# Qubit Regularization of Asymptotic Freedom

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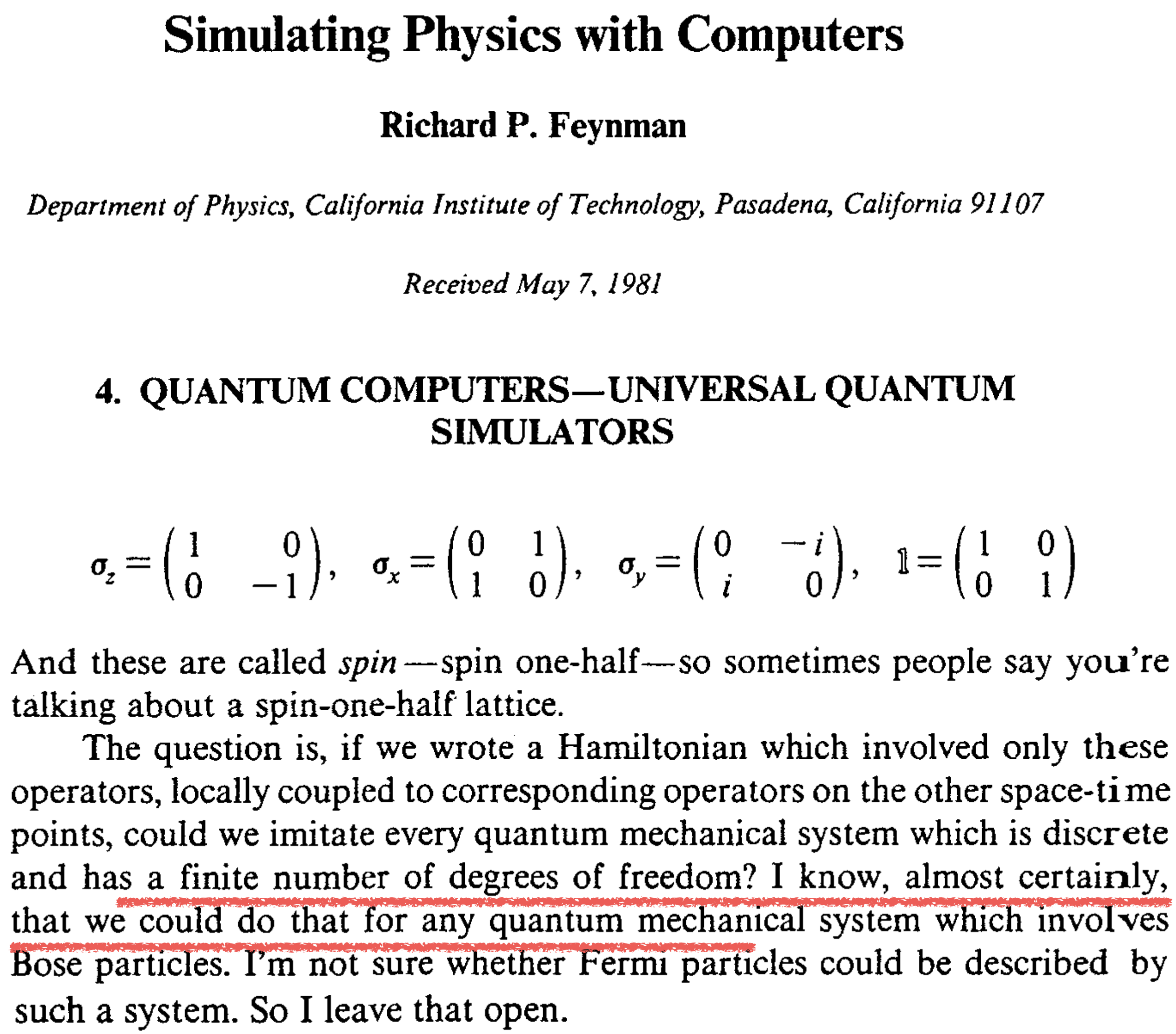
# Motivation



**Simulating Physics with Computers**

**Richard P. Feynman**

*Department of Physics, California Institute of Technology, Pasadena, California 91107*

*Received May 7, 1981*

**4. QUANTUM COMPUTERS—UNIVERSAL QUANTUM SIMULATORS**

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And these are called *spin*—spin one-half—so sometimes people say you're talking about a spin-one-half lattice.

The question is, if we wrote a Hamiltonian which involved only these operators, locally coupled to corresponding operators on the other space-time points, could we imitate every quantum mechanical system which is discrete and has a finite number of degrees of freedom? I know, almost certainly, that we could do that for any quantum mechanical system which involves Bose particles. I'm not sure whether Fermi particles could be described by such a system. So I leave that open.

*Can we formulate bosonic quantum field theories using quantum spin-half particles (qubits) as building block?*

Recent advances in quantum computation suggests that some day in the future answer to such a question could become important.

Challenge: Bosons usually require an infinite dimensional Hilbert space per lattice site due to the canonical commutation relation:

$$[\phi(x), \pi(y)] = i\delta_{x,y}$$

*Is this "local" infinity in the Hilbert space dimension necessary?*

Goal: Construct a lattice Hamiltonian with "n-qubits" on each lattice site, which yields the QFT in the continuum limit.

n-qubits: $2^n$ dimensional Hilbert space on each lattice site.

# Two Ideas to make progress

## Qubit Truncation

Begin with a traditional lattice Hamiltonian, truncate the Hilbert space at every lattice site to construct a n-qubit Hamiltonian.

continuum limit: $a \to 0\,,\ n \to \infty$

Particle Physics point of view.

## Qubit Regularization

Explore the space of n-qubit Hamiltonians to reach the correct continuum limit.

continuum limit: $a \to 0$

Condensed Matter Physics point of view.

# Qubit Truncation

Simplest approach!

*Truncate our "favorite" lattice Hamiltonian to an n-qubit space.*

Usually no guarantee that you are studying the QFT of interest in the continuum limit, unless "n" becomes large!

Important to distinguish between

**Local Lattice Quantities** ⟷ **Universal Quantities**

Easy to check!

Important to compute

$a \to 0\,, \; n \to \infty$

Many QFTs are being formulated and studied using Qubit-Truncation,

Jordan, Lee, Preskill, … : Scalar Fields

Jansen, Kaplan, Kclo, Raychoudhary, Stryker, …. : Gauge Fields

Alexandru, Bedaque, Lamm,….: Discrete subgroups

Banuls, Jansen, Meurice, … : Tensor Networks

There is some evidence of exponential convergence as "n" becomes large for **local lattice quantities**.

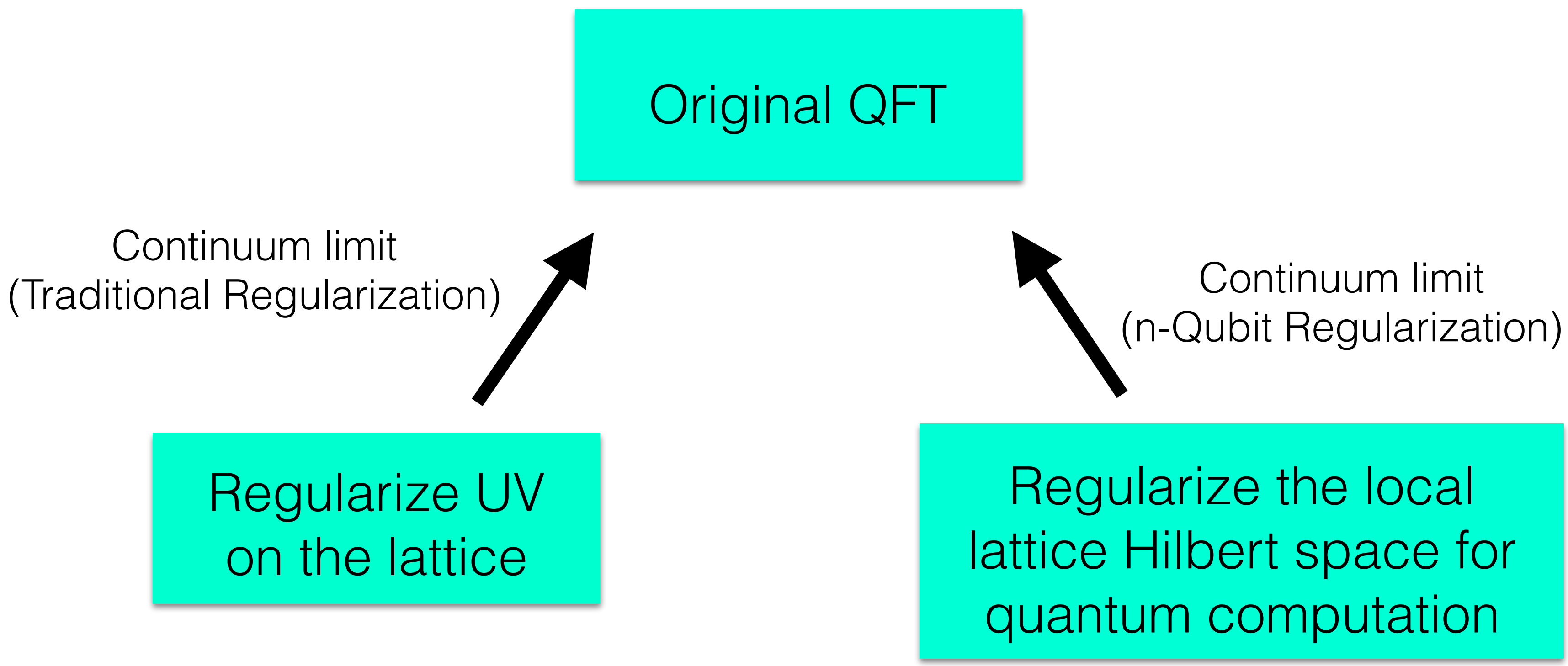
Important to check if this is true also for **universal quantities**.

see for example Paulsen et. al., 2008.09252

# Qubit Regularization of QFTs

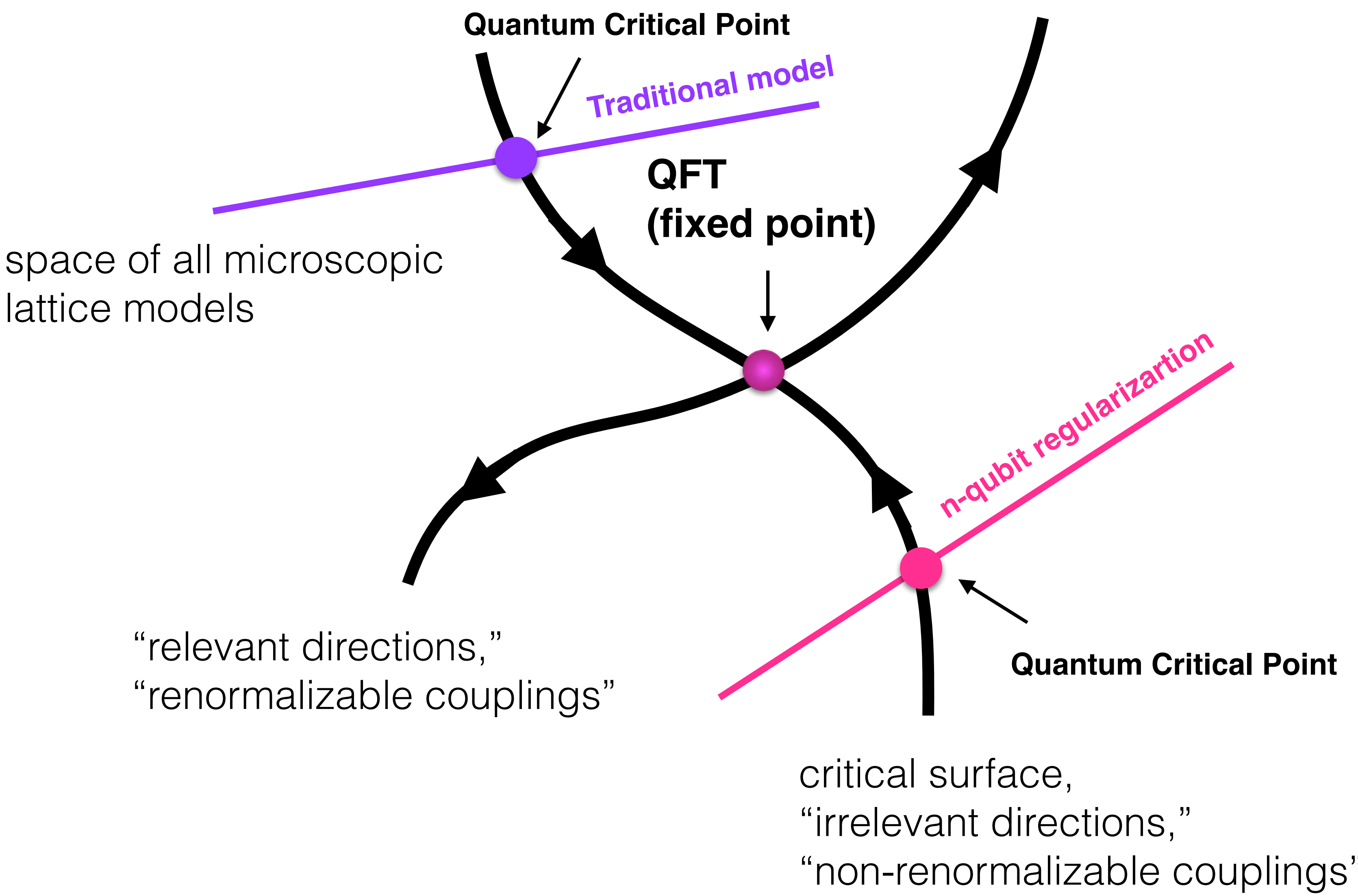
View the truncation of the Hilbert space per lattice site as a new type of regularization for quantum computation.

H. Singh and S.C., PRD 100, 054505 (2019).




*Important: Can we recover the continuum limit with a finite value of "n"?*

Yes if we can find the correct quantum critical point!



# D-Theory

The D-theory is a type of qubit regularization.

Brower, SC, Riederer and Wiese NPB 693,149 (2004)

In D-theory we allow for an extra dimension, which naturally allows for the number of qubits to grow, if necessary.

Asymptotically free theories are typically reproduced by making "n" large, but things again converge exponentially fast in some models.

Beard, Pepe, Riederer and Wiese PRL 94, 010603 (2006)

Can we reproduce asymptotic freedom even with "n" finite and fixed?

# O(3) scalar QFT

Traditional Model: Lagrangian Approach

$$S = -\frac{1}{2g} \sum_{x,\alpha} \vec{\phi}_x \cdot \vec{\phi}_{x+\alpha}$$

constraint:

$$\vec{\phi}_x \cdot \vec{\phi}_x = 1$$

$x$ is a point in d+1 dimensional Euclidean lattice

Hamiltonian Approach: Model of "rotors"

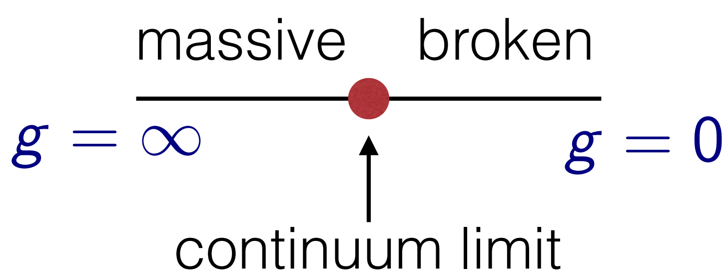
$$H = \frac{g}{2} \sum_{x} \vec{L}_x \cdot \vec{L}_x + \frac{1}{2g} \sum_{x,j} \vec{\phi}_x \cdot \vec{\phi}_{x+j}$$

$x$ is a point in d dimensional spatial lattice

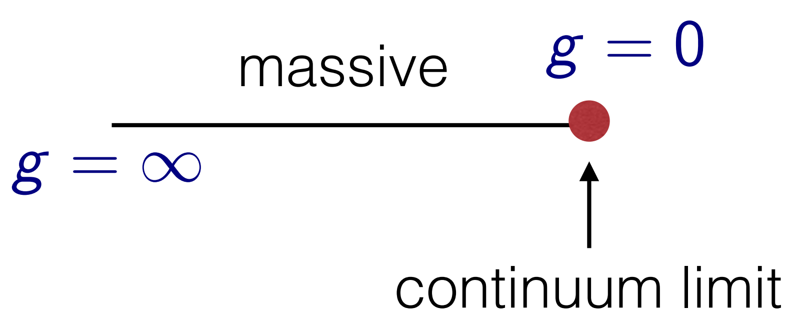
conjugate variables: $\vec{\phi}_x, \; \vec{L}_x$

QFT emerges at the quantum critical point.

Phase diagram:




Wilson-Fisher (d=2) / Gaussian (d=3)




asymptotically free theory(d=1)

# Qubit Truncation

Hilbert space per lattice site of the traditional model:

$$\vec{\phi}_x \equiv (\theta, \varphi)_x$$

Complete basis:

$$\int d\Omega \; |\theta, \varphi\rangle\langle\theta, \varphi| \; = \; I$$

"position basis"

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} | \; \ell, m \; \rangle\langle \; \ell, m \; | \; = \; I$$

"momentum basis"

Minimal truncation:

$\ell = 0$

$$|\ell = 0, m = 0\rangle$$

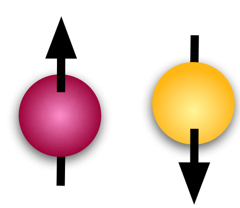
singlet

$\ell = 1$

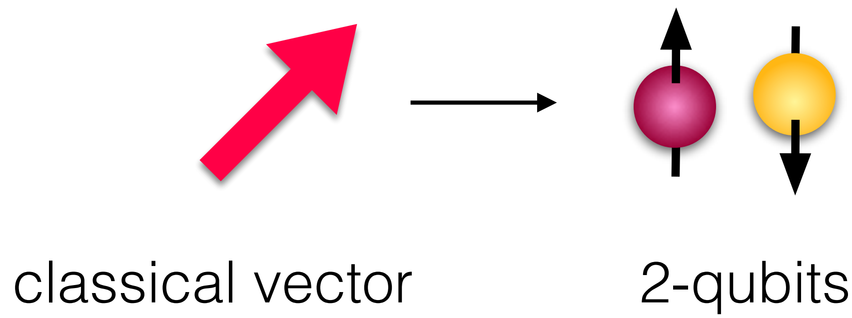
$$|\ell = 1, m\rangle, m = 0, \pm 1$$

triplet

Two cubits per lattice site (n=2):



Are two qubits per lattice site sufficient to recover the traditional O(3) QFT?




*Typically the physics of IR fixed points that arise at quantum critical points separating two phases is easy to recover with a small number of qubits.*

Wilson-Fisher fixed point

Gaussian fixed point

← Easy!

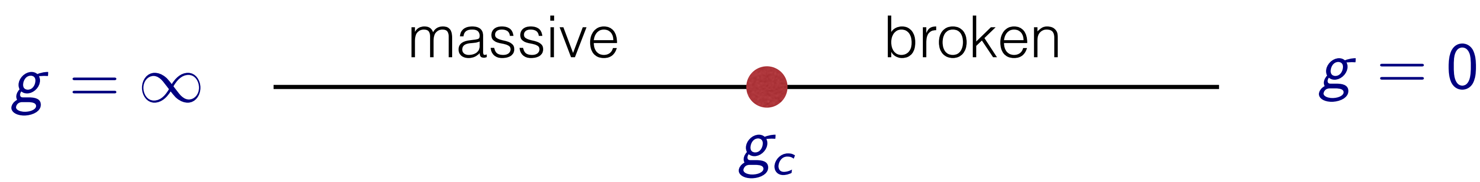
H. Singh and S.C., PRD 100, 054505 (2019).

Asymptotically free fixed point ← Hard!

# Wilson-Fisher Fixed Point

In d=3 the critical point has been studied extensively.

Phase diagram:




Susceptibility:

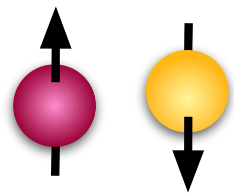
$$\chi = \sum_x \langle \vec{\phi}(x) \cdot \vec{\phi}(0) \rangle \qquad \chi\, L^{\eta-2} \sim f\left((1 - g/g_c)L^{1/\nu}\right)$$

Critical Exponents:

$\nu = 0.7113(11), \quad \eta = 0.0378(6)$

Pelisetto and Vicari Phys. Repts. (2002)

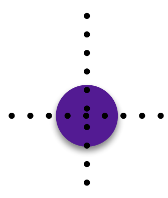
# Qubit Regularization of the WF fixed point:

Two qubits per lattice site: 

$\ell = 0$

$|s, \mathbf{r}\rangle$

singlet

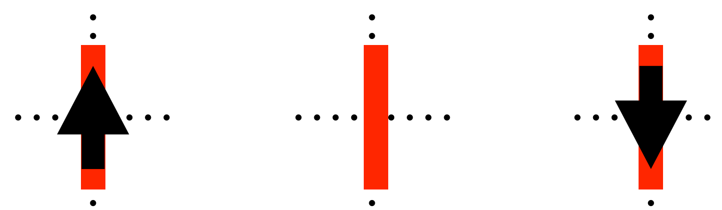


Fock
Vacuum

$\ell = 1$

$|m, \mathbf{r}\rangle$, $m = 0, +1, -1$

triplet

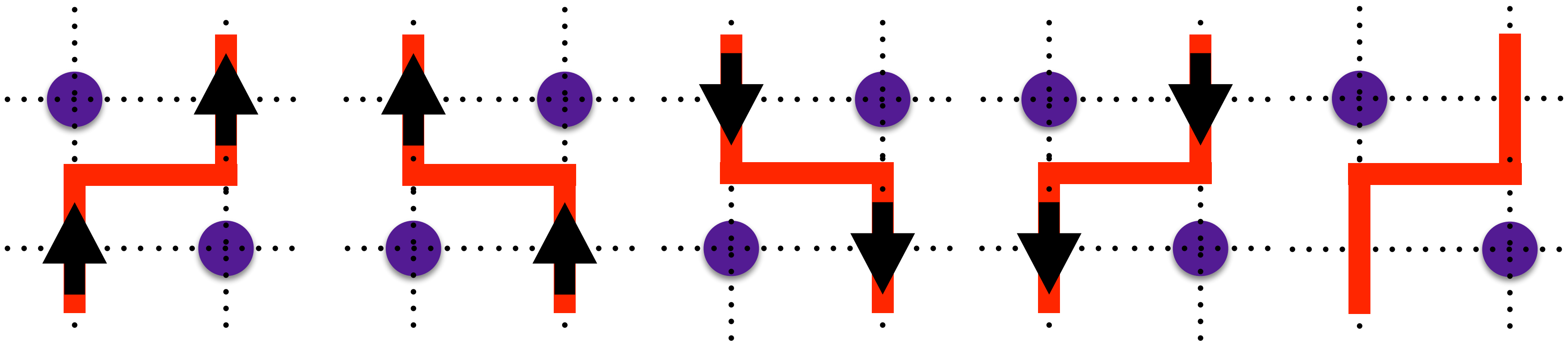


Spin-1 particle

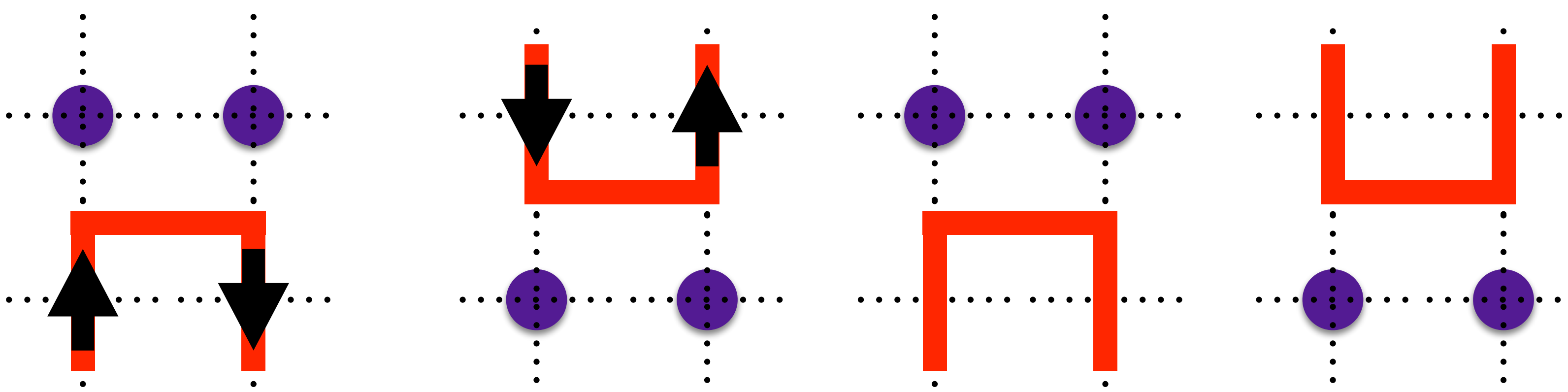
O(3) invariant Hamiltonian

$$H = J_t \sum_{\mathbf{r}} \sum_{m} |m, \mathbf{r}\rangle\langle m, \mathbf{r}| - \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left( J_h H^h_{\mathbf{r},\mathbf{r}'} + J_p H^p_{\mathbf{r},i} \right)$$

Hopping term



Pair Creation/Annihilation term



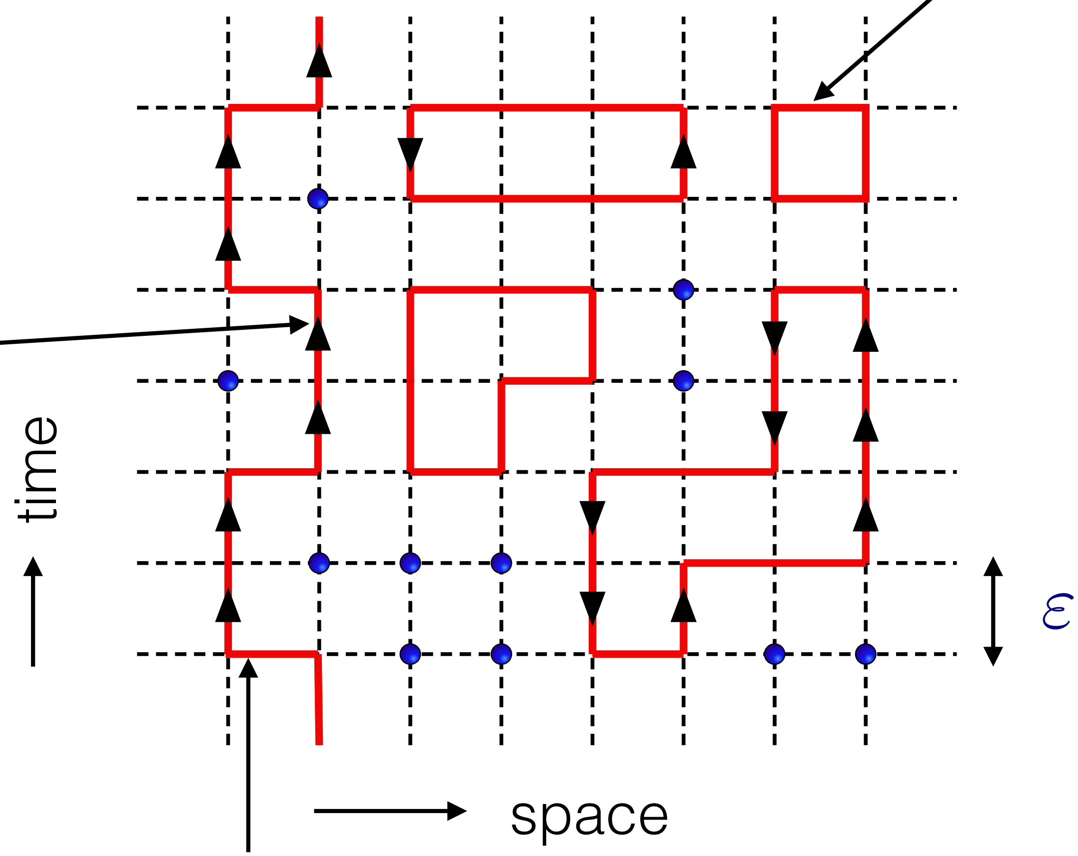
# 2-Qubit Model: Loop Gas

Partition function: $Z = \mathrm{Tr}\left(e^{-\varepsilon H}\, e^{-\varepsilon H} \,\ldots\, e^{-\varepsilon H}\right)$

$$Z = \sum_{[s,m]} \prod_{\langle ij \rangle} W_{\langle ij \rangle}$$

$W_s^p = \varepsilon J_p$

$W_t = e^{-\varepsilon J_t}$




$W_s^h = \varepsilon J_h$

For simplicity we set

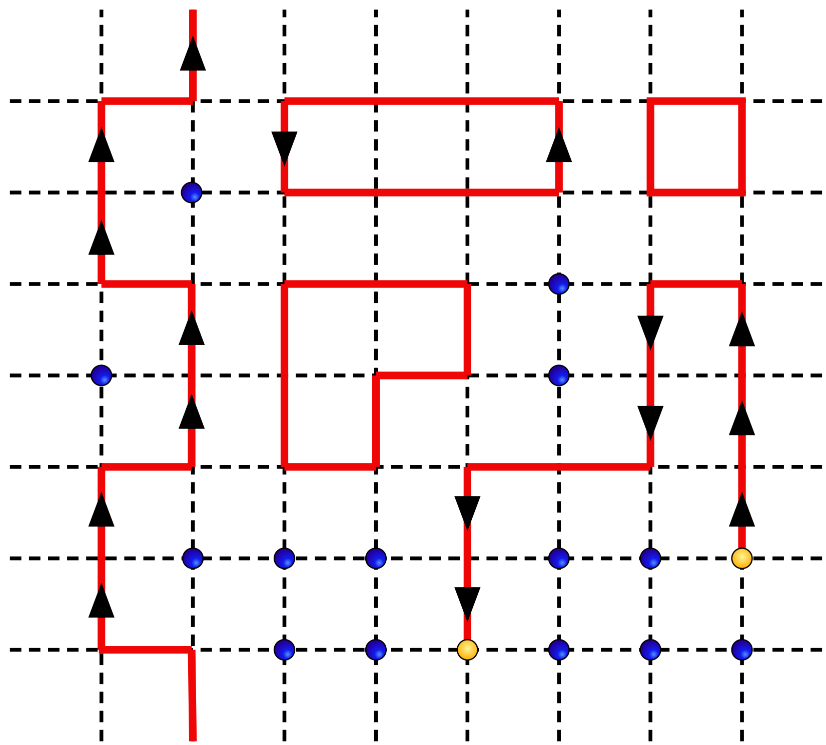
$$J_h = J_p = J$$

$$\lambda = J_t / J$$

# Susceptibility

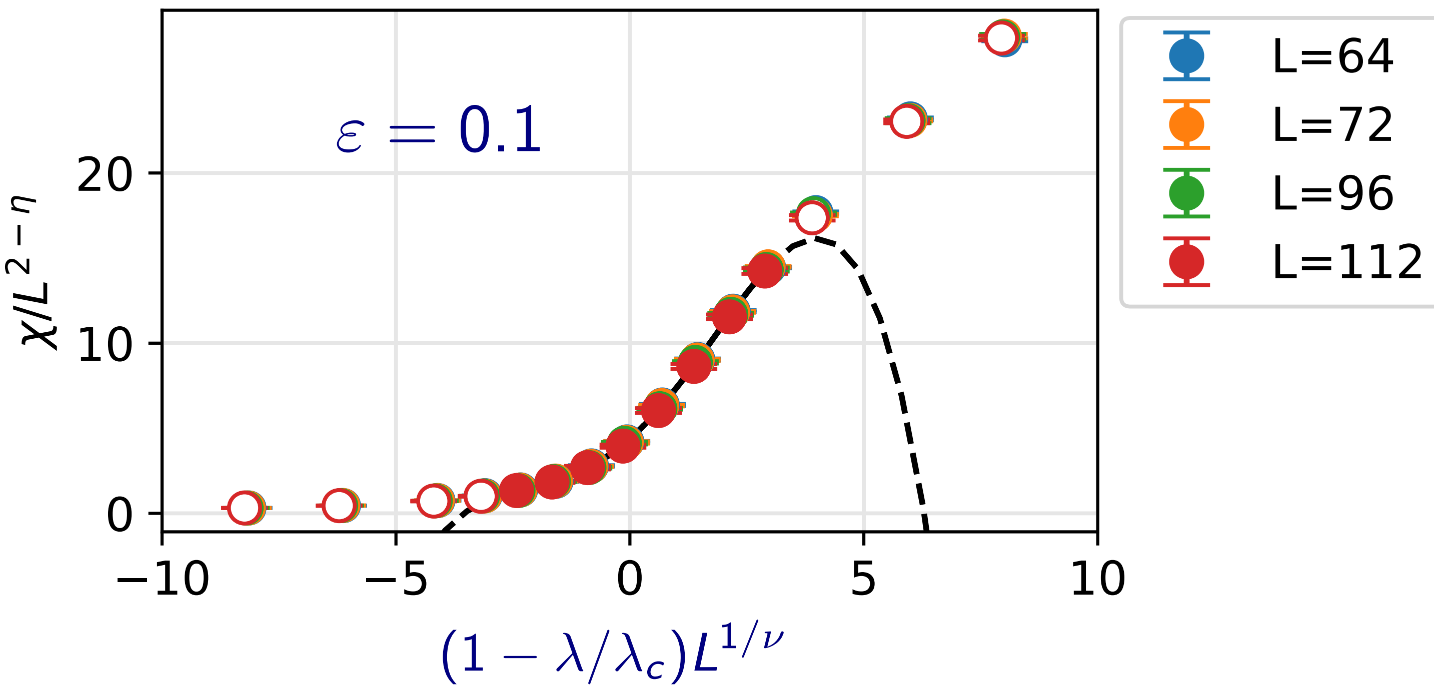
$$\chi = \frac{1}{ZL^d} \sum_{\mathbf{r},\mathbf{r}'} \int dt_0^\beta \; dt \; \mathrm{Tr}\Big(e^{-(\beta-t)H} O_{\mathbf{r},m} e^{-tH} O^\dagger_{\mathbf{r}',m}\Big)$$

$$O^\dagger_{\mathbf{r}',m} \;=\; a^\dagger_{\mathbf{r},m} \;+\; (-1)^m \; a_{\mathbf{r},-m}$$



critical scaling

$$\chi L^{2-\eta} \sim f\left((1-\lambda/\lambda_c)L^{1/\nu}\right)$$




Fit values in the qubit model:

$$\lambda_c = 4.81695(37)$$

$$\nu = 0.693(15), \quad \eta = 0.038(26)$$

H. Singh and S.C., PRD 100, 054505 (2019).

Results in the traditional model:

$$\nu = 0.7113(11), \quad \eta = 0.0378(6)$$

Pelisetto and Vicari Phys. Repts. (2002)

# Asymptotically free Fixed Point

*Can we recover the Asymptotically free QFT with n=2?*

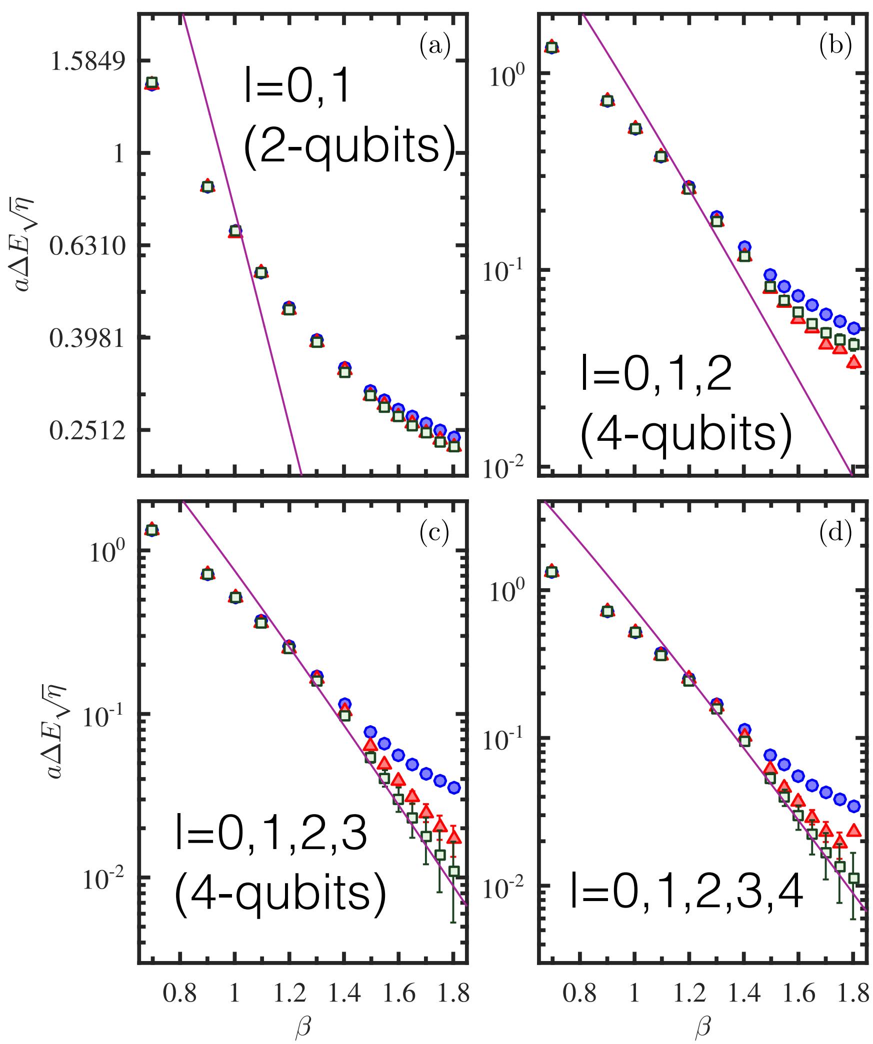
Not with the qubit truncation approach.

Bruckmann, Jansen, Kuhn
PRD 99, (2019) 074501.

What about with qubit regularization?

Yes we can!

Buser, et. al., 2012.02153



# The Challenge of Asymptotic Freedom

*Find the quantum critical point with a marginal coupling*
*that captures the physical scales*
*from the infrared (IR) to the ultraviolet (UV) correctly!*

lattice spacing scale (n-qubits) << physical UV length scales << physical IR length scale (mass gap)

Physics at all scales in this region are universal

How can we study the various physical scales quantitatively?

# Universal Step Scaling Function

Luscher, Wiesz and Wolff, *Nucl.Phys.B* 359 (1991) 221-243.

Define a dimensionless coupling through a finite size mass scale

$$u(L) \;=\; m(L)\; L$$

In an asymptotically free theory

$$u(sL) = \sigma(s, u(L))$$

$\sigma(s, u)$ ↑ Step Scaling function (universal)

Can compute it in perturbation theory:

$$\lim_{u \to 0} \sigma(s, u) = u + \sigma_0(s) u^2 + \sigma_1(s) u^3 + \ldots$$

Asymptotic form:

$$\lim_{u \to \infty} \sigma(s, u) \;=\; s\; u$$

We can also use correlation length instead of the mass gap.

Correlation length using the second moment definition is easy to compute

Caracciolo, Edwards and Sokal, PRL 75, 1891 (1995).

$$G(\mathbf{r}, t) \;=\; \langle \vec{\phi}(\mathbf{r}, t)\; \vec{\phi}(0,0)\rangle$$

$$\tilde{G}_{\mathbf{p},\omega} \;=\; \int \; d^d\mathbf{r}\; dt\; G(\mathbf{r}, t)\; e^{i\mathbf{p}\cdot\mathbf{r}+i\omega t}$$

$$\chi = \tilde{G}_{0,0} \qquad F = \tilde{G}_{2\pi/L,0}$$

$$\xi(L) = \frac{1}{2\sin(\pi/L)}\sqrt{\frac{\chi}{F} - 1}$$

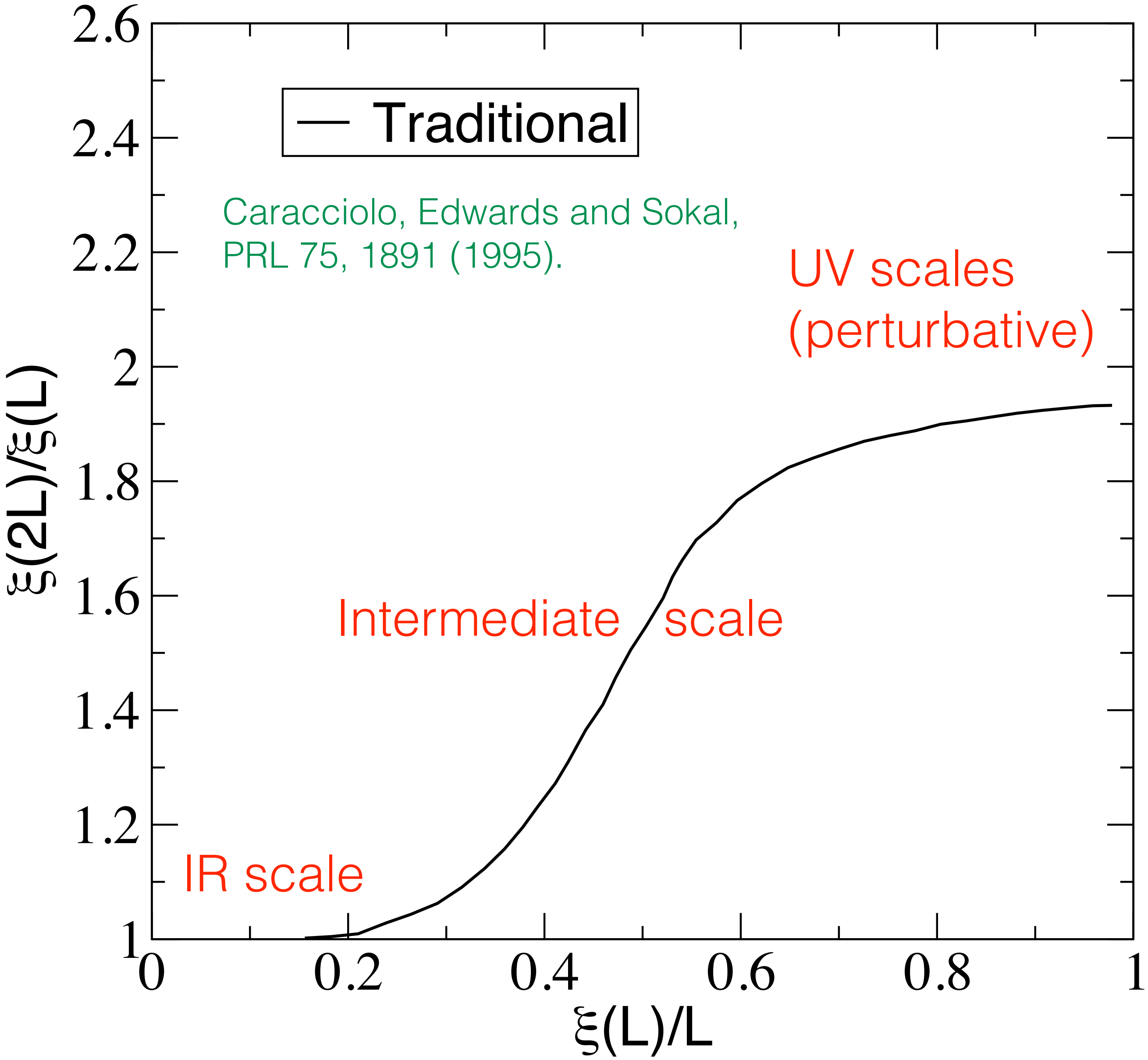
$$u(L) = \frac{L}{\xi(L)}$$

$$\frac{\xi(2L)}{\xi(L)} \;=\; \frac{2u(L)}{u(2L)}$$

$$=\; \frac{2u}{\sigma(2, u)}$$

Plot $\dfrac{\xi(2L)}{\xi(L)}$ vs. $\dfrac{\xi(L)}{L}$

SSF for the traditional O(3) model



# Lagrangian vs. Hamiltonian

Traditional Model is in the Lagrangian approach.

→ Symmetric under space-time rotations!

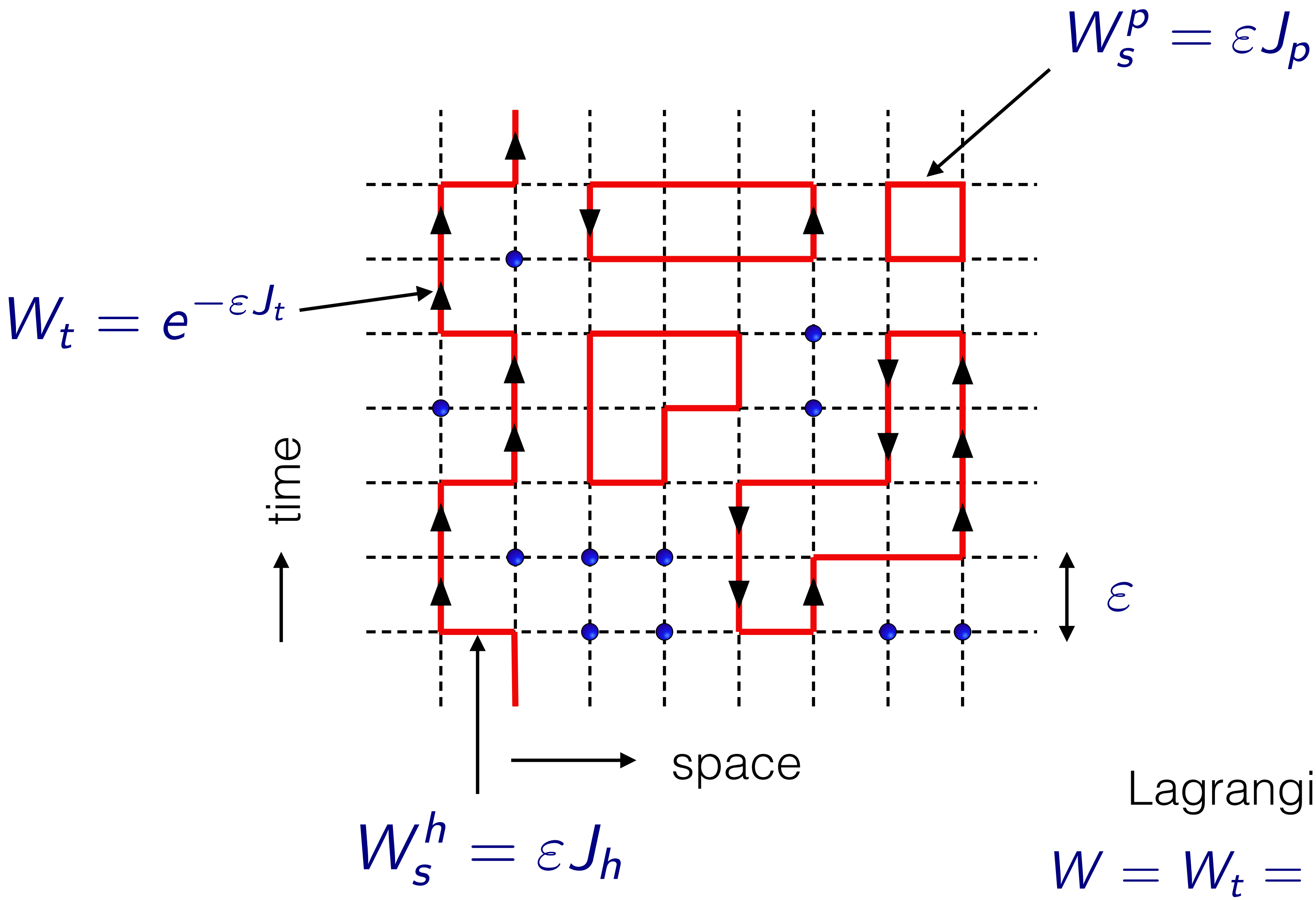
In a Hamiltonian approach this symmetry is lost.

$$F = \tilde{G}_{2\pi/L,0} \qquad F_t = \tilde{G}_{0,2\pi/\beta} \qquad F \neq F_t$$

Assuming the quantum critical point in the qubit model is relativistic, one has to tune inverse temperature β for every L so that $F = F_t$

Sometimes we can take the "Lagrangian limit" and construct qubit space-time lattice models that have an in-built space-time rotation symmetry!

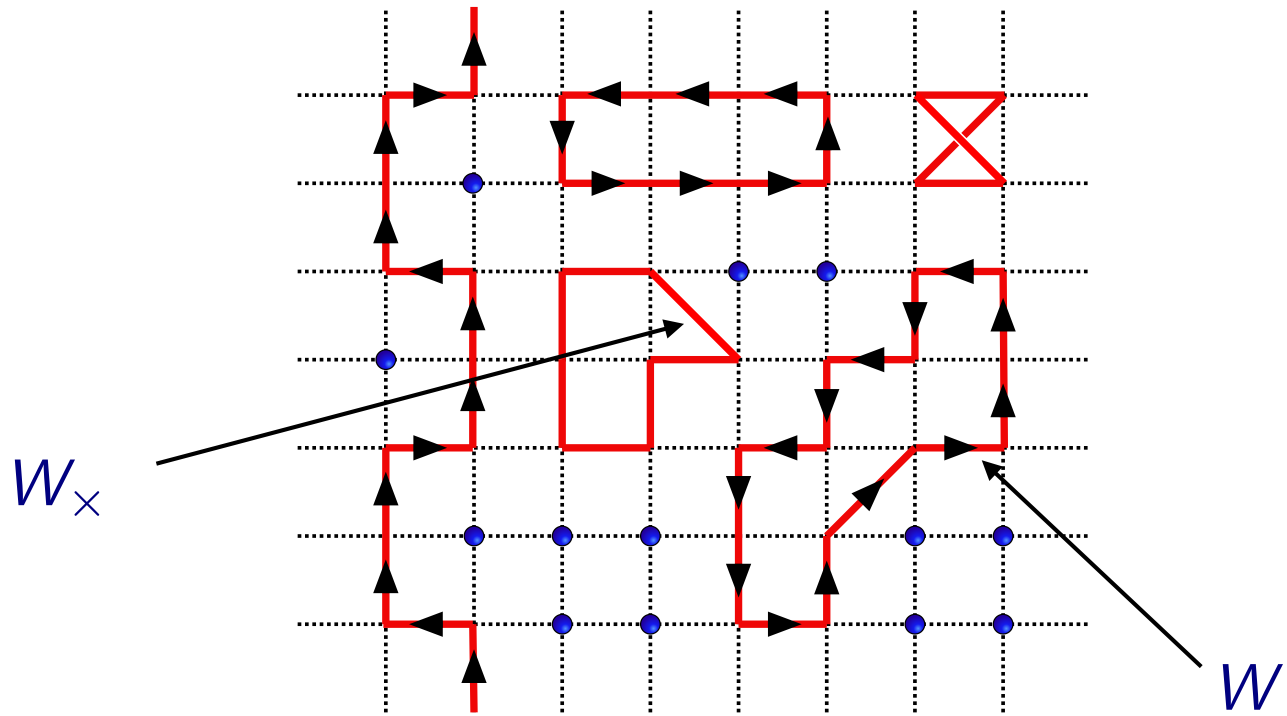
# Example: 2-Qubit Loop Gas




Lagrangian Limit

$$W = W_t = W_s^h = W_s^p$$

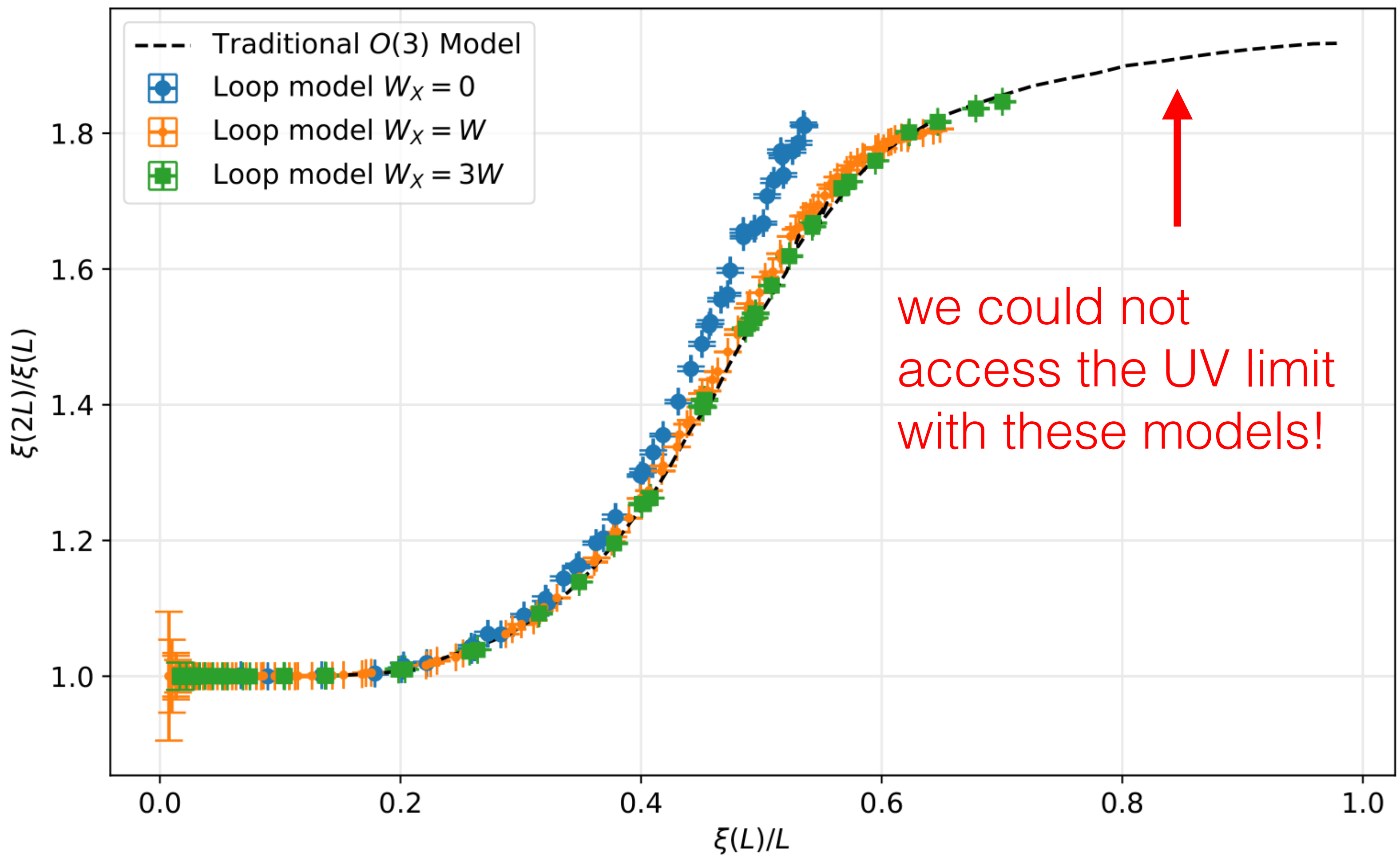
# 2-Qubit loop gas with an additional coupling




The symmetries of the model do not change!

# SSF for the 2-qubit loop gas

Method: Worldline Monte Carlo        Lattice sizes: L= 24,…,512




we could not access the UV limit with these models!

# Exploring more 2-Qubit Models

Phase A: Ground state dominated by spin-singlets, while spin triplet excitations are massive. ← main phase of interest

Phase B: Spin triplets dominate and form a ferromagnet.

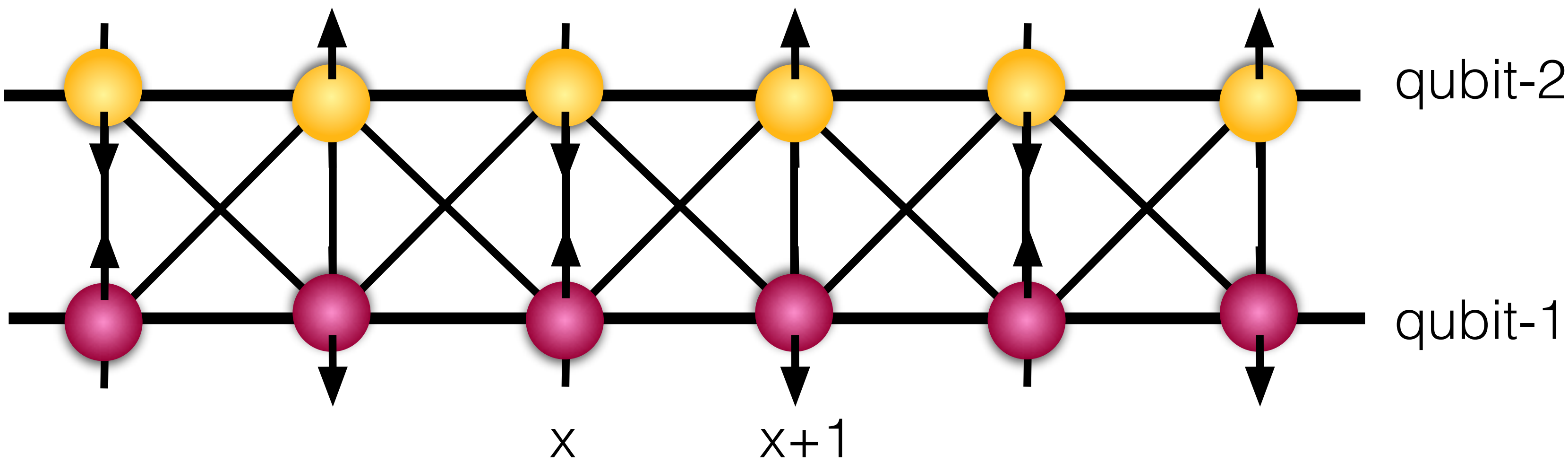
Phase C: Local spin triplets form spin singlet dimers on bonds and break translation invariance spontaneously.

Phase D: Local spin triplets form a massive topological phase, called Haldane phase!

Phase E: critical phase where long distance physics is described by a k=1 SU(3) WZW model.

*Goal: Explore quantum critical points connected to phase A*

General 2-qubit model: “quantum spin-half ladders”.




$$H = \sum_{(x,a),(y,b)} J_{(x,a),(y,b)} \, \mathbf{S}_x^a \cdot \mathbf{S}_y^b$$

# Computing the correlation length

$$G_{k,\omega} = \frac{1}{Z} \int dt \sum_x \mathrm{Tr}\Big(\mathcal{O}(x,t)\ \mathcal{O}(0,0) e^{-\beta H}\Big) e^{i2\pi kx/L + i2\pi\omega t/\beta}$$

$$\mathcal{O}(x,t) = e^{Ht}(-1)^x (S_x^{1z} - S_x^{2z}) e^{-Ht}$$

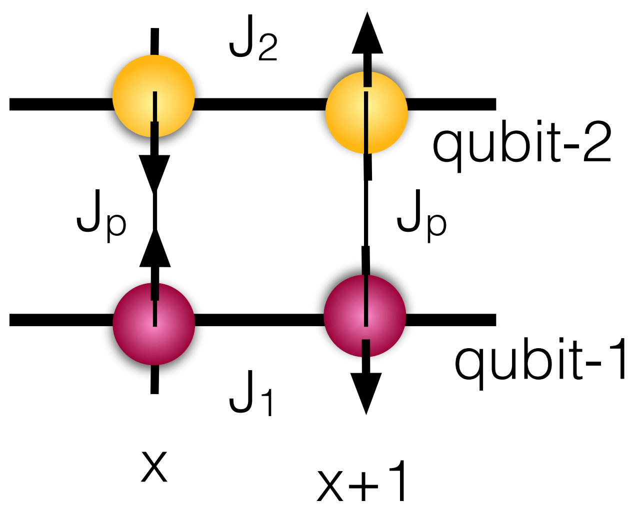
$$\chi = G_{0,0},\ \ F = G_{1,0}, F_t = G_{0,1}$$

We tune β and L so that we obtain $F = F_t$

$$\xi(L) = \frac{1}{2\sin(\pi/L)}\sqrt{\frac{\chi}{F} - 1}$$

# Spin-Ladders

$$H = \sum_x J_1 \, \mathbf{S}_{x,1} \cdot \mathbf{S}_{x,1} + J_2 \, \mathbf{S}_{x,2} \cdot \mathbf{S}_{x,2} + J_p \, \mathbf{S}_{x,1} \cdot \mathbf{S}_{x,2}$$




Symmetric Ladder:

$$J_1 = J_2 = 1$$

Assymmetric Ladder:

$$J_1 = 1, J_2 = 0.5$$

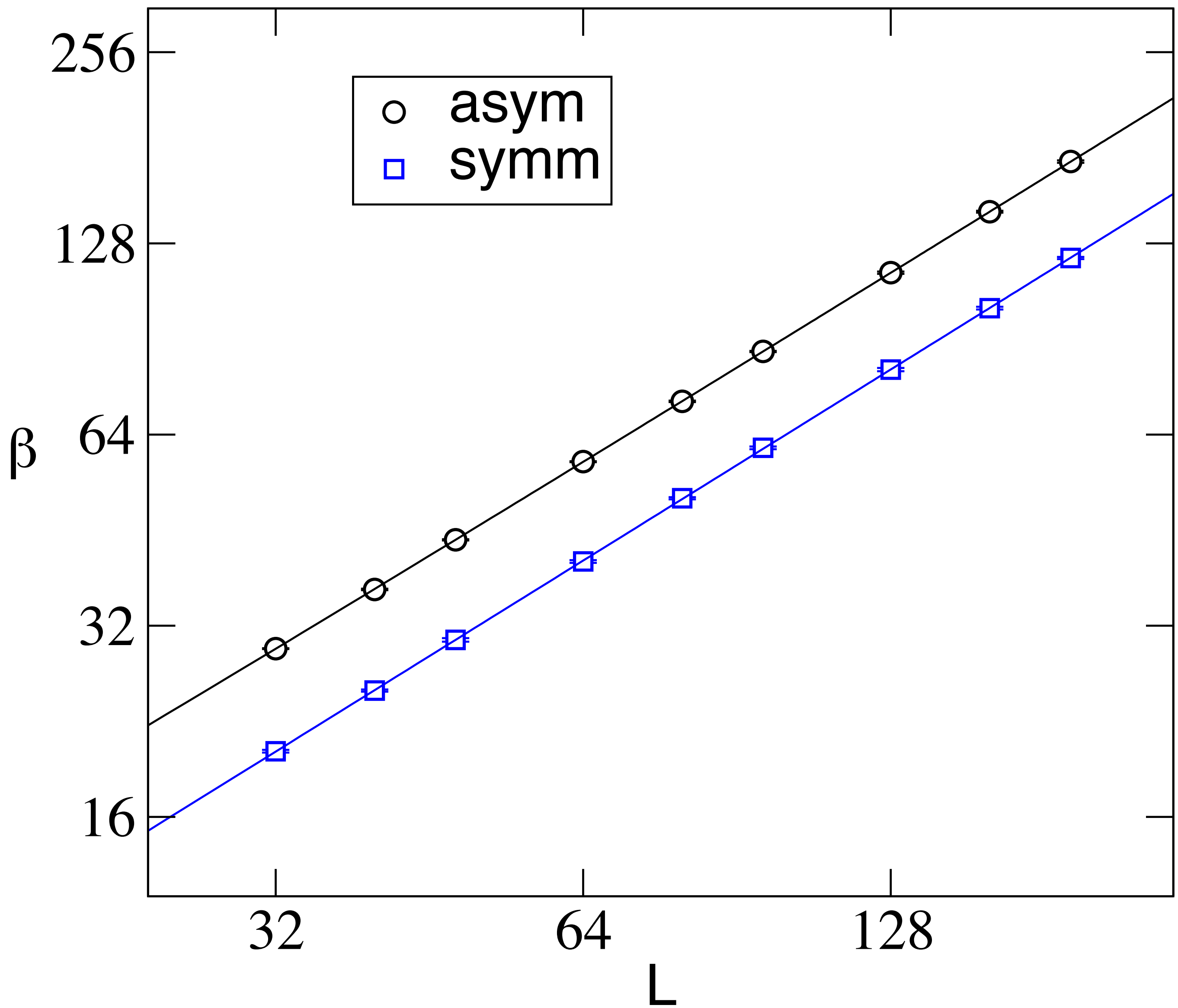
Mass gap generated when

$$J_p \neq 0$$

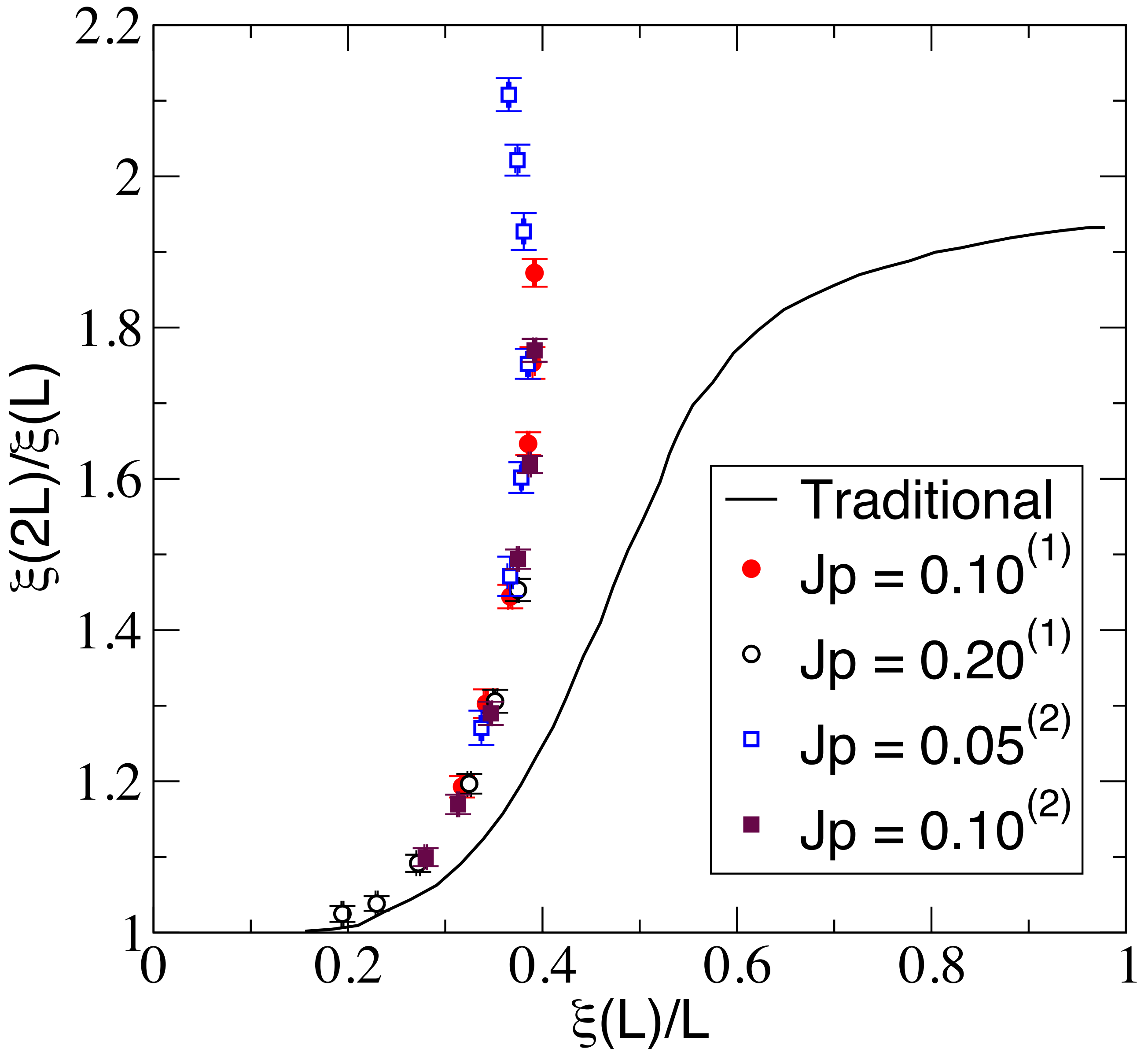
Quantum Critical Point:

$$J_p \to 0$$

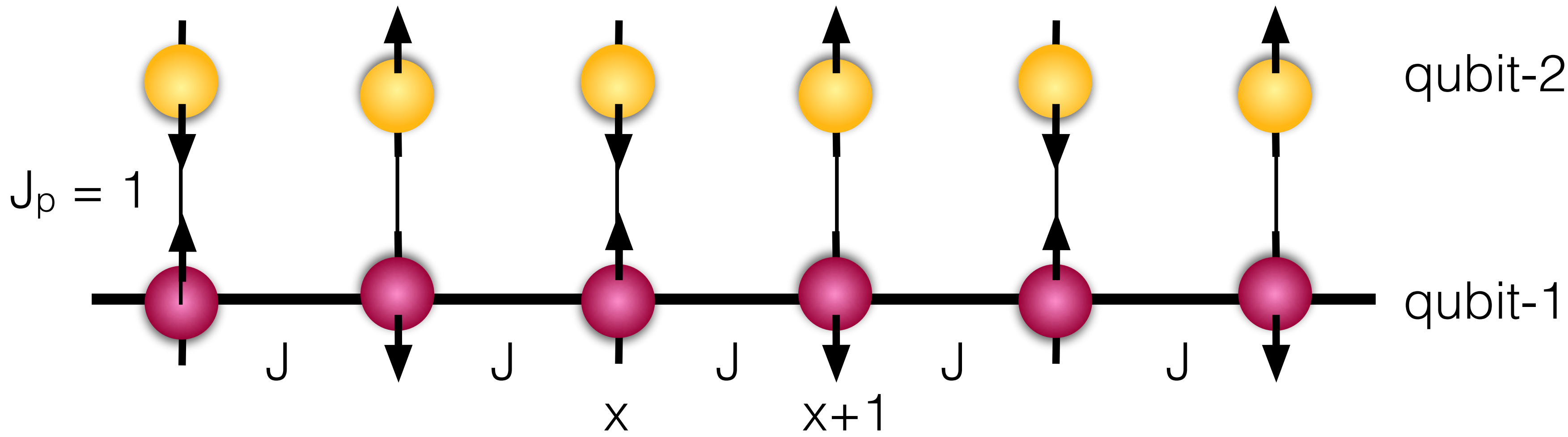
Tuning β as a function of L to obtain $F = F_t$



# Step scaling function



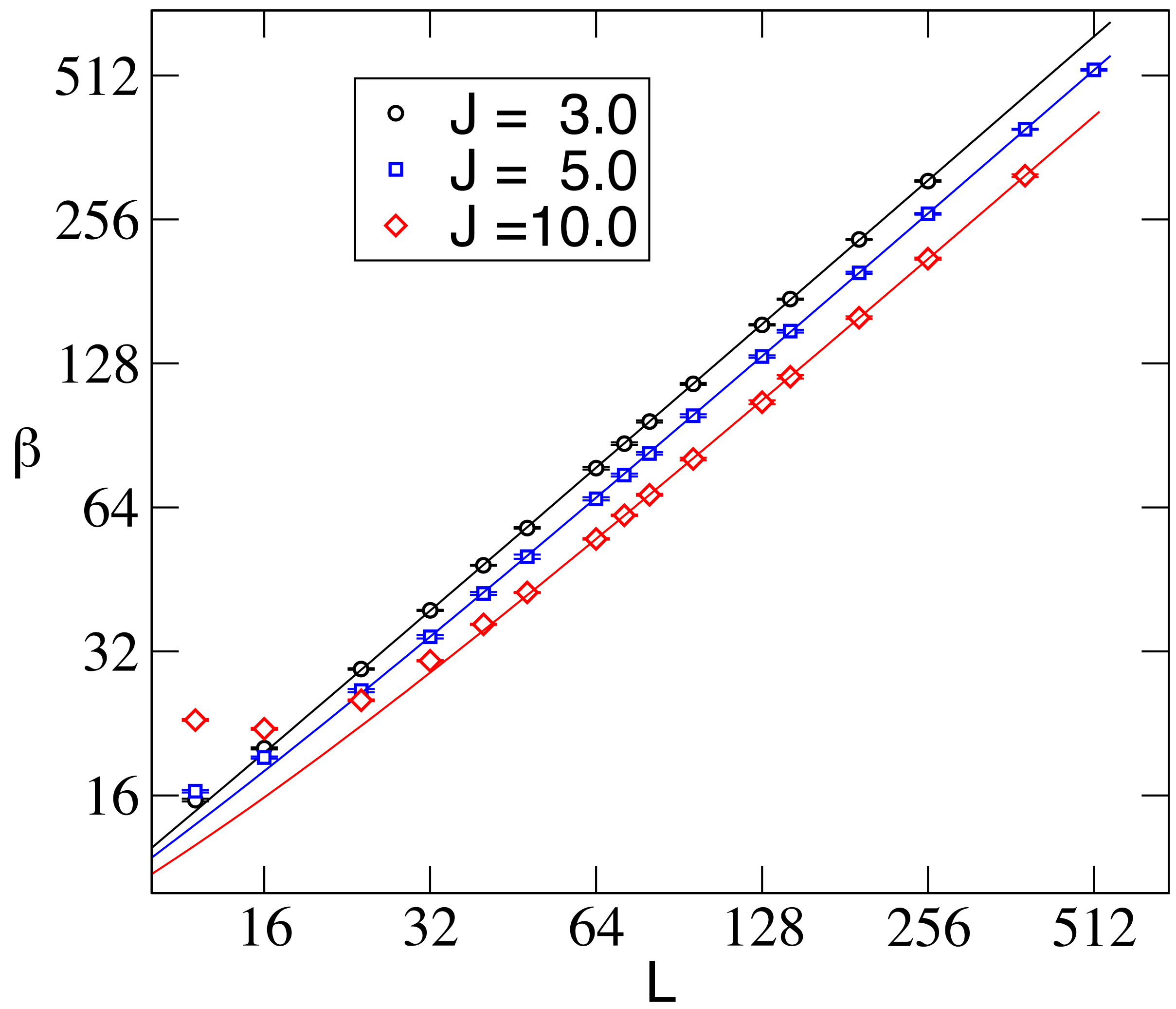
# Heisenberg-Comb



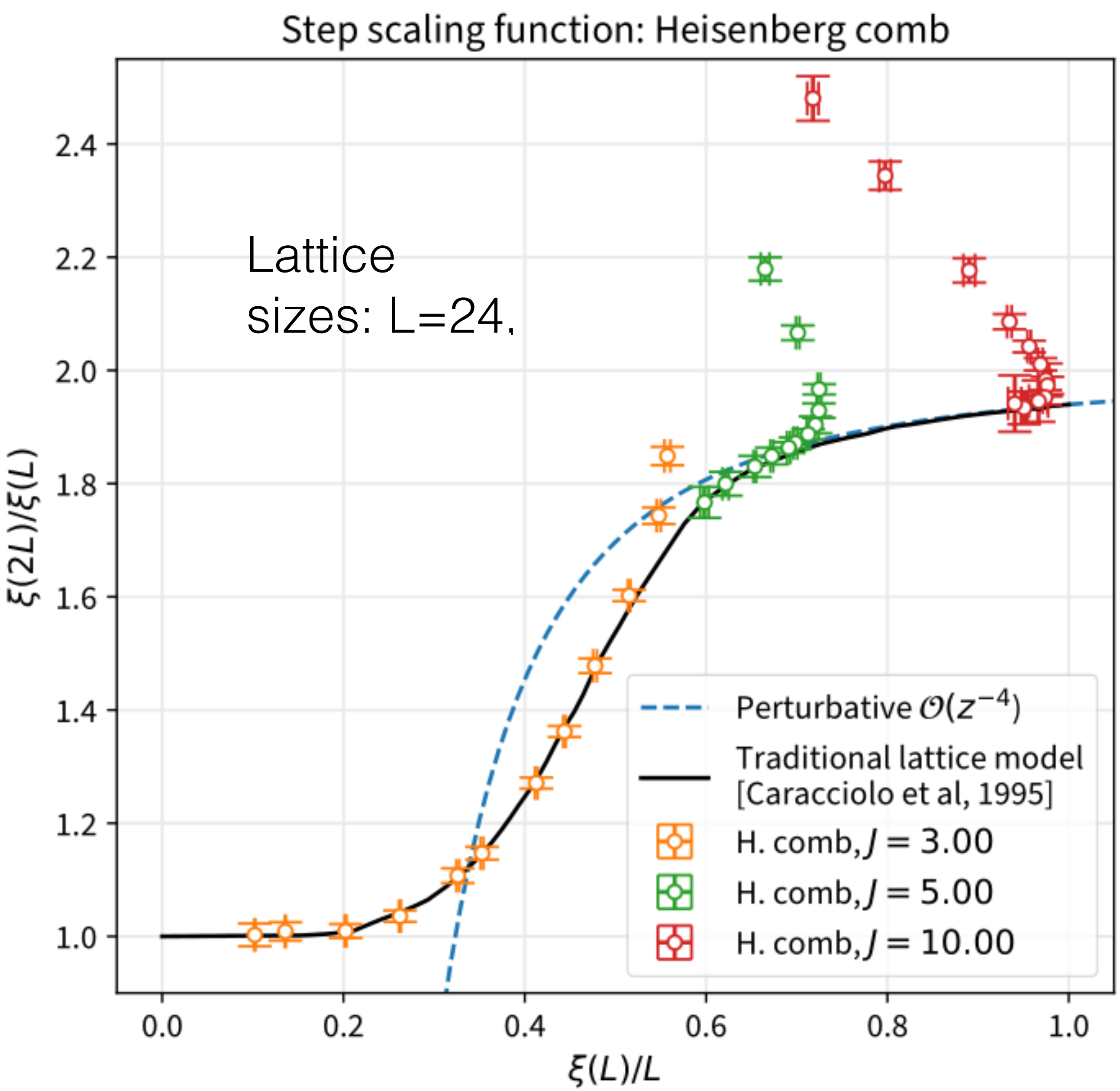

$$H = \sum_x J\, \mathbf{S}_{x,1} \cdot \mathbf{S}_{x,1} + \mathbf{S}_{x,1} \cdot \mathbf{S}_{x,2}$$

Quantum Critical Point: $J \to \infty$

Tuning β as a function of L to obtain $F = F_t$



# Step scaling function

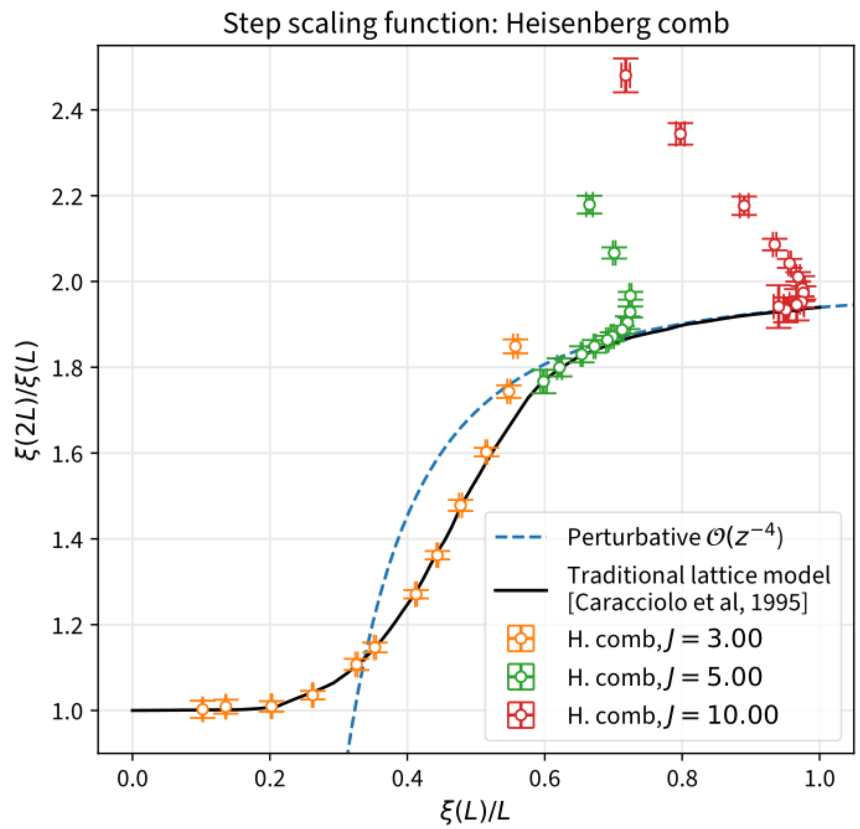


Observations:

For L < $L_{min}$ we do not see scaling

$L_{min}$ grows with J

$L_{min}$ is the UV scale




Infinite L correlation length

$$\lim_{L\to\infty} \xi(L) = \xi_\infty$$

| $J$ | $L_{min}$ | $\xi_\infty$ |
|---|---|---|
| 3 | 30 | 25 |
| 5 | 100 | 600 |
| 10 | 400 | 200,000 |

# Conclusions

Qubit Regularization is a useful idea in the study of QFTs using quantum computation.

Question: What is the smallest number of qubits per lattice site necessary to reproduce the original QFT?

Here we showed that we can reproduce continuum two-dimensional O(3) non-linear sigma models with just 2-qubits!

Asymptotic freedom can also be reproduced!

The d-theory approach is a subset of the more general idea of qubit regularization.