# The moduli space of Calabi–Yau metrics and pseudo-Kähler reduction by stages

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Joint with Mario Garcia-Fernandez

**Calabi–Yau metrics and their moduli spaces**

A *Calabi–Yau* metric on a smooth manifold $M^{2n}$ is a Riemannian metric with

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Moduli spaces of Calabi–Yau metrics have been extensively studied in both the mathematical and physics literature. Our current understanding of them crucially relies on a separation of variables, possible thanks to Yau's solution of the Calabi conjecture.

$$\{\text{CY moduli}\} \cong \{\text{complex moduli}\} \times \{\text{Kähler moduli}\}$$

▶ Shing-Tung Yau. In: *Communications on Pure and Applied Mathematics* 31.3 (1978)

▶ Philip Candelas and Xenia C. de la Ossa. In: *Nuclear Physics B* 355.2 (1991)

More precisely, a Calabi–Yau metric on $M$ can be recast into the data of a tuple $(J, \Omega, \omega)$, where $J$ is a complex structure on $M$, $\Omega$ is a nonvanishing holomorphic volume form on $(M, J)$ and $\omega$ $(= g(J, \cdot))$ is a Kähler metric satisfying

$$\frac{\omega^n}{n!} = i^{n^2} \Omega \wedge \overline{\Omega}. \qquad (*)$$

### Theorem (Yau,'78)

Let $(M^{2n}, J, \Omega)$ be a compact Calabi–Yau n-fold. Then for each Kähler class $c \in H^{1,1}(M, J)$ satisfying $(*)$ at the level of cohomology, there exists a unique Kähler metric $\omega \in c$ satisfying $(*)$.

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$$H^2(M, \mathbb{C}) = H^{1,1}(M, J) \longrightarrow \mathcal{M}_{CY} \longrightarrow \mathcal{T}$$

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In the process to solving this problem, we will encounter two new phenomena.

- A group action by a <u>real</u> Lie group, with <u>complex</u> orbits.
- Higher symmetries.

# Construction of the moduli space

# B-fields

A naïve definition for the moduli space might be

$$\mathcal{M}_{CY} = \{(J, \Omega, \omega)\}/\operatorname{Diff}_0(M).$$

It has been understood for a while that the above definition of $\mathcal{M}_{CY}$ is not the appropriate one. In fact, the above space is not symplectic or complex in any natural way. The standard approach to deal with this issue is by "complexifying" $\omega$ using a *B-field*.

$$\omega \longrightarrow \omega + ib, \quad b \in \Omega^2(M)$$

This adds the directions in our space of parameters necessary for it to support a complex and a symplectic structure.

## Strategy

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Consider $\mathbf{T}M := TM \oplus T^*M$ with the pseudo-euclidean metric

$$\langle X + \xi, X + \xi \rangle = \xi(X).$$

For each $B$-field $b \in \Omega^2(M)$, the map $e^b \colon \mathbf{T}M \to \mathbf{T}M$ defined by

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is an orthogonal automorphism of $(\mathbf{T}M, \langle, \rangle)$. The set $\mathcal{L} = \{(J, \omega + ib)\}$ is bijective to the set of *isotropic liftings* $\bar{\ell}$ of $\mathbf{T}M \otimes \mathbb{C}$ via

$$(J, \omega + ib) \longmapsto \bar{\ell} = e^{i\omega - b}(T_J^{0,1}M).$$

▶ Nigel Hitchin. In: *Quarterly Journal of Mathematics - QUART J MATH* 54 (Sept. 2002)

# Strategy

### Lemma (GF, –)

*The natural complex structure on $\mathcal{L}$ is given by*

$$\mathbf{J}(\dot{J}, \dot{\omega} + i\dot{b}) = \left( J\dot{J}, i(\dot{\omega} + i\dot{b})^{1,1+0,2} - i(\dot{\omega} + i\dot{b})^{2,0} \right).$$

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Now, consider the natural map $(J, \Omega, \omega + ib) \in \mathcal{P} \mapsto (J, \omega + ib) \in \mathcal{L}$. Its fibre above $(J, \omega + ib) \in \mathcal{L}$ corresponds to an open subset of the complex vector space $\Omega^{n,0}(M, J)$, which is naturally a complex manifold in its own right.

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- the map $\mathcal{P} \to \mathcal{L}$ is holomorphic, and
- it induces the natural complex structures on the fibers of $\mathcal{P} \to \mathcal{L}$.

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Let $\mathcal{Z} = \mathcal{P}/\mathbb{C}^*$. Then $\mathcal{Z}$ is an infinite dimensional complex manifold, and the natural map $\mathcal{P} \to \mathcal{Z}$ is a holomorphic $\mathbb{C}^*$-bundle on $\mathcal{Z}$.

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$$(J, \Omega, \omega + ib) \longmapsto \int_M \|\Omega\|_\omega \frac{\omega^n}{n!}$$

determines a Hermitian metric $\mathbf{h}$ on $\mathcal{P} \to \mathcal{Z}$. Let

$$\omega = iF_{\mathbf{h}} = -dd^c \log \mathbf{h} \in \Omega^2(\mathcal{Z}).$$

▶ Mario Garcia-Fernandez, Roberto Rubio, Carlos Shahbazi, and Carl Tipler. In: *Proc. Lond. Math. Soc.* 125.3 (2022). eprint: 1803.01873

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## Proposition (GF, –)

*The real $(1, 1)$-form $\boldsymbol{\omega}$ is nondegenerate and indefinite.*

## The integrability condition

Roughly speaking, from a physical perspective, Calabi–Yau manifolds arise by imposing $N = 1$ supersymmetry (SUSY) in heterotic string theory. The SUSY equations are divided into two families:

1. the *D-term equations*: $\mu = 0$,
2. the *F-term equations*: $\delta \mathcal{W} = 0$.

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1. the *D-term equations*: $\mu = 0$,
2. the *F-term equations*: $\delta\mathcal{W} = 0$.

In our situation, the F-term equations are

$$N_J = 0, \quad H_0 - db + d^c\omega = 0$$

where $H_0$ (the *NS flux*) is a fixed closed three-form on $M$.
Equivalently, $\bar{\ell} = e^{i\omega - b}(T_J^{0,1}M) \leq E \otimes \mathbb{C}$ is involutive with respect to the *twisted Dorfman bracket*

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y \iota_X H_0.$$

## Remark

The bracket

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - \iota_Y d\xi + \iota_Y \iota_X H_0$$

together with the pseudo-Euclidean metric $\langle, \rangle$ on $\mathbf{T}M$ and the canonical projection $\pi \colon \mathbf{T}M \to TM$ satisfy the following axioms

$$[u_1, [u_2, u_3]] = [[u_1, u_2], u_3] + [u_2, [u_1, u_3]]$$
$$[u_1, f u_2] = f[u_1, u_2] + \pi(u_1)(f) u_2$$
$$\pi(u_1)(\langle u_2, u_3 \rangle) = \langle [u_1, u_2], u_3 \rangle + \langle u_2, [u_1, u_3] \rangle$$
$$\pi([u_1, u_2]) = [\pi(u_1), \pi(u_2)]$$
$$[u_1, u_2] + [u_2, u_1] = \pi^* d(\langle u_1, u_2 \rangle)$$

We say $E = (\mathbf{T}M, \langle, \rangle, [,], \pi)$ is a *Courant algebroid*.

## Symmetries

To eliminate the dependency on the $B$-field, the symmetry group $\mathrm{Diff}_0(M)$ needs to be replaced by an extension

$$0 \longrightarrow \Omega^2_{\mathrm{closed}}(M) \longrightarrow \mathcal{G} \longrightarrow \mathrm{Diff}_0(M) \longrightarrow 1$$

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A group that fits the bill is $\mathcal{G} = \mathrm{Aut}_0(E)$, the identitiy component of the group of *Courant automorphisms* of $E$, i.e the set of automorphisms of $E$ preserving the metric $\langle , \rangle$, bracket $[ , ]$ and anchor $\pi \colon E \to TM$.

## Symmetries

The Lie algebra $\operatorname{Lie}\mathcal{G}$ is the vector space of *Courant derivations*, i.e first order differential operators $D\colon \Gamma(E) \to \Gamma(E)$ satisfying

$$\langle Du_1, u_2\rangle + \langle u_1, Du_2\rangle = X_D(\langle u, v\rangle)$$
$$[Du_1, u_2] + [u_1, Du_2] = D[u_1, u_2]$$
$$\pi(Du) = [X_D, \pi(u)]$$

for some $X_D \in \Gamma(TM)$.

The *inner derivations* $[u, ]$, $u \in \Gamma(E)$ form an ideal in $\operatorname{Lie}\mathcal{G}$. We let $\mathcal{H} \leq \mathcal{G}$ be the connected subgroup of $\mathcal{G}$ integrating this ideal.

$$0 \longrightarrow \Omega^2_{\text{exact}}(M) \longrightarrow \mathcal{H} \longrightarrow \operatorname{Diff}_0(M) \longrightarrow 1$$

# The moment map

Via $(J, \omega + ib) \leftrightarrow \bar{\ell} \leq E \otimes \mathbb{C}$, $\mathcal{G}$ naturally acts on $\mathcal{P}$ and $\mathcal{Z}$ by holomorphic isometries, preserving the subset $\mathcal{Z}_0$ of solutions of the F-term equations.

## Theorem (Ashmore, Strickland-Constable, Waldram '20; GF, –)

*The action $\mathcal{G} \curvearrowright \mathcal{Z}$ is Hamiltonian with moment map $\boldsymbol{\mu} \colon \mathcal{Z} \to (\mathrm{Lie}\,\mathcal{G})^*$. Moreover, if $\tilde{\boldsymbol{\mu}} := \mathrm{restr}_{\mathrm{Lie}\,\mathcal{H}} \circ \boldsymbol{\mu}_{|_{\mathcal{Z}_0}} \colon \mathcal{Z}_0 \to (\mathrm{Lie}\,\mathcal{H})^*$, then*

$$\tilde{\boldsymbol{\mu}}(J, \Omega, \omega + ib) = 0 \iff \begin{cases} \bar{\partial}\Omega & = 0, \\ d^\star\omega - d^c \log \|\Omega\|_\omega & = 0. \end{cases}$$

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Explicitly, we have

$$\langle \boldsymbol{\mu}(\mathbf{p}), \zeta \rangle = (d^c \log \mathbf{h})(\zeta \cdot \mathbf{q})$$

where $\mathbf{q} \in \mathcal{P}_{\mathbf{p}}$ is arbitrary and $\zeta \cdot$ denotes the infinitesimal action.

## The moment map

We have

$$d^{\star}\omega - d^c \log \|\Omega\|_{\omega} = 0 \iff d\omega_0^{n-1} = 0$$

where $\omega_0 := \|\Omega\|_{\omega}^{1/(n-1)}\omega$. We say $\omega_0$ is a *balanced* metric and $\omega$ is *conformally balanced*.

Being conformally balanced is generally a much weaker condition than being Kähler. In this situation, however, the F-terms imply the additional constraint

$$dd^c\omega = 0.$$

In this situation, [Ivanov, Papadopoulos, AIM '13] show that $\omega$ must be a Kähler metric and $\|\Omega\|_{\omega}$ must be constant. Thus $\mathrm{Hol}(g) \leq \mathrm{SU}(n)$, although we can't rule out the possibility of $\mathrm{Hol}(g)$ being a proper subgroup of $\mathrm{SU}(n)$.

**The moduli metric**

# Towards the moduli space metric

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Let $\mathbf{p} \in \tilde{\boldsymbol{\mu}}^{-1}(0)$. Then $T_{[\mathbf{p}]}\mathcal{M}$ is naturally (pre)symplectic with symplectic structure induced by the symplectic structure on $T_{(J,\Omega,\omega+ib)}\mathcal{Z}$.

## Towards the moduli space metric

Consider a (possibly infinite dimensional) pseudo-Kähler Hamiltonian $G$-space $(Z, \omega_Z, \mu_Z)$. For $p \in \mu_Z^{-1}(0)$, we have a complex

$$\text{Lie } G \xrightarrow{\ P\ } T_p Z \xrightarrow{\ L\ } (\text{Lie } G)^*$$

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1. $(Z, \omega_Z)$ is finite dimensional Kähler,
2. $P \colon \text{Lie } G \to T_p Z$ is a differential operator, $P^* P \colon \text{Lie } G \longrightarrow (\text{Lie } G)^*$ is elliptic of index zero, and $\dim \ker P^* P = 0$.

# Higher symmetries

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$[,] \colon \Gamma(E) \to$ Lie $\mathcal{H}$. This map fits into a complex

$$\mathscr{C}^\infty(M) \xrightarrow{\ 2d\ } \Gamma(E) \xrightarrow{\ [,]\ } \mathrm{Lie}\,\mathcal{H}.$$

Higher geometry suggests we should think of the Lie algebra of $\mathcal{H}$ as the
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arrow $\mathscr{C}^\infty(M) \xrightarrow{2d} \Gamma(E)$. In accordance to this, we replace our Lie $\mathcal{H}$-valued
moment map $\tilde{\boldsymbol{\mu}}$ with a *higher moment map* $\hat{\boldsymbol{\mu}}: \mathcal{Z}_0 \to \Gamma(E)$, determined by

$$\langle \tilde{\boldsymbol{\mu}}(p), [u,] \rangle = \langle\!\langle \hat{\boldsymbol{\mu}}(p), u \rangle\!\rangle_{\mathbf{p}}$$

where $\langle\!\langle , \rangle\!\rangle_{\mathbf{p}}: \Gamma(E) \times \Gamma(E) \to \mathbb{R}$ is an explicit nondegenerate pairing.

▶ Dmitry Roytenberg and Alan Weinstein. In: *Letters in Mathematical Physics* 46.1 (1998)
▶ Henrique Bursztyn, Gil R. Cavalcanti, and Marco Gualtieri. In: *Advances in Mathematics* 211.2 (2007)

# Higher symmetries

Let $\mathbf{p} = (J, \Omega, \omega + ib) \in \tilde{\boldsymbol{\mu}}^{-1}(0) = \hat{\boldsymbol{\mu}}^{-1}(0)$. Consider the complex

$$\Gamma(E) \xrightarrow{\ \mathbf{P}\ } T_{\mathbf{p}}\mathcal{Z}_0 \xrightarrow{\ \mathbf{L}\ } \Gamma(E) \qquad (S^*)$$

where $\mathbf{P} \colon \Gamma(E) \to T_{\mathbf{p}}\mathcal{Z}_0$ is the composition of $[,] \colon \Gamma(E) \to \operatorname{Lie} \mathcal{H}$ with the infinitesimal action of $\mathcal{H}$ at $\mathbf{p}$ and $\mathbf{L} \colon T_{\mathbf{p}}\mathcal{Z}_0 \to \Gamma(E)$ is the differential of $\hat{\boldsymbol{\mu}}$ at $\mathbf{p}$. Then $H^1(S^*)$ models the tangent space of $\mathcal{M}$ at the class of $\mathbf{p}$.

### Remark

While $T_{\mathbf{p}}\mathcal{Z}_0$ is not the space of sections of a vector bundle on $M$ itself, it is (a quotient of) the set of solutions of a first order differential equation defined on the vector bundle underlying $T_{\mathbf{p}}\mathcal{P}$.

## Infinitesimal complex orbits

Let
$$V_+ := (\ell \oplus \bar{\ell}) \cap E = e^{-b}\{X + g(X),\ X \in TM\}$$

and define $J_+ : V_+ \to V_+$ by

$$J_+ u := iu_\ell - iu_{\bar{\ell}}.$$

Then, by involutivity of $\bar{\ell}$, for $u \in \Gamma(V_+)$ we have

$$\mathbf{P}J_+ u = \mathbf{J}\mathbf{P}u.$$

Note the similarity with the infinitesimal action of $\mathrm{Diff}_0(M)$ on the space of (integrable) complex structures on $M$.

$$L_{JX}J = JL_X J$$

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### Fact
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# Infinitesimal complex orbits

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In particular, we have

$$\mathbf{P}\Gamma(V_+) \leq \operatorname{Im}\mathbf{P} \cap (\operatorname{Im}\mathbf{P})^{\perp}.$$

But $\mathbf{P}\Gamma(V_+) \neq \{0\}$. So $\operatorname{Im}\mathbf{P}$ and $(\operatorname{Im}\mathbf{P})^{\perp}$ are *never* in direct sum.

## Upside-down reduction by stages

The previous slide suggests a reduction by stages approach.
Consider the orthogonal decomposition

$$E = V_+ \oplus V_-$$

where $V_\pm = e^{-b}\{X \pm g(X), \ X \in TM\}$. Then the analogy between the infinitesimal action of $\Gamma(V_+)$ and the infinitesimal action of diffeomorphisms on the space of complex structures on $M$ suggests we should think of $\Gamma(V_+) \cong_\pi \Gamma(TM)$ as the Lie algebra of $\mathrm{Diff}_0(M)$. As such,

$$0 \longrightarrow \Gamma(V_-) \longrightarrow \Gamma(E) \longrightarrow \Gamma(V_+) \longrightarrow 0$$

can be suggestively thought of as a higher analogue of the Lie algebra sequence associated to $1 \to \Omega^2_{\mathrm{exact}}(M) \to \mathcal{H} \to \mathrm{Diff}_0(M) \to 1$.

## Upside-down reduction by stages

For $\mathbf{p} = (J, \Omega, \omega + ib) \in \mathcal{Z}_0$, we have

$$\hat{\boldsymbol{\mu}}(\mathbf{p}) = e^{-b}[g^{-1}(\eta + \overline{\eta}) + g^{-1}(d^\star \omega - d^c \log \|\Omega\|_\omega) - (d^\star \omega - d^c \log \|\Omega\|_\omega)]$$

where $\eta$ is the $(0, 1)$-form determined by $\bar{\partial}\Omega = i\eta \wedge \Omega$. Thus, if

$$\hat{\boldsymbol{\mu}}_{\pm}(\mathbf{p}) \in \Gamma(V_{\pm})$$

are the components of $\hat{\boldsymbol{\mu}}(\mathbf{p})$ under the decomposition $E = V_+ \oplus V_+$ then, identifying $V_{\pm} \cong_\pi TM \cong_g T^*M$, we have

$$\hat{\boldsymbol{\mu}}_+ \sim \eta + \overline{\eta}, \quad \hat{\boldsymbol{\mu}}_- \sim \eta + \overline{\eta} - (d^\star \omega - d^c \log \|\Omega\|_\omega).$$

Observe in particular that $\hat{\boldsymbol{\mu}}_+(\mathbf{p}) = 0$ if and only if $\bar{\partial}\Omega = 0$, which is a holomorphic equation on $\mathcal{Z}_0$.

## Upside-down reduction by stages

Differentiating, for $\mathbf{p} \in \tilde{\boldsymbol{\mu}}^{-1}(0)$, we obtain two complexes

$$\Gamma(V_+) \xrightarrow{\mathbf{P}_+} T_p\mathcal{Z}_0 \xrightarrow{\mathbf{L}_+} \Gamma(V_+) \qquad (S_+^*)$$

$$\Gamma(V_-) \xrightarrow{\mathbf{P}_-} H^1(S_+^*) \xrightarrow{\mathbf{L}_-} \Gamma(V_-) \qquad (S_-^*)$$

such that $H^1(S_-^*) \cong H^1(S^*)$.

The cohomology $H^1(S_+^*) = \ker \mathbf{L}_+ / \operatorname{Im} \mathbf{P}_+$ is naturally endowed with a metric and complex structure by passage to the quotient.

The operator $\mathbf{P}_-^* \mathbf{P}_- : \Gamma(V_-) \to \Gamma(V_-)$ is not elliptic, but can be modified to be made elliptic by taking the higher symmetries into account. In fact, we can extend $(S^*)$ to a complex

$$\mathscr{C}^\infty(M) \xrightarrow{2d} \Gamma(E) \xrightarrow{\mathbf{P}} T_{\mathbf{p}}\mathcal{Z}_0 \xrightarrow{\mathbf{L}} \Gamma(E). \qquad (S^*)$$

Accordingly, we can also extend $(S_-^*)$ to

$$\mathscr{C}^\infty(M) \xrightarrow{d_-} \Gamma(V_-) \xrightarrow{\mathbf{P}_-} H^1(S_+^*) \xrightarrow{\mathbf{L}_-} \Gamma(V_-) \qquad (S_-^*)$$

where $d_- = e^{-b}(d - g^{-1}d)$ is the composition of $2d$ with the orthogonal projection $E \to V_-$. Then

$$\Delta_- := \mathbf{P}_-^* \mathbf{P}_- + d_- d_-^*.$$

is a second-order elliptic differential operator on $\Gamma(V_-)$.

There is a natural Hermitian metric $G$ and a natural holomorphic structure $\bar{\partial}^{V_- \otimes \mathbb{C}}$ on $V_- \otimes \mathbb{C}$ such that under an identification

$$V_- \otimes \mathbb{C} \cong T_{1,0}^* M \oplus T^{1,0} M$$

we have

$$G = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}, \quad \bar{\partial}^{V_- \otimes \mathbb{C}} = \begin{pmatrix} \bar{\partial} & \partial \omega \\ 0 & \bar{\partial} \end{pmatrix}$$

Let $\Delta_{\mathrm{Dolb}} \colon \Gamma(V_- \otimes \mathbb{C}) \to \Gamma(V_- \otimes \mathbb{C})$ be the corresponding Dolbeault Laplacian operator associated to $\omega_0$.

Proposition

$\Delta_- = \Delta_{\mathrm{Dolb}}$.

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In our situation, $\tilde{\boldsymbol{\mu}}(\mathbf{p}) = 0$ implies that $\partial\omega = 0$. So the formula for $\bar{\partial}^{V_- \otimes \mathbb{C}}$ on $V_- \otimes \mathbb{C} \cong T^*_{1,0}M \oplus T^{1,0}M$ collapses to

$$\bar{\partial}^{V_- \otimes \mathbb{C}} = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \bar{\partial} \end{pmatrix}$$

that is, $V_- \otimes \mathbb{C} \cong T^*_{1,0}M \oplus T^{1,0}M$ is, in fact, a splitting of holomorphic vector bundles. Moreover, by Ricci-flatness of $\omega$ and the Weitzenböck identity, one can show that under $V_- \otimes \mathbb{C} \cong_\pi TM \otimes \mathbb{C} \cong_g T^*M \otimes \mathbb{C}$,

$$\Delta_{\text{Dolb}} = \Delta_{\text{Hodge}} = d^*d + dd^*.$$

Thus $\Delta_- = \Delta_{\text{Hodge}}$.

# Construction of the moduli space metric

## Theorem (GF, −)

At each point $\mathbf{p} \in \tilde{\boldsymbol{\mu}}^{-1}(0)$, there is an orthogonal decomposition

$$H^1(S_+^*) = \operatorname{Im} \mathbf{P}_- \oplus (\operatorname{Im} \mathbf{P}_-)^\perp$$

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### Corollary (GF, −)

*At each point $\mathbf{p} \in \tilde{\boldsymbol{\mu}}^{-1}(0)$, $H^1(S^*)$ inherits a complex structure and a metric compatible with its natural symplectic structure.*

In fact, there is a natural isomorphism of complex vector spaces

$$H^1(S^*) \cong H^{n-1,1}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$$

such that $\langle , \rangle_{\mathcal{M}}$ is given by

$$\langle \phi + \mathfrak{a}, \phi + \mathfrak{a} \rangle_{\mathcal{M}} = \|\phi\|_{WP}^2 + \frac{1}{n}|\Lambda\mathfrak{a}|^2 + Q(\mathfrak{a}_0^{1,1}) + Q(\mathfrak{a}_0^{0,2}).$$

Here $\|\cdot\|_{WP}$ is the Weil-Petersson metric, $\Lambda$ is the contraction operator, $\mathfrak{a}_0 = \mathfrak{a} - \frac{1}{n}(\Lambda\mathfrak{a})[\omega]$ and $Q$ is the Hodge-Riemann form.

$$\|\phi\|_{WP}^2 = \frac{2}{\int_M i^{n^2}\Omega \wedge \overline{\Omega}} \int_M \phi \wedge \star\overline{\phi},$$

$$Q(\mathfrak{a}_0^{p,q}) = -\frac{1}{(n-2)!\,\mathrm{Vol}(\omega)} \mathfrak{a}_0^{p,q} \frown \overline{\mathfrak{a}_0^{p,q}} \frown [\omega]^{n-2}$$

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In particular, if **p** determines a true Calabi–Yau metric, i.e the holonomy is exactly equal to $\mathsf{SU}(n)$, then $H^{0,2}(M) = 0$ and $\langle, \rangle_{\mathcal{M}}$ is positive definite. This is the Candelas-de la Ossa metric, obtained from the Hermitian metric $\mathcal{P} \to \mathbb{R}$ defined by

$$(J, \Omega, \omega + ib) \longmapsto \mathsf{Vol}(\omega)^{1/2} \left( \int_M i^{n^2} \Omega \wedge \overline{\Omega} \right)^{1/2}$$

# The Hull–Strominger system

## Setting

Generalized geometry is particularly well-equipped to handle and interpret equations in the context of torsional geometries, in particular non-Kähler geometry. An important example is the *Hull–Strominger system*, an analogue of the Calabi–Yau equations for non-Kähler manifolds.

Let $K$ be a compact Lie group, $\langle , \rangle_{\mathfrak{k}}$ a symmetric nondegenerate invariant pairing on its Lie algebra and $P \to M$ principal $K$-bundle with vanishing first Pontryaguin class

$$p_1(P) := -[\langle F_A \wedge F_A \rangle_{\mathfrak{k}}] \in H^4(M).$$

Fix a connection $A_0$ on $P$ and a three-form $H_0$ on $M$ such that

$$dH_0 = \langle F_{A_0} \wedge F_{A_0} \rangle_{\mathfrak{k}}.$$

This data determines a Courant algebroid $E$ over $M$ with underlying vector bundle $TM \oplus \operatorname{ad} P \oplus T^*M$.

## The Hull–Strominger system

For $(J, \Omega, \omega + ib, A)$, with $(J, \Omega, \omega + ib)$ as before and $A$ a principal connection on $A$, the *Hull–Strominger system* is the system of equations

$$N_J = 0, \qquad\qquad \bar{\partial}\Omega = 0,$$
$$H_0 - db + \mathsf{CS}(A_0, A) + d^c\omega = 0, \qquad d^\star\omega - d^c \log \|\Omega\|_\omega = 0,$$
$$F_A^{0,2} = 0, \qquad\qquad F_A \wedge \omega^{n-1} = 0.$$

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$$F_A^{0,2} = 0, \qquad\qquad F_A \wedge \omega^{n-1} = 0.$$

Let $\mathcal{Z} = (J, \Omega, \omega + ib, A)$ and $\mathcal{L} = \{(J, \omega + ib, A)\}$. Then $\mathcal{Z}$ and $\mathcal{L}$ are, again, infinite dimensional complex manifolds, the complex structure on $\mathcal{L}$ coming from a one-to-one correspondence

$$(J, \omega + ib, A) \in \mathcal{L} \longleftrightarrow \bar{\ell} \le E \otimes \mathbb{C}.$$

Moreover, the first column of the Hull–Strominger system (the *F-term equations*) asserts the involutivity of $\bar{\ell}$.

$\mathcal{Z}$ admits a pseudo-Kähler structure $\omega$ determined by the functional

$$(J, \Omega, \omega + ib, A) \longmapsto \int_M \|\Omega\|_\omega \frac{\omega^n}{n!}.$$

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There is a subgroup $\mathcal{G} \leq \mathrm{Aut}_0(E)$ fitting into an exact sequence

$$0 \longrightarrow \Omega^2_{\mathrm{closed}}(M) \longrightarrow \mathcal{G} \longrightarrow \mathrm{Diff}^K_0(P) \longrightarrow 1$$

acting on $(\mathcal{Z}, \omega)$ by holomorphic isometries preserving the subset of solutions $\mathcal{Z}_0$ to the F-term equations.

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acting on $(\mathcal{Z}, \omega)$ by holomorphic isometries preserving the subset of solutions $\mathcal{Z}_0$ to the F-term equations. The action is Hamiltonian with moment map $\boldsymbol{\mu} \colon \mathcal{Z} \to (\mathrm{Lie}\,\mathcal{G})^*$, and if $\tilde{\boldsymbol{\mu}} = \mathrm{restr}_{\mathrm{Lie}\,\mathcal{H}} \circ \boldsymbol{\mu}_{|\mathcal{Z}_0}$,

$$\tilde{\boldsymbol{\mu}} = 0 \iff \bar\partial \Omega = 0, \quad d^\star \omega - d^c \log \|\Omega\|_\omega = 0, \quad F_A \wedge \omega^{n-1} = 0.$$

▶ Mario Garcia-Fernandez, Roberto Rubio, and Carl Tipler. In: *Journal of Differential Geometry* 128.1 (2024). eprint: 2004.11399

▶ Anthony Ashmore, Charles Strickland-Constable, David Tennyson, and Daniel Waldram. In: *Journal of High Energy Physics* 2020.11 (Nov. 2020)

Let $\mathbf{p} \in \tilde{\boldsymbol{\mu}}^{-1}(0)$ be a solution of the Hull–Strominger system. Let $\bar{\ell}$ be the corresponding isotropic, and consider the orthogonal decomposition

$$E = V_+ \oplus V_-$$

with $V_+ = (\ell \oplus \bar{\ell}) \cap E$, $V_- = V_+^\perp$. Then we may, again, construct two complexes

$$\Gamma(V_+) \xrightarrow{\mathbf{P}_+} T_p \mathcal{Z}_0 \xrightarrow{\mathbf{L}_+} \Gamma(V_+) \qquad (S_+^*)$$

$$\mathscr{C}^\infty(M) \xrightarrow{d_-} \Gamma(V_-) \xrightarrow{\mathbf{P}_-} H^1(S_+^*) \xrightarrow{\mathbf{L}_-} \Gamma(V_-) \qquad (S_+^*)$$

and define a second-order differential operator $\Delta_- \colon \Gamma(V_-) \to \Gamma(V_-)$ by

$$\Delta_- = \mathbf{P}_-^* \mathbf{P}_- + d_- d_-^*.$$

On the other hand, the complex vector bundle $V_- \otimes \mathbb{C}$ admits a natural pseudo-Hermitian metric $G$ and a natural holomorphic structure $\bar{\partial}^{V_- \otimes \mathbb{C}}$ such that under an identification

$$V_- \otimes \mathbb{C} \cong T^*_{1,0} M \oplus (\text{ad } P \otimes \mathbb{C}) \oplus T^{1,0}$$

we have

$$G = \begin{pmatrix} g^{-1} & 0 & 0 \\ 0 & \langle \cdot, \bar{\cdot} \rangle_{\mathfrak{k}} & 0 \\ 0 & 0 & g \end{pmatrix}, \quad \bar{\partial}^{V_- \otimes \mathbb{C}} = \begin{pmatrix} \bar{\partial} & \langle \cdot, F_A \rangle_{\mathfrak{k}} & \partial \omega \\ 0 & \bar{\partial}^A & F_A \\ 0 & 0 & \bar{\partial} \end{pmatrix}$$

Let $\Delta_- : \Gamma(V_- \otimes \mathbb{C}) \to \Gamma(V_- \otimes \mathbb{C})$ be the corresponding Dolbeault Laplacian. Then

$$\Delta_- \propto \Delta_{\text{Dolb}}.$$

## Theorem (GF, −)

Assume that $\dim \ker \Delta_{\mathrm{Dolb}} = b_1(M)$. Then we have an orthogonal decomposition

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## Corollary (GF, −)

Assume that $\dim \ker \Delta_{\text{Dolb}} = b_1(M)$, $h^{0,2} = 0$ and the $\partial\bar{\partial}$-lemma holds. Then the tangent space of $\mathcal{M}$ at $[\mathbf{p}]$ is naturally isomorphic to $H^1(V_- \otimes \mathbb{C})$.

▶ Xenia De la Ossa and Eirik Svanes. In: *Journal of High Energy Physics* 2014 (Feb. 2014)

▶ Anthony Ashmore, Xenia de la Ossa, Ruben Minasian, Charles Strickland-Constable, and Eirik Eik Svanes. In: *Journal of High Energy Physics* 2018.10 (Oct. 2018)