# Gauge Theory, Toda spaces & Coulomb branches

Physics and recently mathematics understanding:

3D (topological) gauge theory
is controlled by
Hyperkähler spaces closely related to Toda system

Pattern: Low energy behavior of a QFT should be
$\Longleftrightarrow$ sigma-model in moduli space of vacua $\mathcal{M}$

Some landmarks for gauge theory w/ linear matter:

- Seiberg and Witten on 3D pure gauge thy for SU(2)
- Argyres - Faraggi, Warner - generalization to SU(n)
- Seiberg - Intriligator on 3D mirror symmetry
- Witten - Hanany on Poincaré series for Coulomb branches

* Bezrukavnikov, Finkelberg, Mirkovic:
Topological description of Toda space from affine Grassmannian

- — Gromov-Witten boundary conds $\Longleftrightarrow$ holo. Lagrangians
- Bullimore - Dimofte - Gaiotto - abelian Coulomb branches
- Braverman - Finkelberg - Nakajima: Chiral rings for polarized reps
- Braverman et al: proposal for quaternionic reps
- — construction of chiral ring

In SUSY gauge theories 3D & higher: have
Coulomb and Higgs branches $\mathcal{M}_C, \mathcal{M}_H$ of $\mathcal{M}$
For 3D $X/G$, $X$ hyperkähler
Higgs: $X///G$ ; Coulomb: Toda + quantum corrections

## 2. The Toda spaces $\mathcal{C}_{3,4}(G;0)$

$\uparrow$ homology $\uparrow$ K-theory

- Hyperkähler manifolds;
- in one complex structure, completely integrable abelian gps over:

$$\mathcal{C}_3 \longrightarrow \mathfrak{g}_{\mathbb{C}} /\!/ G_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}/W \;;\; \mathcal{C}_4 \longrightarrow G_{\mathbb{C}} /\!/ G_{\mathbb{C}} = T_{\mathbb{C}}/W$$

- Abelian cases: $\mathcal{C}_3 = \dfrac{T^* T_{\mathbb{C}}^{\vee}}{W}$, $\mathcal{C}_4 = \dfrac{T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}}{W}$ resonatelim

- General cases: affine blow-ups of Weyl quotients

- BFM: $\mathbb{C}[\mathcal{C}_3(G;0)] = H_*^G(\Omega G)$ Pontryagin product
  $\mathbb{C}[\mathcal{C}_4(G;0)] = K_*^G(\Omega G)$ & homology co-product

  $\Longrightarrow$ Hopf algebras over $H_*^G$, $K_*^G$

- Thm (-) Some boundary conditions for 3D topological gauge thy correspond to bundles of categories w/ Lagrangian support on $\mathcal{C}_{3,4}$ (Kapustin - Rozansky - Saulina 2-category)

Eg from symplectic mfolds with Hamiltonian $G$ action:

- Symplectic cohomologies of certain open mfolds
- Quantum cohomologies of compact mfolds

Examples: - a point (Verlinde formulas)
- a cx. representation (Generalized & Coulomb br.)
- Compact Fanos (tomorrow)

## 3. Gauged point with a bulk deformation

$$W = \frac{h}{2}\cdot \xi^2, \quad \xi \in \mathfrak{g}_{\mathbb{C}} \quad (\text{invariant quadratic form})$$
$$h \in H^4(BG)$$

The exponentiated graph $\Gamma(dW)$ meets the unit section of the Toda groups at lattice points in $\mathfrak{t}_{\mathbb{C}}/W$, $T_{\mathbb{C}}/W$.

The Hessian determinants are the structure constants for a Frobenius algebra. This is the 2D TQFT "$*/G$" with bulk deformation $W$.

## 4. Complex representation $V$

Noncompact $\Rightarrow$ use $\mathbb{C}^\times$ scaling to render things finite

Equivariant parameter $\mu$ (complex mass in physics)
$\in H^2(B\mathbb{C}^\times)$

The associated Lagrangian is again $\Gamma(\exp(dW))$ for the GLSM superpotential in $H_*$ and $K_*$

$$\mathfrak{t}_{\mathbb{C}} \ni \xi \longmapsto \prod_{\text{wts}\cdot\nu} (\mu + \langle \nu | \xi \rangle)^\nu$$
$$\in T_{\mathbb{C}}^\vee \qquad \text{Toda sections}$$
$$T_{\mathbb{C}} \ni x \longmapsto \prod_\nu (1 - m^{-1} x^{-\nu})^\nu$$

open Q: extend to moduli of curves

**Thm** The associated TQFT computed by intersecting with the unit section is the Gromov-Witten gauged theory $V/G$ (w/ Chris Woodward, generalizing Witten)

# 5. Main Theorem on Chiral Rings $\mathcal{C}_{3,4}(G;E)$

$G$ = compact connected Lie gp; $E$ = quaternionic rep;
"polarized" means $E = V \oplus V^*$

Nakajima; Bullimore–Dimofte–Gaiotto; yours truly;
Braverman–Finkelberg–Nakajima;

1. There exist[4] constructible, equivariant coefficient systems $\mathcal{H}_E$, $\mathcal{K}_E$ over the loop Grassmannian $G_{\mathbb{C}}[[z]]\backslash G_{\mathbb{C}}((z))/G_{\mathbb{C}}[[z]]$

$G \backslash \Omega G = G \backslash {}^{L}G/G$

2. They are $E_2$-multiplicative under Pontryagin products and their equivariant cohomologies $\mathcal{C}[\mathcal{C}_{3,4}(G;E)]$ are $E_3$ ("Poisson structures of degree $-2$")

3. They are multiplicative in $E$, $\mathcal{H}_E \otimes \mathcal{H}_F \longrightarrow \mathcal{H}_{E\oplus F}$
so $\mathcal{C}_{3,4}(G;E) \underset{\text{Toda}}{\times} \mathcal{C}_{3,4}(G;F) \longrightarrow \mathcal{C}_{3,4}(G;E\oplus F)$

4. Non-polarized $E$ require the removal of obstructions

5. $H^G_*(\Omega G; \mathcal{H}_E)$ and $K^G_*(\Omega G; \mathcal{K}_E)$ are birational to $\mathcal{C}_3$, $\mathcal{C}_4$ and are expected to be the chiral rings for $E/G$

6. (Abelianization) $\mathcal{C}_{3,4}(G;E) \cong \mathcal{C}_{3,4}(T; E-\mathfrak{g}_{\mathbb{H}})/W$
if $E$ contains the roots of $\mathfrak{g}$. [–]

7. Polarized case: construction from GLSM boundary cond.
[–]

## 6. Construction in the Polarized case

(Physics; Nakajima; B-F-N; BDG)

Morally Choose a polar half $V$ of $E$

Get an index bundle "$H^0 - H^1$"$(\mathbb{P}^1; P \times_G V \otimes \sqrt{K})$ along $\mathbb{P}^1$ over $Bun_{G_\mathbb{C}}(\mathbb{P}^1) \sim {}_G\backslash \Omega G = {}_G\backslash LG/G$

Build the associated linear space Spec Sym (dual sheaf)

Coefficient systems $\mathcal{H}_E$, $\mathcal{K}_E$ are cohomologies with compact vertical supports

Morally $\mathcal{C}_{3,4}(G;E) = \operatorname{Spec} H^*_G\, K^*_G(\Omega G; \mathcal{H}_E, \mathcal{K}_E)$
with Pontryagin products.

unit = volume form $\Rightarrow$ difficult to make precise

Product structure should come from 3D pair of pants by solving a gauged Dirac equation w/ prescribed boundary conditions

$$\begin{tikzcd}[column sep=small] & \left(\begin{array}{c}\text{moduli space}\\ \text{of solutions}\end{array}\right) \arrow[dl] \arrow[dr] & \\ Bun_G(\mathbb{P}^1) \times Bun_G(\mathbb{P}^1) & & Bun_G(\mathbb{P}^1)\end{tikzcd}$$

## 7. Algebraic Geometry Rewording (B F N)

The splitting $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ reduces the 3D Dirac equation to the $\bar\partial$ equation (and TQFT $\Rightarrow$ constant in $t$)

$\Rightarrow$ complex geometry can be used:

Use "tiny sphere" —:— click with doubled origin

"Hecke correspondence diagram"

$$\begin{array}{ccc} & \mathrm{Bun}(—:—) & \\ \swarrow & & \searrow \\ \mathrm{Bun}(—:—) \times \mathrm{Bun}(—:—) & & \mathrm{Bun}(—:—) \end{array}$$

top sheets, bottom sheets, outer sheets

The correspondence diagram is now well-defined and gives an $E_3$ multiplication on $H^G_*(\Omega G; \mathcal{H}_E, \mathcal{K}_E)$.

## 8. Global construction from GLSM

$\mathcal{C}_{3,4}(G;E)$ arises by gluing two copies of the Toda space along the vertical shear by $\exp(dW)$ from GLSM.

Equivalently: The chiral ring for $E$ is the subring of functions on the Toda space which survive $\exp(dW)$ translation

Reformulation (Pomerleano): This is the subring of functions that preserve the lattice $QH^*_G(V) \subset SH^*_G(V)$ (including its bulk deformations).

# 9. Non-polarized case: $E \neq V \oplus V^*$

- I don't have a good interpretation in terms of Gromov-Witten boundary conditions.

Given: in terms of $G \underset{T}{\times} V$ ($E$ is a double over $T$)

the formula I have is not 'clean' though

Caution: check paper linked from my website; the arXiv version has many calculational mistakes

Problem: Invoking the construction for $E$ instead of $V$ leads to $\mathcal{C}_{3,4}(G; E \oplus E)$.

Need to extract "square roots" of the $\mathcal{H}, \mathcal{K}$

Method: check real structures.

$V \mapsto V \oplus V^*$ skew quaternionic

Investigate:

$\in KO^{-1}(pt)$

$$
\begin{tikzcd}
 & & & KO \arrow[r, hook] & KU \arrow[d] & \\
BG \arrow[r, "E"] & BSp \arrow[r, squiggly, "\Omega^2"] & \Omega G \arrow[rr, "\Omega^2 E", "G\text{-map}"'] \arrow[urr, dashed] & & \Sigma^2 KO \arrow[r, "\eta"] & \Sigma KO
\end{tikzcd}
$$

Polarization of $E$ would lift $\Omega^2 E$ to $KU$

Obstructed by $\eta \circ \Omega^2 E \in KO^1$

In any case: want an $E_2$ lift so obstruction really is

$$BG \xrightarrow{E} BSp \xrightarrow{\eta} \Sigma^3 KO$$

Seems unhelpful until we recall that

we don't need a complete lift!
Just enough to build the coefficient systems.

So the obstruction is the image, via $\Sigma^4 J$, into
$\Sigma^4 GL_1(H\mathbb{Z})$ or $\Sigma^4 GL_1(KU)$ (or $\Sigma^4 GL_1(KO)$)

For cohomology: obstruction class in $H^4(BG;\mathbb{Z}/2)$
$(W_1)$ and is $c_2(E) \bmod 2 = w_4(E)$

For KO-theory: a secondary obstruction $\sigma \in H^5(BG;\mathbb{Z}/2)$
$(W_2)$ is defined if $w_4(E) = 0$

For KU-theory: the $2^{nd}$ obstruction is $B\sigma \in H^6(BG;\mathbb{Z})$
$(W_3)$ (Essentially $\frac{1}{2}c_3(E)$)

**Theorem** (nasty calculation)

If $G$ is connected and $w_4(E) = 0$, then $B\sigma = 0$.

(Fails for disconnected groups)

**Improvements** · One can weaken the obstruction to
$w_4$ is the square of a class in $H^2(BG;\mathbb{Z})$

Witten: obstruction is im $\pi_4 G \xrightarrow{E} \pi_4 Sp$

· One can even reduce to the obstruction predicted by Ed Witten $\Longleftarrow$ $w_4$ has a square root $\in H^2(BG;\mathbb{Z}/2)$

at the price of collapsing the cohomology grading mod 2:

$$BG \xrightarrow{E} BSp \xrightarrow{\eta} \left.\begin{array}{cc} 0 & \\ \mathbb{Z}/2 & 5 \\ \mathbb{Z}/2 & 4 \\ \mathbb{Z} & 3 \\ 0 & \end{array}\right\} \Sigma^3 KO$$

homology grading $\longrightarrow$