\propto \exp(-\Delta U/k_B T)$

large barrier $1 \ll \dfrac{\Delta U}{k_B T} = \dfrac{4\sqrt{2}}{3}\dfrac{\omega}{D_{\mathrm{rot}}}\varepsilon^{3/2} + O(\dots)$

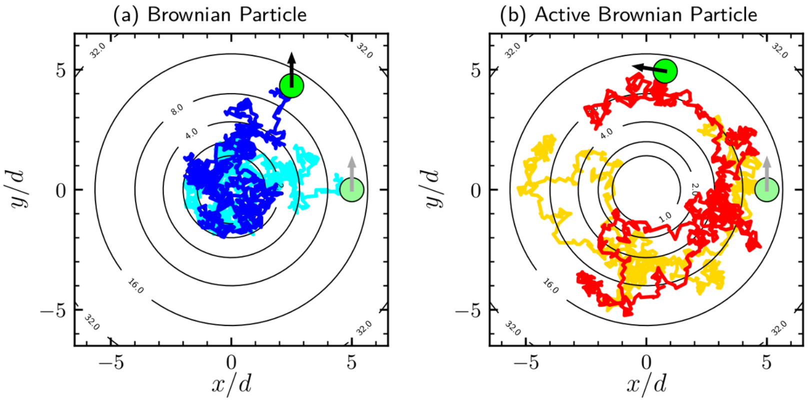
define **reduced rotational diffusion coefficient**
$\hat{D}_{\mathrm{rot}} := |\varepsilon|^{-3/2} D_{\mathrm{rot}}/\omega$
→ harmonic picture should hold for $\varepsilon > 0$ and $\hat{D}_{\mathrm{rot}} \ll 1$

maximal enhancement $D_{\mathrm{max}}/D_0 \propto |\varepsilon|^{-1} \propto D_{\mathrm{rot}}^{-2/3}$





Active Brownian Particle – harmonic well



- passive particle is most likely in the center, Boltzmann distribution $\propto \exp(-U(\mathbf{r})/k_B T)$
- **active particle** is off-center, non-equilibrium process

Stochastic differential equations

Active Brownian particle

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = -\mu k \mathbf{r} + v\,\mathbf{u} + \eta(t), \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \zeta(t)$$

harmonic restoring force

fixed velocity

orientation $\mathbf{u} = (\cos\vartheta, \sin\vartheta)$

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t'), \qquad \langle\zeta(t)\zeta(t') = 2D_{\mathrm{rot}}\delta(t-t')$$

independent Gaussian white noise

- Einstein relation $D = \mu k_B T$ defines temperature of the bath
- thermal oscillator length $d = \sqrt{k_B T/k}$, trap relaxation time $\tau = 1/\mu k$
- dimensionless parameters

rotationality $D_{\mathrm{rot}}\tau$ Péclet number $\mathrm{Pe} = vd/D$

Fokker-Planck equation

conditional probability density $\mathbb{P}(\mathbf{r}, \vartheta, t | \mathbf{r}_0, \vartheta_0)$ (Green function)

Fokker-Planck equation (Markov process)

$$\partial_t \mathbb{P} = \Omega \mathbb{P} := \boldsymbol{\nabla} \cdot (\mu k \mathbf{r} \mathbb{P}) + D \boldsymbol{\nabla}^2 \mathbb{P} + D_{\text{rot}} \partial_\vartheta^2 \mathbb{P} - v \mathbf{u} \cdot \boldsymbol{\nabla} \mathbb{P}$$

passive particle in harmonic well — orientational diffusion — active propulsion

- Fokker-Planck equation for non-equilibrium dynamics
- coupling between translational motion and orientational diffusion via active propulsion $\mathbf{u} = (\cos \vartheta, \sin \vartheta)$

steady state solution by Malakar *et al.*, PRE **101**, 022610 (2020)

Solution strategy

- formal solution $\mathbb{P}(\mathbf{r}, \vartheta, t | \mathbf{r}_0, \vartheta_0) = e^{\Omega t} \delta(\mathbf{r} - \mathbf{r}_0) \delta(\vartheta - \vartheta_0)$
- Boltzmann equilibrium without self-propulsion

$$p^{\text{eq}}(\mathbf{r}, \vartheta) = \frac{\exp(-r^2/2d^2)}{4\pi d^2}, \qquad \int \mathrm{d}\mathbf{r} \int_0^{2\pi} \mathrm{d}\vartheta \; p^{\text{eq}}(\mathbf{r}, \vartheta) = 1$$

  thermal oscillator length $d = \sqrt{k_B T / k}$
- splitting off the equilibrium reference state

$$\Omega[\psi(\mathbf{r}, \vartheta) p^{\text{eq}}(\mathbf{r}, \vartheta)] =: p^{\text{eq}}(\mathbf{r}, \vartheta) \mathcal{L} \psi(\mathbf{r}, \vartheta)$$

- decompose $\mathcal{L} = \mathcal{L}_0 + \text{Pe} \mathcal{L}_1$ with Péclet number $\text{Pe} := vd/D$
  in polar coordinates $\mathbf{r} = r(\cos\varphi, \sin\varphi)$

$$\mathcal{L}_0 \psi = \frac{1}{\tau} \left[\frac{d^2}{r} \partial_r (r \partial_r \psi) + \frac{d^2}{r^2} \partial_\varphi^2 \psi + D_{\text{rot}} \tau \partial_\vartheta^2 \psi - r \partial_r \psi \right]$$

$$\mathcal{L}_1 \psi = \frac{d}{\tau} \left[-\cos(\chi) \partial_r \psi - \frac{1}{r} \sin(\chi) \partial_\varphi \psi + \frac{r}{d^2} \cos(\chi) \psi \right]$$

  with relative angle $\chi = \angle(\mathbf{u}, \mathbf{r}) = \vartheta - \varphi$

Hilbert space formulation

- Kubo scalar product

$$\langle\phi|\psi\rangle := \int \mathrm{d}\mathbf{r} \int_0^{2\pi} \mathrm{d}\vartheta\, p^{\mathrm{eq}}(\mathbf{r},\vartheta)\phi(\mathbf{r},\vartheta)\psi(\mathbf{r},\vartheta)$$

  reference operator is Hermitian $\langle\phi|\mathcal{L}_0\psi\rangle = \langle\mathcal{L}_0\phi|\psi\rangle$

  eigenvalue problem $\mathcal{L}_0\psi = -\lambda\psi$

  → eigenvalues are real

- Hilbert space basis $|\psi_\Lambda\rangle$

$$\langle\psi_\Lambda|\psi_M\rangle = \delta_{\Lambda M}$$

$$\sum_\Lambda p^{\mathrm{eq}}(\mathbf{r},\vartheta)\psi_\Lambda(\mathbf{r},\vartheta)\psi_\Lambda(\mathbf{r}_0,\vartheta_0)^* = \delta(\mathbf{r}-\mathbf{r}_0)\delta(\vartheta-\vartheta_0)$$

- generalized position-orientation eigenstates $|\mathbf{r},\vartheta\rangle$ with $\psi(\mathbf{r},\vartheta) = \langle\mathbf{r}\vartheta|\psi\rangle$ to make isomorphism $|\psi\rangle \leftrightarrow \psi(\mathbf{r},\vartheta)$ manifest

  orthogonal $\quad p^{\mathrm{eq}}(\mathbf{r},\vartheta)\langle\mathbf{r},\vartheta|\mathbf{r}_0,\vartheta_0\rangle = \delta(\mathbf{r}-\mathbf{r}_0)\delta(\vartheta-\vartheta_0)$

  complete $\quad \int \mathrm{d}\mathbf{r}\mathrm{d}\vartheta\, p^{\mathrm{eq}}(\mathbf{r},\vartheta)|\mathbf{r},\vartheta\rangle\langle\mathbf{r},\vartheta| = \mathbb{I}$

Formal expression for propagator

- use orthogonality and completeness relations...

$$\mathbb{P}(\mathbf{r},\vartheta,t|\mathbf{r}_0\vartheta_0) = e^{\Omega t}\delta(\mathbf{r}-\mathbf{r}_0)\delta(\vartheta-\vartheta_0) = e^{\Omega t}\sum_\Lambda p^{\text{eq}}(\mathbf{r},\vartheta)\psi_\Lambda(\mathbf{r},\vartheta)\psi_\Lambda(\mathbf{r}_0,\vartheta_0)^*$$

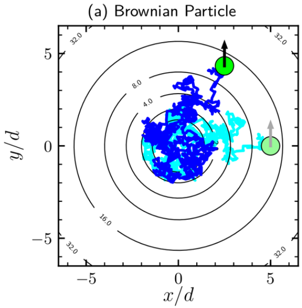
$$= p^{\text{eq}}(\mathbf{r},\vartheta)e^{\mathcal{L}t}\sum_\Lambda \psi_\Lambda(\mathbf{r},\vartheta)\psi_\Lambda(\mathbf{r}_0,\vartheta_0)^* = p^{\text{eq}}(\mathbf{r},\vartheta)e^{\mathcal{L}t}\sum_\Lambda \langle\mathbf{r},\vartheta|\psi_\Lambda\rangle\langle\psi_\Lambda|\mathbf{r}_0,\vartheta_0\rangle$$

$$= p^{\text{eq}}(\mathbf{r},\vartheta)\sum_\Lambda \langle\mathbf{r},\vartheta|e^{\mathcal{L}t}\psi_\Lambda\rangle\langle\psi_\Lambda|\mathbf{r}_0,\vartheta_0\rangle$$

$$\mathbb{P}(\mathbf{r},\vartheta,t|\mathbf{r}_0\vartheta_0) = p^{\text{eq}}(\mathbf{r},\vartheta)\langle\mathbf{r},\vartheta|e^{\mathcal{L}t}\mathbf{r}_0\vartheta_0\rangle$$

→ propagates initial state to final state

Symmetries



- rotation of the position $\mathbf{r}$ around the center

  generator $L = -i\partial_\varphi$ 'orbital momentum'

- rotation of the orientation $\mathbf{u}$

  generator $S = -i\partial_\vartheta$ 'spin'

- simultaneous rotation of position and orientation

  generator $J = -i\partial_\varphi - i\partial_\vartheta = L + S$ 'total angular momentum'

→ total angular momentum J conserved for active Brownian particle,
'good quantum number'
for passive particles orbital momentum L and spin S are conserved separately

Passive Brownian particle

- eigenvalue problem $\mathcal{L}_0 \psi = -\lambda\psi$
  separation ansatz $\psi(r,\varphi,\vartheta) = \exp(i\ell\varphi)\exp[i(j-\ell)\vartheta]R(r)$ $\qquad j = l + s$ reflects conservation laws

$$\rightarrow \frac{d^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) - \frac{d^2\ell^2}{r^2}R(r) - D_{\mathrm{rot}}\tau(j-\ell)^2 R(r) - r\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \lambda\tau R(r) = 0$$

- orbital angular momentum barrier dominates for $r \to 0 \rightarrow R(r) \propto r^{|\ell|}$
  thermal oscillator length d sets the scale

ansatz $\qquad R(r) = r^{|\ell|} L(\rho^2), \qquad$ with $\rho = r/d\sqrt{2}$

$$xL''(x) + (1+|\ell|-x)L'(x) + [\lambda\tau/2 - D_{\mathrm{rot}}\tau(j-\ell)^2/2 - |\ell|/2]L(x) = 0$$

solutions are **associated Laguerre polynomials** $\mathrm{L}_n^{|\ell|}(x)$

eigenvalue $\qquad \lambda_{n\ell j} = \frac{1}{\tau}(2n + |\ell|) + D_{\mathrm{rot}}(j-\ell)^2$

Passive Brownian particle – cont'd

- Laguerre polynomials

orthogonal $$\int_0^\infty x^k e^{-x} \mathrm{L}_m^k(x) \mathrm{L}_n^k(x) \mathrm{d}x = \delta_{mn} \frac{(n+k)!}{n!}$$

complete $$x^k e^{-x} \sum_{n=0}^\infty \frac{n!}{(n+k)!} \mathrm{L}_n^k(x) \mathrm{L}_n^k(x_0) = \delta(x - x_0)$$

- eigenfunctions

$$\langle \mathbf{r}\vartheta | \psi_{n,\ell,j} \rangle = \psi_{n,\ell,j}(\mathbf{r},\vartheta) = \sqrt{\frac{n!}{(n+|\ell|)!}} \left(\frac{r}{d\sqrt{2}} \right)^{|\ell|} \mathrm{L}_n^{|\ell|} \left(\frac{r^2}{2d^2} \right) e^{i\ell\varphi} e^{i(j-\ell)\vartheta}$$

$$\langle \psi_{n',\ell',j'} | \psi_{n,\ell,j} \rangle = \delta_{nn'} \delta_{\ell\ell'} \delta_{jj'}$$

$$p^{\mathrm{eq}}(\mathbf{r},\vartheta) \sum_{n=0}^\infty \sum_{\ell=-\infty}^\infty \sum_{j=-\infty}^\infty \psi_{n,\ell,j}(\mathbf{r},\vartheta) \psi_{n,\ell,j}(\mathbf{r}_0,\vartheta_0)^* = \delta(\mathbf{r}-\mathbf{r}_0)\delta(\vartheta-\vartheta_0)$$

propagator $$\mathbb{P}_0(\mathbf{r},\vartheta,t|\mathbf{r}_0,\vartheta_0) = p^{\mathrm{eq}}(\mathbf{r},\vartheta) \sum_{n=0}^\infty \sum_{\ell=-\infty}^\infty \sum_{j=-\infty}^\infty e^{-\lambda_{n,\ell,j}t} \psi_{n,\ell,j}(\mathbf{r},\vartheta) \psi_{n,\ell,j}(\mathbf{r}_0,\vartheta_0)^*$$

Active Brownian particle – spectrum

- eigenvalues in perturbation theory – **action** of the active propulsion

$$\mathcal{L}_1|\psi_{n,\ell,j}\rangle = \frac{1}{\sqrt{2\tau}}\begin{cases}\sqrt{n+\ell+1}\,|\psi_{n,\ell+1,j}\rangle - \sqrt{n+1}\,|\psi_{n+1,\ell-1,j}\rangle & \text{if } \ell>0,\\ \sqrt{n+1}\,|\psi_{n,\ell+1,j}\rangle + \sqrt{n+1}\,|\psi_{n,\ell-1,j}\rangle & \text{if } \ell=0,\\ \sqrt{n-\ell+1}\,|\psi_{n,\ell-1,j}\rangle - \sqrt{n+1}\,|\psi_{n+1,\ell+1,j}\rangle & \text{if } \ell<0.\end{cases}$$

rotational invariance → j is unchanged ✓
activity **couples** different orbital momenta
one of the quantum numbers $n, |\ell|$ increases by 1
→ sort states such that matrix $\langle\psi_{n',\ell',j}|\mathcal{L}_1|\psi_{n,\ell,j}\rangle$ is **strictly lower-diagonal !!!**

eigenvalues independent of Péclet number – harmonic oscillator is **isospectral**

Propagator

- formal expression for propagator

$$\mathbb{P}(\mathbf{r},\vartheta,t|\mathbf{r}_0,\vartheta_0) = p^{\text{eq}}(\mathbf{r},\vartheta)\langle\mathbf{r},\vartheta|e^{\mathcal{L}t}\mathbf{r}_0\vartheta_0\rangle \qquad \text{insert completeness} \qquad \sum_{n,\ell,j}|\psi_{n,\ell,j}\rangle\langle\psi_{n,\ell,j}| = \mathbb{I}$$

$$= p^{\text{eq}}(\mathbf{r},\vartheta)\sum_{n,\ell,j}\langle\mathbf{r}\vartheta|\psi_{n,\ell,j}\rangle \underbrace{\langle\psi_{n,\ell,j}|e^{\mathcal{L}t}\mathbf{r}_0\vartheta_0\rangle}_{=:M_{n,\ell,j}(\mathbf{r}_0,\vartheta_0,t)}$$

expansion in eigenfunctions $\qquad \mathbb{P}(\mathbf{r},\vartheta,t|\mathbf{r}_0,\vartheta_0) = p^{\text{eq}}(\mathbf{r},\vartheta)\sum_{n,\ell,j} M_{n,\ell,j}(\mathbf{r}_0,\vartheta_0,t)\psi_{n,\ell,j}(\mathbf{r},\vartheta)$

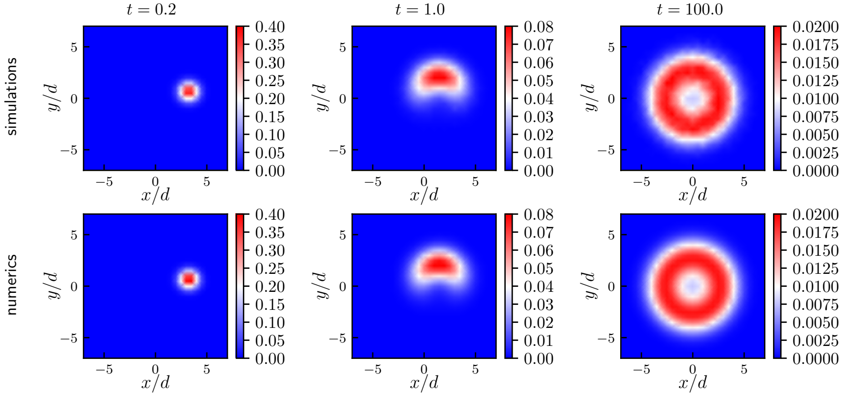
- Dyson equation (perturbartion identity)

$$e^{\mathcal{L}t} = e^{\mathcal{L}_0 t} + \text{Pe}\int_0^t \mathrm{d}s\, e^{\mathcal{L}(t-s)}\mathcal{L}_1 e^{\mathcal{L}s}$$

- **recursion relation** (use angular momentum conservation)

$$M_{n,\ell,j}(\mathbf{r}_0,\vartheta_0,t) = e^{-\lambda_{n,\ell,j}t}\langle\psi_{n,\ell,j}|\mathbf{r}_0,\vartheta_0\rangle + \text{Pe}\int_0^s \mathrm{d}s\, e^{-\lambda_{n,\ell,j}(t-s)}\sum_{n',\ell'}\langle\psi_{n,\ell,j}|\mathcal{L}_1|\psi_{n',\ell',j}\rangle M_{n',\ell',j}(\mathbf{r}_0,\vartheta_0,s)$$

Propagator – cont'd



- analytics corroborated by simulation ✓
- stationary distribution **non-Gaussian** – pile-up of probability on the edge

Correlation functions

- Perturbation $\mathcal{L}_1$ 'upper diagonal'
  → no chains, no term appears twice
  → scheme terminates for simple correlation functions: **integrable**
- Positional autocorrelation function (PAF)

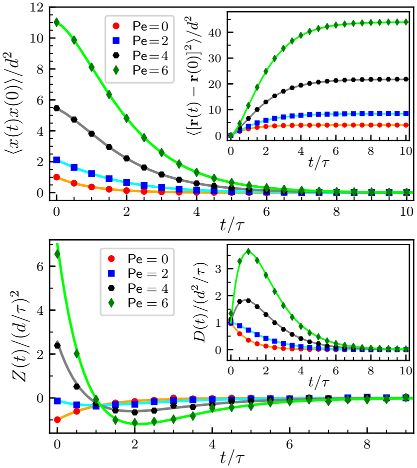
$$\langle x(t)x(0)\rangle = d^2\left[e^{-t/\tau} - \frac{\mathrm{Pe}^2}{2}\,\frac{D_{\mathrm{rot}}\tau e^{-t/\tau} - e^{-D_{\mathrm{rot}}t}}{1-(D_{\mathrm{rot}}\tau)^2}\right]$$

eigenvalues are unchanged!

- Velocity autocorrelation function (VACF)

$$Z(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle x(t)x(0)\rangle$$

not a completely monotone function → fingerprint of **non-equilibrium dynamics**



M. Caraglio and T. Franosch, PRL (2022)

Summary and Conclusions

Circle swimmer

- exact solution in terms of generalized Mathieu functions
- ISF displays non-trivial plateau due to circular motion

Gravitaxis

- formal solution of Fokker-Planck equation
- generates low-order moments
- **resonance** close to classical bifurcation for small noise
- rationalized in terms of **harmonic approximation**

Active Brownian particle in a harmonic well

- **isospectral:** eigenvalues remain unchanged
- perturbative scheme for **propagator**
- **closed expressions** for low-order correlation functions