Non-reductive group actions and quotients

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Geometric Structures and Stability
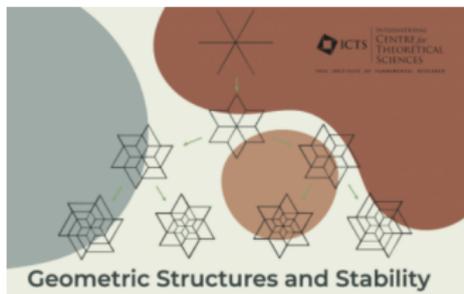
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$$\mathrm{Lie}(K_{\mathbb{C}}) \cong \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}.$$

E.g. $K = U(n)$ and $K_{\mathbb{C}} = GL(n)$ or $K = SU(n)$ and $K_{\mathbb{C}} = SL(n)$.

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Alternative definition: G is reductive $\Leftrightarrow$ every representation of G is the direct sum of irreducibles.

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$\mathbb{P}(1,1,2) = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ with weights 1,1 and 2, is

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where

- $R \cong GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* \cong GL(2)$ reductive
- $U \cong (\mathbb{C}^+)^3$ unipotent acting via

$$(x, y, z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2) \text{ for } (\lambda, \mu, \nu) \in \mathbb{C}^3.$$

$$\mathrm{Aut}(\mathbb{P}(1,1,2)_2) = \left(\begin{array}{ccc|c} & & & \nu \\ & \mathrm{Sym}^2 GL(2) & & \mu \\ & & & \lambda \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$$

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- Topologically: $X /\!/ G = X^{ss}/\sim$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.
  As a set: $X /\!/ G = \{\text{closed orbits of } G \text{ on } X\}$.

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Let $T = (\mathbb{C}^*)^r$ be a maximal torus of G, choose homogeneous coordinates on $\mathbb{P}^n$ so that T acts diagonally with weights $\alpha_0, \ldots, \alpha_n \in \mathrm{Lie}(T)^*$, that is

$$[x_0 : \ldots : x_n] \mapsto [t^{\alpha_0} x_0 : \ldots : t^{\alpha_n} x_n] \text{ for some } \alpha_i \in \mathbb{C}^r.$$

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Then $x = [x_0 : \cdots : x_n] \in X$ is **semistable (respectively stable) for the action of T** iff 0 lies in (respectively lies in the interior of) the convex hull in $\mathrm{Lie}(T)^*$ of the set

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Moreover x is **semistable (respectively stable) for the action of G** iff gx is semistable (respectively stable) for the action of T for **every** $g \in G$:

$$X^{ss, G} = \cap_{g \in G} g X^{ss, T}$$

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   Example: $\mathbb{C}^+ = \{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \}$ acts on $X = \mathrm{Sym}^2 \mathbb{C}^2$. Let

   $x_0 = y^2, x_1 = xy, x_2 = x^2$ be basis of X. Then $A(X)^{\mathbb{C}^+} = \mathbb{C}[x_0, x_1^2 - x_0 x_2]$ and the orbit space looks like:
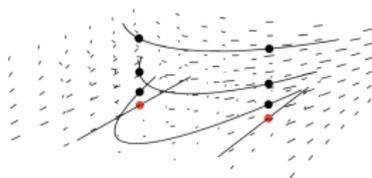
The pairs of lines $\{x_0 = 0, x_1 = \pm a\}$ for any given $a \neq 0$ are not distinguished.

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The enveloping quotient is defined as

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\phi : X^{ss} \to X//_e H = \bigcup_{f \in I^{ss}} \mathrm{Spec}((A(X)^H)(f)) \subseteq \mathrm{Proj}(A(X)^H),
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We say that $H = U \rtimes R$ is graded if there is a central $\mathbb{C}^* \subset R$ such that

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In most applications the acting non-reductive group is graded.

1) Moduli spaces of toric hypersurfaces

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = R \ltimes U \text{ where } R = GL(2) \text{ and } U = (\mathbb{C}^+)^3$$

$$(x, y, z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2) \text{ for } (\lambda, \mu, \nu) \in (\mathbb{C}^+)^3$$

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The central one-parameter subgroup $\mathbb{C}^* = \left(\begin{array}{cccc} t^2 & & & \\ & t^2 & & \\ & & t^2 & \\ 0 & 0 & 0 & 1 \end{array} \right)$ of $R \cong GL(2)$ acts

on $Lie(U)$ with weight 2.

Application: K-stability: For certain hypersurfaces K-stability = NRGIT stability

2) **Reparametrisation groups**

- $J_k(1, n) = \{k$-jets of germs of holomorphic maps $(\mathbb{C}, 0) \to (\mathbb{C}^n, 0)\} =$
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$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

$$= (f', \ldots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} = \mathbb{C}^* \rtimes \mathbf{U}_k$$

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Applications This group plays a central role in many applications:

- Enumerative geometry of Hilbert schemes of points: tautological intersection theory, curve counting
- Global singularity theory: Thom polynomials and multisingularity locus of maps
- Hyperbolicity questions and in particular the Kobayashi and Green-Griffiths-Lang conjecture
- Jet and arc spaces, Igusa zeta function and the Monodromy Conjecture

3) **GIT stratification** Given a linear G-action, X has a stratification $X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$ into locally closed subsets S_β indexed by

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Applications Construction of moduli of unstable objects: Quotient of S_β by G is thje quotient of $\overline{Y}_\beta$ by the parabolic subgroup P_β, which is a positively graded group.

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Application: Algebraic statistics, MLE estimation of Gaussian graphical group models. Popov-Pommerening conjecture (1970): for any affine H_S-variety X the algebra of invariants $k[X]^{H_S}$ is finitely generated.

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3. (**Hilbert-Mumford criterion**) $X^{s,\hat{U}} = X^{ss,\hat{U}} = \bigcap_{u \in U} uX^{s,\lambda(\mathbb{C}^*)} = X_{\min} \setminus UZ_{\min}$.

Remarks

- Theorem works for graded linear groups in general.
- Nice VGIT picture for $H = U \rtimes R$ action when R contains higher dimensional grading torus, that is, the grading $\mathbb{C}^*$ can be moved inside R.

Nagata counterexample:

$$\begin{pmatrix} 1 & t_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & t_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & t_{16} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$t_i = \sum_{m=1}^{13} b_{im} u_m, \quad \text{with } B \in \mathrm{Mat}_{13x16} \text{ suff. generic}$$

NRGIT Theorem:

$$\begin{pmatrix} t^{15\epsilon} & t_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^{14\epsilon} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^{13\epsilon} & t_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^{12\epsilon} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^{\epsilon} & t_{16} \\ 0 & 0 & 0 & 0 & \cdots & 0 & t^{-\epsilon} \end{pmatrix}$$

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3. Then one can deduce that

$$p : X_{\min} \to Z_{\min}(X) \text{ induces } p : X_{\min}/U \to Z_{\min}(X)$$

   which is $\lambda(\mathbb{C}^*)$-invariant, and has affine fibers F_Z over $z \in Z_{\min}(X)$ acts on each F_Z with positive weights. Hence $X_{\min} \setminus UZ_{\min} \subseteq X^{s,\hat{U}}$.

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which is $\lambda(\mathbb{C}^*)$-invariant, and has affine fibers F_Z over $z \in Z_{\min}(X)$ acts on each F_Z with positive weights. Hence $X_{\min} \setminus UZ_{\min} \subseteq X^{s,\hat{U}}$.

4. Thus, we have

$$p : (X_{\min} \setminus UZ_{\min})/\hat{U} \to Z_{\min}(X)$$

with weighted projective spaces as fibers. The source of this map is projective, hence the invariants are finitely generated and $X /\!/_e \hat{U} = .(X_{\min} \setminus UZ_{\min})/\hat{U}$.

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- For $H = U \rtimes R$ more candidates.

Example 1 Blowing up at $X_{\min}^{d_{\max}}$ does not work

- $$H = \left\{ \begin{pmatrix} t_1 & u_2 & u_3 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{C}^*, u_1, u_2 \in \mathbb{C} \right\} \subset \mathrm{GL}(3, \mathbb{C})$$

  acts on the space of projectivized $3 \times n$ matrices $X = \mathbb{P}(r_1, r_2, r_3)$

- $$\lambda(\mathbb{C}^*) = \left\{ \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t^1 & 0 \\ 0 & 0 & t^{-3} \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\}$$

  so $Z_{\min} = \mathbb{P}(0, 0, r_3)$, and $d_{\max}(Z_{\min}) = 0$, $d_{\max}(X_{\min}) = 1$.

- $X_{\min}^1 = \{[M] : \mathrm{rank}(r_2, r_3) = 1 \ (r_1 \text{ is arbitrary})\}$
  $UZ_{\min} = \{[M] : \mathrm{rank}(r_1, r_3) = 1, r_2 = 0, r_3 \neq 0\}$

- Blow-up at $X_{\min}^1$:

  $$\hat{X} = \left\{ ([M], [N]) \in \mathbb{P}(\mathrm{Mat}_{3 \times n}) \times \mathbb{P}(\mathrm{Mat}_{1 \times n}) \,\middle|\, \begin{array}{l} N = r_2^{\perp}(r_3) \text{ if } \mathrm{rank}(r_2, r_3) = 2 \\ N \in r_2^{\perp} \text{ if } \mathrm{rank}(r_2, r_3) = 1 \end{array} \right\}.$$

  $$h \cdot ([M], [N]) = ([hM], [N]) \text{ for } h \in H$$

  and hence the action on the blow-up space is the same as on the original, we do not gain anything by blowing-up this way:

  $$\hat{d}_{\max}(\hat{X}_{\min}) = d_{\max}(X) = 1$$

Example 2 Blowing up at $UZ_{\min}^{d_{\max}}$ does not work.

- $H = \hat{U} = \left\{ \begin{pmatrix} t & 0 & s & v \\ 0 & t & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{C}^*, s, v \in \mathbb{C} \right\} \subset \mathrm{GL}(4, \mathbb{C})$ acts on the space

  of projectivized $4 \times n$ matrices $X = \mathbb{P}(r_1, r_2, r_3, r_4)$

-
$$Z_{\min} = \{r_1 = r_2 = 0\} \supset Z_{\min}^{d_{\max}} = \{r_1 = r_2 = r_3 = 0\}$$

  and $d_{\max} = 1$ where

$$\mathrm{Stab}_U(y) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : s \in \mathbb{C} \right\} \subset U \text{ for } y \in Z_{\min}^1$$

- We blow up at
$$UZ_{\min}^{d_{\max}} = \{[r_1, r_2, r_3, r_4]^T : r_2 = r_3 = 0, \mathrm{rank}(r_1, r_4) = 1\}$$

  Take the point $(z, \tilde{z})$ where

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Z_{\min}^{U'} \text{ and } \tilde{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \end{bmatrix},$$

  where the latter matrix represents a point over z. Check: $\mathrm{Stab}_U(z, \tilde{z}) = U'$.

Theorem (B-Kirwan, 2025) For iterated blow-up along $UZ_{\min}^{d_{\max}}$ the lexicographic pair $(d^{\max}, \dim(UZ_{\min}^{d_{\max}}))$ is a delayed Lyapunov (ranking) function for non-reductive GIT blow-ups, with delay 3. Hence blowing-up terminates.

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Thank you for your attention.