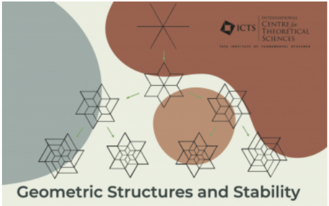
Non-reductive group actions: jet moduli spaces

Gergely Bérczi
Aarhus University



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2026

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3. **Cohomology of $X//H$:** Our theory works when $s = ss$. Is there a (stacky) integration theory in the presence of stabilisers?

Moduli of jets

Motivation: X-complex manifold. (Jets of) holomorphic maps $f : \mathbb{C} \to X$ and their moduli space lie at the heart of classical problems in enumerative, complex and arithmetic geometry.

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- We call $J_k(1,n)/\mathrm{Diff}_k$ the **moduli of k-jets of germs**, or **jet moduli space of order k**

Jet bundles

- Let $f:(\mathbb{C},0)\to(X,x)$, $\quad f(t)=(f_1(t),f_2(t),\dots,f_n(t))$ be a curve written in some local holomorphic coordinates t on $\mathbb{C}$ and $(z_1,\dots,z_n)$ on X. The k-jet bundle is
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- Diff_k acts fiberwise with fibration $J_k X / \mathrm{Diff}_k \to X$

Jet spaces, Arc spaces:

$$\Delta_m = \mathrm{Spec}\, \mathbb{C}[t]/(t^{m+1}) \quad \text{and} \quad \Delta = \mathrm{Spec}\, \mathbb{C}[[t]].$$

There are obvious inclusion maps

$$\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_m \subset \cdots \subset \Delta. \tag{1}$$

Definition[Jets on a scheme] Let X be a scheme over $\mathbb{C}$. Set-theoretically, the space of m-th order jets on X consists of the morphisms of schemes over $\mathbb{C}$:

$$X_m := \mathrm{Hom}(\Delta_m, X), \quad X_\infty := \mathrm{Hom}(\Delta, X).$$

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 & J_k(1,n)/\mathrm{Diff}_k \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Curvilinear Hilbert scheme} & \text{NRGIT quotient} & \text{Demailly-Semple bundle}
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$\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) = \overline{\{I \subset \mathfrak{m} : \mathfrak{m}/I \simeq t\mathbb{C}[t]/t^k\}}$

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Curvilinear Hilbert scheme

$$\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) = \overline{\{I \subset \mathfrak{m} : \mathfrak{m}/I \simeq t\mathbb{C}[t]/t^k\}}$$

NRGIT quotient

$$\mathbb{P}(\mathbb{C} \oplus J_k(1,n))//\mathrm{Diff}_k$$

Demailly-Semple bundle

$$X_k \to \ldots \to X_1 = \mathbb{P}^{n-1}$$

- singular projective
- endowed with tautological bundle,
- tautological intersection theory encodes enumerative geometry problems
- **enumerative geometry**
- **arithmetic geometry**

- nonsingular projective
- JK-type integral formula for intersection theory
- **hyperbolicity conjectures**

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$R = \mathbb{C}[x_1, \dots, x_n]$.

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On surfaces:

- $\mathrm{Hilb}^k(\mathbb{C}^2) = \mathrm{GHilb}^k(\mathbb{C}^2)$ is smooth of dimension $2k$
- $\mathrm{Hilb}^k_0(\mathbb{C}^2) = \mathrm{CHilb}^k(\mathbb{C}^2)$ is irreducible, singular of dimension $(k-1)$ (Briancon) and moreover, a complete intersection: $\mathrm{CHilb}^k(\mathbb{C}^2) = s^{-1}(0)$ for a section of $B = \mathcal{O}^{[k]}/\mathcal{O}$ (Haiman).

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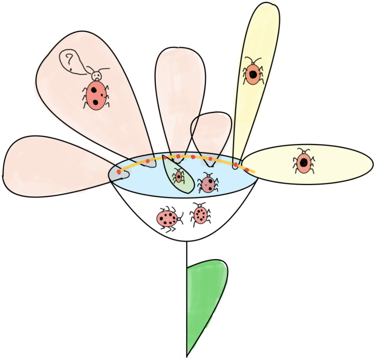
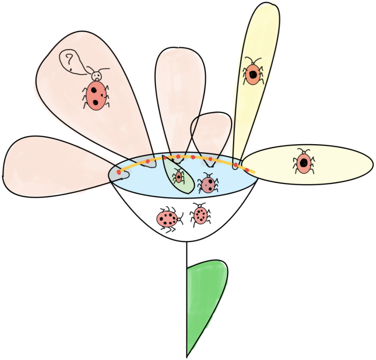
For $n > 2$ $\mathrm{Hilb}^k(\mathbb{C}^n)$ has pathological behaviour, Murphy's law (Vakil). Components, singularities, deformation theory is hard (out of reach).

The refined Bellis Hilbertis (following J. Jelisiejew)





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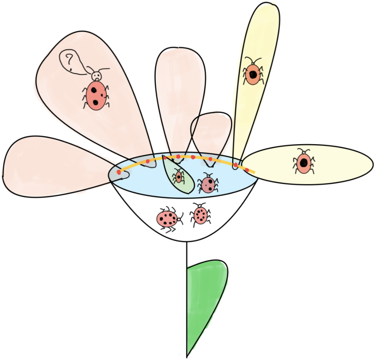

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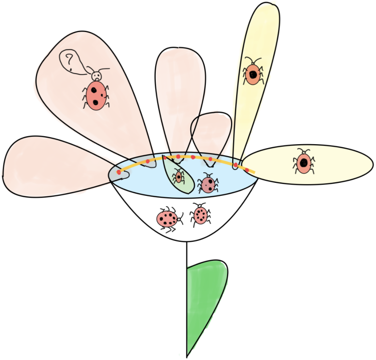

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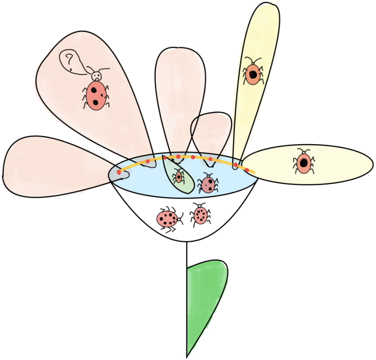

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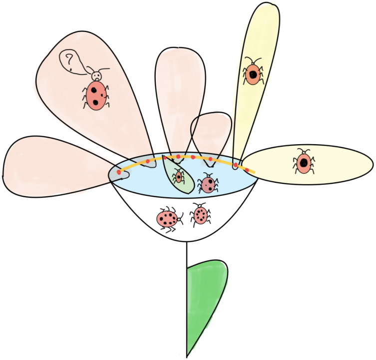

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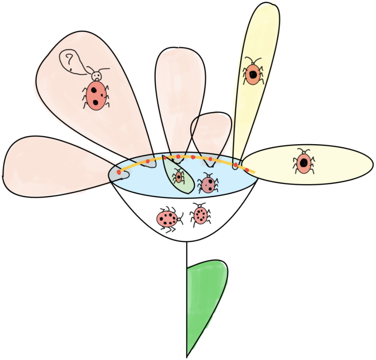

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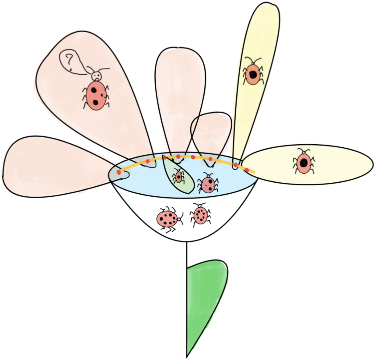

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- **Theorem (B-Svendsen '23):** $\mathrm{CHilb}^k(\mathbb{C}^n)$ contains all torus fixed points (red dots). That is: monomial ideals deform to a smooth jet of curve.

Tautological intersection theory of Hilbert schemes

Tautological bundles: $V \to X$ rank r bundle $\rightsquigarrow V^{[k]} \to \mathrm{Hilb}^k(X)$ rank rk bundle

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Geometric subsets: $A_1, \ldots, A_s$ be finite dimensional quotient algebras of $\mathbb{C}[x_1, \ldots, x_n]$ of dimension $\dim_{\mathbb{C}}(A_i) = k_i$ with $k = k_1 + \ldots + k_s$.

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Theorem, B-Szenes 2021 If f is sufficiently generic (stable) then

$$m_k = \int_{\mathrm{GHilb}^{[k]}(M)} \mathrm{Euler}((f^*TM)^{[k]}/f^*TM))$$

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Göttsche '95: S surface, $A_1 = \ldots = A_s = \mathbb{C}[x,y]/(x^2, xy, y^2)$, and $\mathrm{Hilb}^{A_1,\ldots,A_s}(S) \subset \mathrm{Hilb}^{3s}(S)$.

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The same exponential expansion holds for any multiplicative characteristic class, and in particular Chern series. Similar formula for Verlinde numbers. Higher dimensional Segre-Verlinde duality?

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Nimma gamanakke dhanyavādagalu.

p-adic integrals

5. Igusa zeta function and Monodromy Conjecture

- p prime, then any $q \in \mathbb{Q}$ uniquely $x = p^m \frac{a}{b}$ with some $m, a, b \in \mathbb{Z}$. Order and the norm of x: $\mathrm{ord}_p(x) = m$ and $|x|_p = \frac{1}{p^m}$.

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- For measurable h one can integrate: $\int_{Q_p, Z_p} h d\mu$ Change of variables:

$$\phi : U \to V \text{ then } \int_{\phi(U)} h d\mu = \int_U (h \circ \phi) \cdot \mathrm{Jac}(h) d\mu$$

.

- More generally: A p-adic field K is a finite extension of Q_p. The ring of integers $O_K \subset K$ is the integral closure of Z_p in K. This is a local ring with maxiomal ideal $\mathfrak{m}_K$ and $O_K/\mathfrak{m} = F_q$ for some $q = p^r$.

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Idea of proof: Let $V_m = \{O_K^d : |f(x)| \leq \frac{1}{q^m}\}$ so the level set is $V_m \setminus V_{m-1}$. Then

$$\mu(V_m) = N_m \frac{1}{q^{md}}$$

Monodromy Conjecture (Igusa) Let s be a pole of $Z(f,s)$. Then $e^{2\pi is}$ is an eigenvalue of the monodromy action on some Milnor fiber $H^i(M_{f,x})$ at some point of $x \in f^{-1}(0)$
B-Rossinelli, in progress: Recursive formula for the Igusa zeta function without resolutions. The Igusa zeta function is the generating function of the number of points on certain components of $\mathrm{Hilb}^k(f)$.