Non-reductive moment maps and cohomology of quotients

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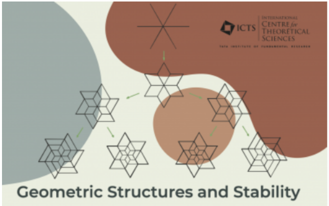

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2. (Hilbert -Mumford) $X^{s,\hat{U}} = X^{ss,\hat{U}} = \bigcap_{u \in U} uX^{s,\lambda(\mathbb{C}^*)} = X_{\min} \setminus UZ_{\min}$.

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- For $H = U \rtimes R$ more candidates.

Toy examples: why candidate blow-ups do not work?

Example 1 Blowing up at $X_{\min}^{d_{\max}}$ does not work

- $$H = \left\{ \begin{pmatrix} t_1 & u_2 & u_3 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{C}^*, u_1, u_2 \in \mathbb{C} \right\} \subset \mathrm{GL}(3, \mathbb{C})$$

  acts on the space of projectivized $3 \times n$ matrices $X = \mathbb{P}(r_1, r_2, r_3)$

- $$\lambda(\mathbb{C}^*) = \left\{ \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t^1 & 0 \\ 0 & 0 & t^{-3} \end{pmatrix} \middle| t \in \mathbb{C} \right\}$$

  so $Z_{\min} = \mathbb{P}(0, 0, r_3)$, and $d_{\max}(Z_{\min}) = 0, d_{\max}(X_{\min}) = 1$.

- $X_{\min}^1 = \{[M] : \mathrm{rank}(r_2, r_3) = 1 \ (r_1 \text{ is arbitrary})\}$
  $UZ_{\min} = \{[M] : \mathrm{rank}(r_1, r_3) = 1, r_2 = 0, r_3 \neq 0\}$

- Blow-up at $X_{\min}^1$:

  $$\hat{X} = \left\{ ([M], [N]) \in \mathbb{P}(\mathrm{Mat}_{3xn}) \times \mathbb{P}(\mathrm{Mat}_{1xn}) \ \middle| \ \begin{array}{l} N = r_2^{\perp}(r_3) \text{ if } \mathrm{rank}(r_2, r_3) = 2 \\ N \in r_2^{\perp} \text{ if } \mathrm{rank}(r_2, r_3) = 1 \end{array} \right\}.$$

  $$h \cdot ([M], [N]) = ([hM], [N]) \text{ for } h \in H$$

  and hence the action on the blow-up space is the same as on the original, we do not gain anything by blowing-up this way:

  $$\hat{d}_{\max}(\hat{X}_{\min}) = d_{\max}(X) = 1$$

Example 2 Blowing up at $UZ_{\min}^{d_{\max}}$ does not work.

- $H = \hat{U} = \left\{ \begin{pmatrix} t & 0 & s & v \\ 0 & t & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{C}^*, s, v \in \mathbb{C} \right\} \subset \mathrm{GL}(4, \mathbb{C})$ acts on the space of projectivized $4 \times n$ matrices $X = \mathbb{P}(r_1, r_2, r_3, r_4)$
-
$$Z_{\min} = \{r_1 = r_2 = 0\} \supset Z_{\min}^{d_{\max}} = \{r_1 = r_2 = r_3 = 0\}$$
  and $d_{\max} = 1$ where
$$\mathrm{Stab}_U(y) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : s \in \mathbb{C} \right\} \subset U \text{ for } y \in Z_{\min}^1$$
- We blow up at
$$UZ_{\min}^{d_{\max}} = \{[r_1, r_2, r_3, r_4]^T : r_2 = r_3 = 0, \mathrm{rank}(r_1, r_4) = 1\}$$
  Take the point $(z, \tilde{z})$ where
$$z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Z_{\min}^{U'} \text{ and } \tilde{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \end{bmatrix},$$
  where the latter matrix represents a point over z. Check: $\mathrm{Stab}_U(z, \tilde{z}) = U'$.

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   - $d_{\max} = 0$ hence $s = ss$.

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- If a complex projective variety $X \subset \mathbb{P}^n$ is acted on by $G = K^{\mathbb{C}}$ via a representation $\rho : G \to \mathrm{GL}(n+1, \mathbb{C})$ such that $\rho(K) \subset U(n+1)$ and $\omega = \omega_{\mathrm{FS}}$ is the Fubini-Study Kähler form, then the K-action on X is Hamiltonian with moment map
$$\mu([x_0 : \cdots : x_n]).a = \frac{x\,\rho^*(a)\,x^\top}{2\pi i\,||x||^2}$$
for all $a \in \mathfrak{k}$ and $x = (x_0, \ldots, x_n)$.
- Fix $a \in \mathfrak{k}$. The component
$$\mu_a : X \to \mathbb{R} \text{ given by } x \mapsto \mu(x) \cdot a$$
  - is a Hamiltonian function of the vector field $x \mapsto a_x$ on X induced by a.
  - If X is Kähler then $\omega(\xi, a_x) = g(\xi, ia_x)$ hence the gradient flow of μ_a is given by
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(c) We use the gradient trick.

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Theorem (Kirwan) $f = ||\mu||^2 : X \to \mathbb{R}$ is a Morse function, and Morse stratification= GIT stratification. The stratification is equivariantly perfect, hence Morse-equalities combined with Kirwan surjectivity gives formula for cohomology of $X//G$.

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2. The Ω-moment map for G acting on Y is a smooth G-equivariant map

$$m = m_{\Omega,Y,G} : \Omega \times Y \to \mathfrak{g}^*$$

   such that for each $(K, \omega) \in \Omega$, the map

$$x \mapsto m((K, \omega), x)$$

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$$Y \xrightarrow{\mu_{(K,\omega)}} \mathfrak{k}^* \hookrightarrow \mathfrak{g}^*,$$

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Non-reductive symplectic reduction

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Apply this with $g(x) = \mu^{H}_{(K,\omega)}(x) \cdot a$ and $f(x) = \mu^{\lambda(\mathbb{C}^*)}_{(K,\omega)}(x) \cdot a$.

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2. the equivariant Euler class of the normal bundle to $UZ_{\min}(X)$ in $X^0_{\min}$ is not a zero divisor in $H^*_{S^1}(UZ_{\min}(X))$.

Two points and a line in $\mathbb{P}^2$

- Let $X = (\mathbb{P}^2)^2 \times (\mathbb{P}^2)^*$ acted on by $\hat{U} = \begin{pmatrix} t & a & b \\ 0 & t^2 & c \\ 0 & 0 & t^3 \end{pmatrix}$
- We represent:
  - points in $\mathbb{P}^2$ by column vectors, with the action of a matrix A given by pre-multiplication by A;
  - lines
    $$L_{(a,b,c)} = \{[x:y:z] \in \mathbb{P}^2 | ax + by + cz = 0\}$$
    by row vectors (a, b, c) and the action of A is given by post-multiplication by A^{-1}.
  - The weights of the action of the maximal torus T of B on X lie in an irregular hexagon (which is their convex hull) in the dual of the Lie algebra of the maximal compact subgroup of T.
- $\lambda : \mathbb{C}^* \to T, \lambda(t) = (t, t^2, t^3)$ gives positively graded extension $\hat{U} = \mathbb{C}^* \rtimes U$. The minimal weight corresponds to the T-fixed point
  $$X_{\min}^{\mathbb{C}^*} = z_{\min} = ([0,0,1],[0,0,1], L_{(1,0,0)})$$
- The stabiliser in U of $z_{\min}$ is trivial, and the closure of its U-orbit is
  $$\overline{Uz_{min}} = \{(p,q,L) \in X | p = q \in L\},$$
  $$X_{\min} = \{([x_1,y_1,z_1],[x_2,y_2,z_2], L_{(a,b,c)}) \in X | z_1 \neq 0 \neq z_2, a \neq 0\}$$
- $\hat{U}$-Theorem: $X^{ss,\hat{U}} = X_{\min} \setminus Uz_{\min} = X_{\min} \setminus \overline{Uz_{\min}}$ has a projective geometric quotient $X//\hat{U} = (X_{\min} \setminus Uz_{\min})/\hat{U}$ when the linearisation is twisted by a rational character χ/c in the interior of the hexagon near to the T-weight for $z_{\min}$.

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Betti numbers: $P_t(X//\hat{U}) = P_t(Z_{\min})\frac{1-t^{2d}}{1-t^2} = \frac{1-t^4}{1-t^2} = 1+t^2+t^4$.
Note: this way we can also describe Hodge numbers of the quotient.

Reductive abelianisation: The following diagram of Shaun Martin relates $X//G$ and $X//T_{\mathbb{C}}$ through a fibering and an inclusion:

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Classical results:

- (SM,'00) $H^*(X//G,\mathbb{Q}) \simeq \frac{H^*(X//T,\mathbb{Q})^W}{ann(e)}$ where $\mathrm{ann}(e) = \{c \in H^*(X//T,\mathbb{Q})^W | c \cup e = 0\}$.

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Non-reductive case:

$$\begin{array}{ccccc} \mu_K^{-1}(0)/S^1 & \xhookrightarrow{j} & \mu_{\hat{U}}^{-1}(0)/S^1 = X//\hat{U} & \xhookrightarrow{i} & \mu_{S^1}^{-1}(0)/S^1 = X//\mathbb{C}^* \\ \big\downarrow \pi & & & & \\ \mu_K^{-1}(0)/K = X//G & & & & \end{array} \quad (2)$$

The description of the corresponding normal bundles is similar using the line bundles corresponding to the roots of U.

Theorem (B-Kirwan '19)

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$$H^*(X//\hat{U},\mathbb{Q}) \simeq \frac{H^*(X//\mathbb{C}^*,\mathbb{Q})}{ann(e(V_{\mathfrak{u}}))}$$

where $e(V_{\mathfrak{u}}) \in H^*(X//\mathbb{C}^*)$ is the Euler class of the bundle $V_{\mathfrak{u}}$ and

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- For any Chern polynomial $\phi(V) \in \mathbb{C}[c_1(V), c_2(V), \ldots]$ whose degree is the dimension of $X//\hat{U}$ we have

$$\int_{X//\hat{U}} \phi(V) = n_{\mathbb{C}^*} \mathrm{Res}_{z=\infty} \int_{Z_{\min}} \frac{i^*_{Z_{\min}}(\phi(V) \cup e(V_{\mathfrak{u}}))dz}{e^T(\mathcal{N}_{Z_{\min}})}$$