Tight-binding model subject to conditional resets at random times

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Work done with Anish Acharya (TIFR)

Anish Acharya and Shamik Gupta, Phys. Rev. E 108, 064125 (2023)

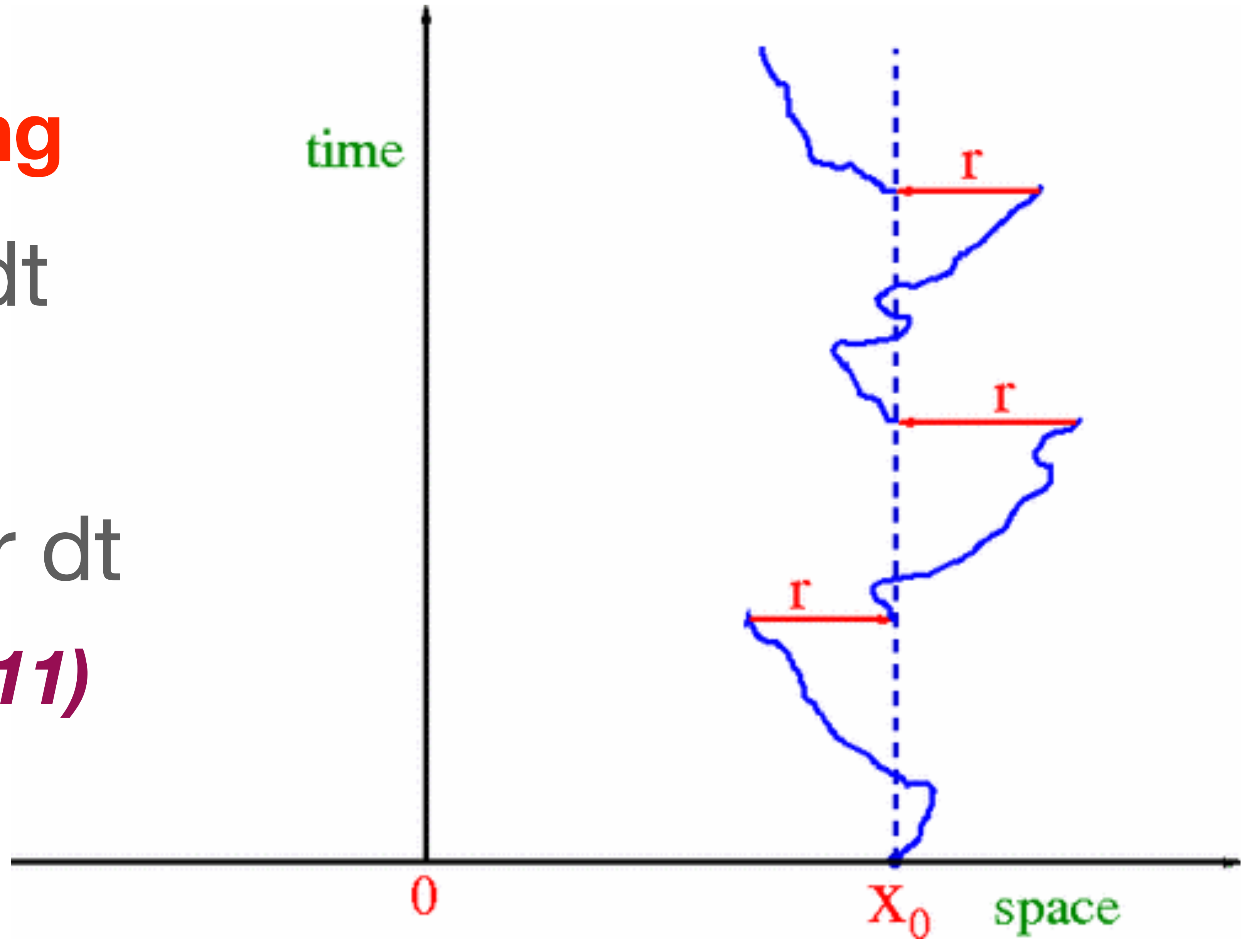
Stochastic resetting

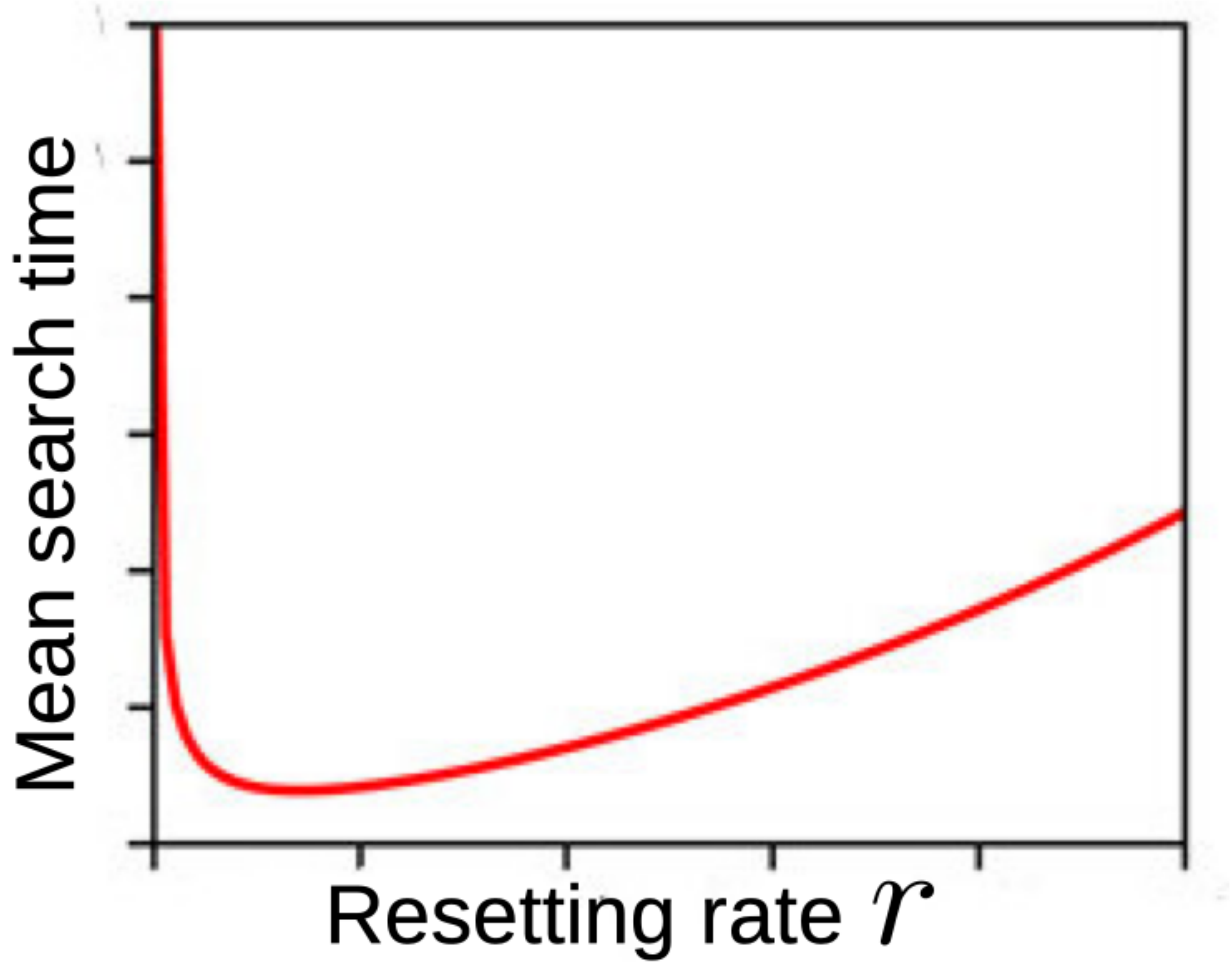
Classical:

Brownian motion with resetting

$$x(t+\mathrm{d}t) = x_0 \text{ with prob. } r\,\mathrm{d}t$$
$$= x(t) + \eta(t)\mathrm{d}t \text{ with prob. } 1-r\,\mathrm{d}t$$

Evans and Majumdar (2011)





Quantum:

$$|\psi(t+dt)\rangle = |\psi(0)\rangle \text{ with prob. } r\,\mathrm{d}t$$
$$= [1 - iH(t)\,\mathrm{d}t]\,|\psi(t)\rangle \text{ with prob. } 1 - r\,\mathrm{d}t$$

Mukherjee, Sengupta and Majumdar (2018)

Reset is unconditional:
Reset is to a given state, irrespective of the current state

Stochastic resetting:
A very (very) active area of research across domains with many contributors around the world

Review: Evans, Majumdar, Schehr, J. Phys. A: Math. Theor. 53 193001 (2020)

Conditional stochastic resetting

Reset is conditional:
Reset is to a set of states, whose choice depends on the current state

Perfetto, Carollo, Magoni, Lesanovsky (2021)

Quantum model:

Transverse-field Ising chain:

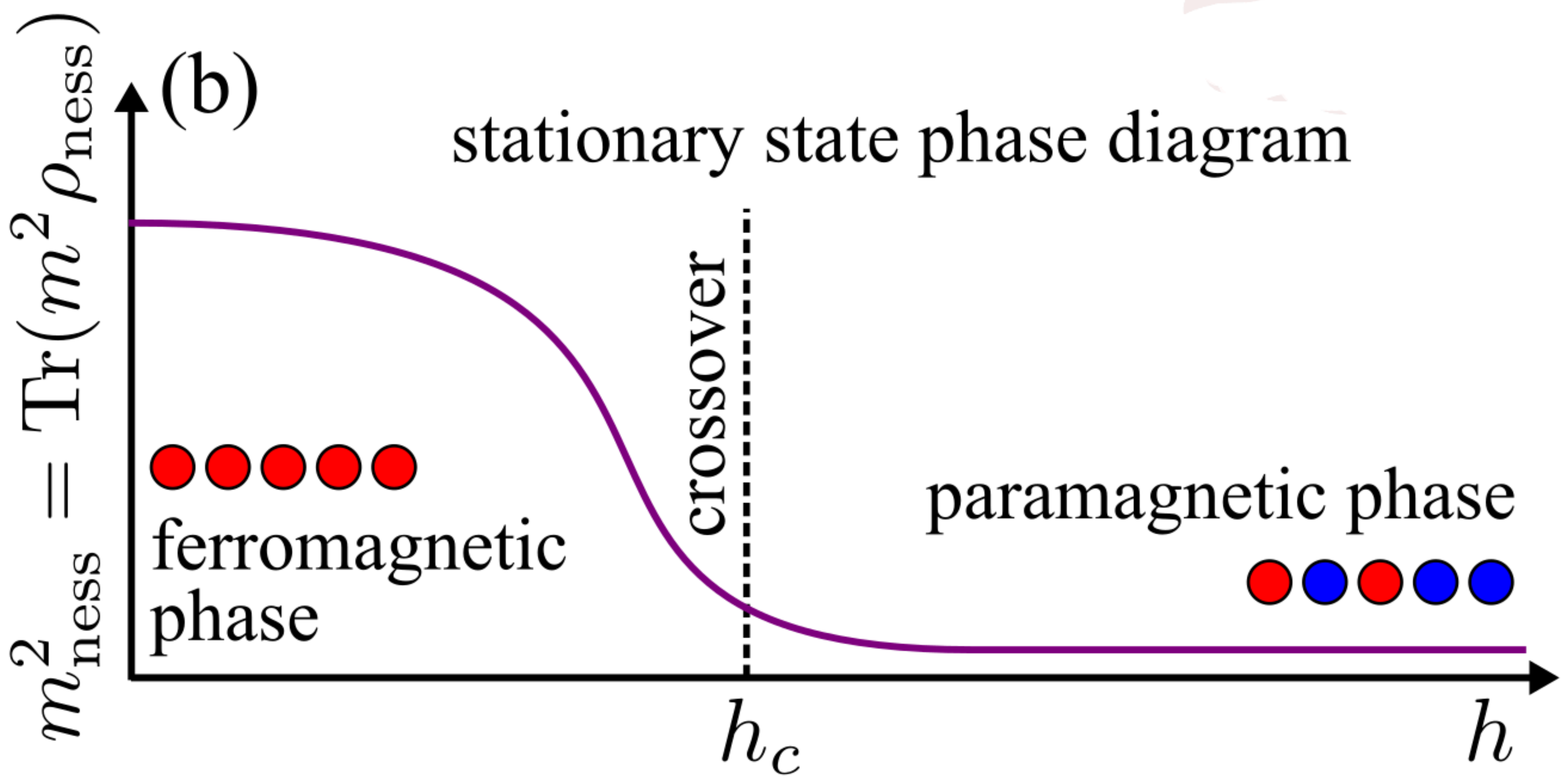
$$H = -J \sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + h\sigma_n^z)$$

Magnetisation $m = \frac{1}{N} \sum_{n=1}^{N} \sigma_n^x$

Measure the magnetisation after a random time:

(a) $m > 0$: Reset to $| \uparrow \uparrow \ldots \uparrow \rangle$

(b) $m < 0$: Reset to $| \downarrow \downarrow \ldots \downarrow \rangle$



Quantum phase transition of bare dynamics visible as a crossover in reset-induced steady state

Conditional stochastic resetting

Reset is conditional:
Reset is to a set of states, whose choice depends on the current state
We ask:

- Effects of conditional resetting when the bare dynamics does not have a steady state by itself ?
- Does Conditional Resetting always lead to a steady state (as does usual resetting) ?
- What is the nature of the steady state that emerges ?
- Can one characterise it analytically (Conditional Resetting results in non-Markovian evolution, as we will see later in the talk) ?

We choose a paradigmatic quantum system:
The tight-binding model

Tight-binding model

$$H = -\frac{\gamma}{2} \sum_{n=-\infty}^{\infty} \left(|n\rangle\langle n+1| + |n+1\rangle\langle n| \right)$$

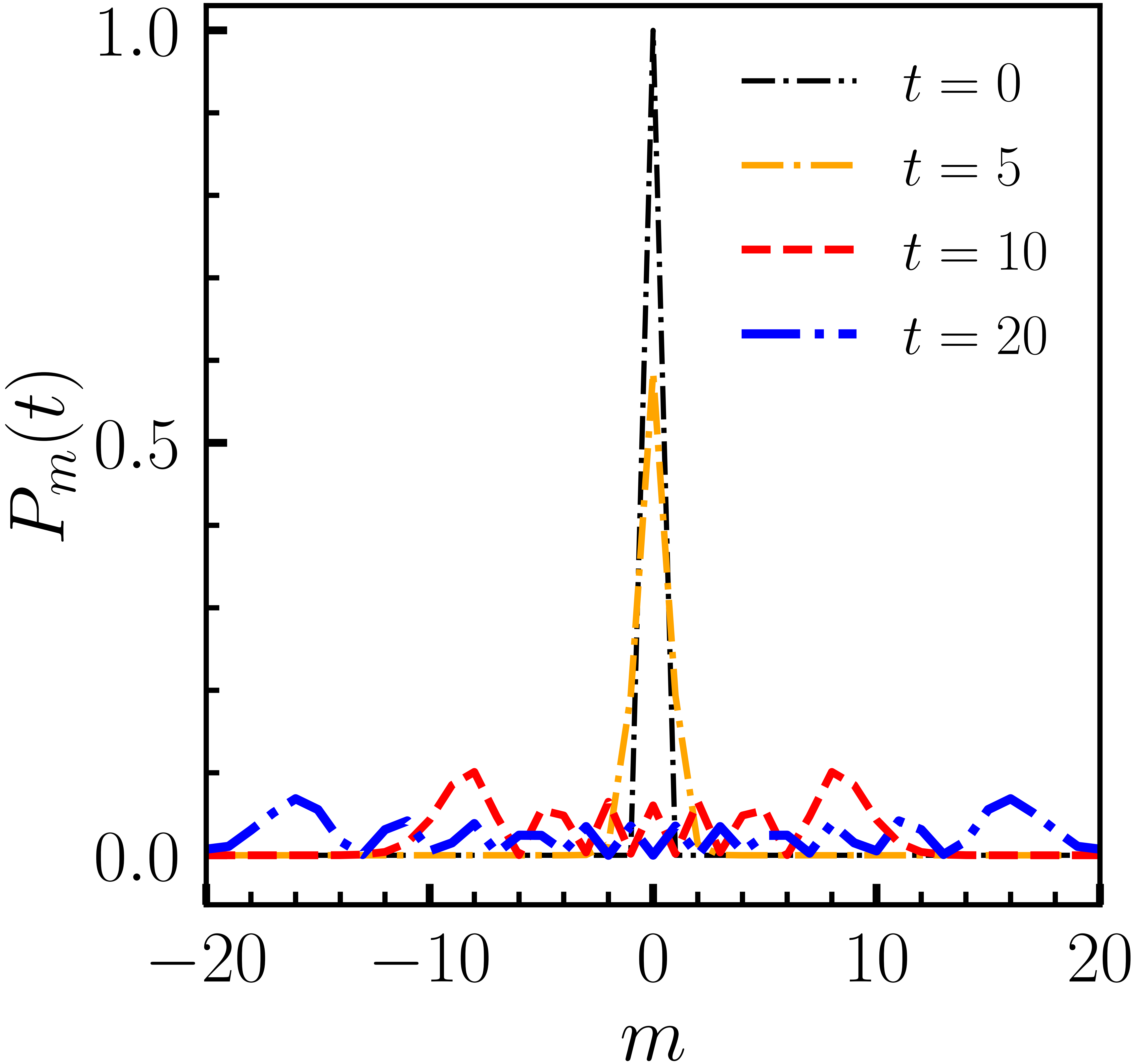
Initial location = n_0

Mean displacement:

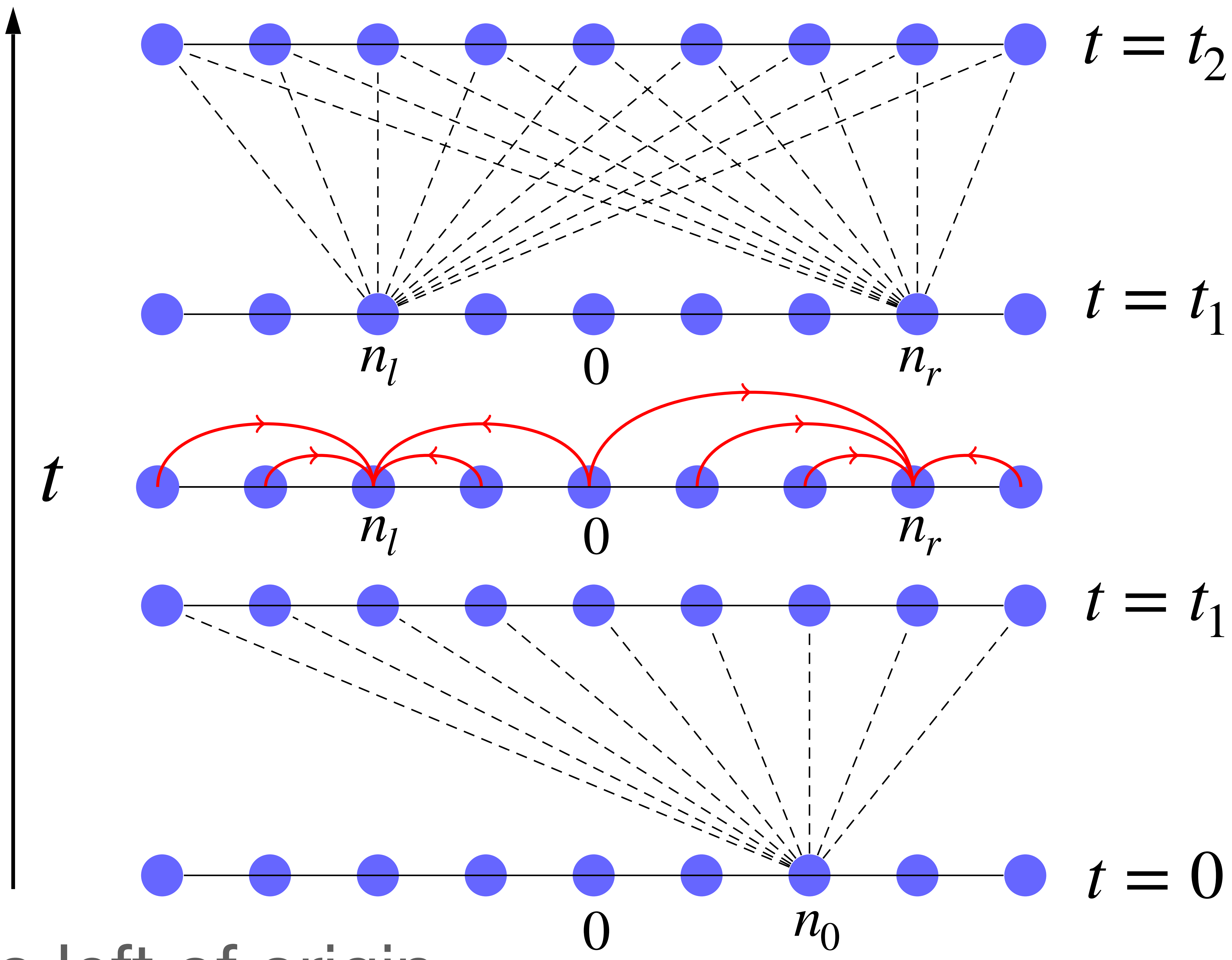
$$\langle m - n_0 \rangle = 0$$

Mean-squared displacement:

$$S(t) = \langle (m - n_0)^2 \rangle = \frac{\gamma^2 t^2}{2}$$



Conditional resetting protocol



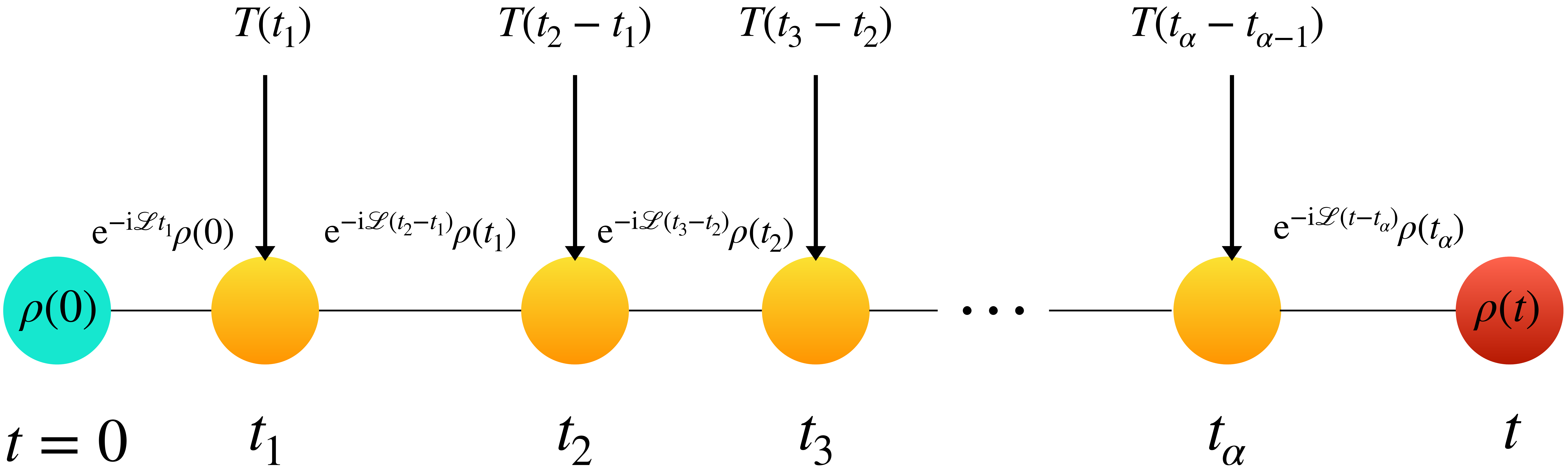
Prob to reset to site n_l to left of origin

$$\Omega_{n_l}^{n_0}(\gamma\tau_1) = \sum_{j=-1}^{-\infty} |\langle j | \mathrm{e}^{-\mathrm{i}H\tau_1} | n_0 \rangle|^2 + \frac{1}{2} \left| \langle 0 | \mathrm{e}^{-\mathrm{i}H\tau_1} | n_0 \rangle \right|^2$$

Prob to reset to site n_r to right of origin

$$\Omega_{n_r}^{n_0}(\gamma\tau_1) = \sum_{j=1}^{\infty} |\langle j | \mathrm{e}^{-\mathrm{i}H\tau_1} | n_0 \rangle|^2 + \frac{1}{2} |\langle 0 | \mathrm{e}^{-\mathrm{i}H\tau_1} | n_0 \rangle|^2$$

Unitary evolution interspersed with conditional resetting



$e^{-\mathrm{i}\mathscr{L}t}$: Implements unitary evolution

$T(t)$: Implements instantaneous conditional resetting

Generalised Lindblad dynamics with non-markovian evolution

$\overline{\rho}(t)$: **Density operator averaged over dynamical realisations involving different reset times**

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho}(t) = -\,\mathrm{i}\mathscr{L}(t)\overline{\rho}(t) + \lambda \sum_{\alpha=1}^{\infty} H_\alpha(t) - \lambda\overline{\rho}(t)$$

$$H_1(t) \equiv T(t)\ \mathrm{e}^{-\lambda t}\ \mathrm{e}_+^{-\mathrm{i}\int_0^t \mathrm{d}t'\ \mathscr{L}(t')}\rho(0)$$

$$H_\alpha(t) \equiv \lambda \int_0^t \mathrm{d}t_{\alpha-1} T(t-t_{\alpha-1})\mathrm{e}^{-\lambda(t-t_{\alpha-1})}\mathrm{e}_+^{-\mathrm{i}\int_{t_{\alpha-1}}^t \mathrm{d}t'\ \mathscr{L}(t')} H_{\alpha-1}(t_{\alpha-1});\ \ \alpha \geq 2$$

Usual Lindblad for unconditional resetting (Markovian):

$$\frac{\mathrm{d}\overline{\rho}(t)}{\mathrm{d}t} = -\,\mathrm{i}\mathscr{L}(t)\overline{\rho}(t) + \lambda T\overline{\rho}(t) - \lambda\overline{\rho}(t)$$

$$\frac{\mathrm{d}\overline{\rho}(t)}{\mathrm{d}t} = -\,\mathrm{i}\mathscr{L}(t)\rho(t) + \gamma\left(O\rho(t)O^\dagger - \frac{1}{2}\{O^\dagger O, \rho(t)\}\right);\quad \gamma = \lambda;\quad O\rho O^\dagger = T\rho$$

We obtain exact analytical results demonstrating effects of non-Markovian evolution in the context of open quantum systems

Exponential resetting:

Time interval between reset satisfies

$$p(\tau) = \lambda e^{-\lambda \tau}$$

Resetting locations symmetric with respect to origin; Initial location at origin

$$n_0 = 0$$

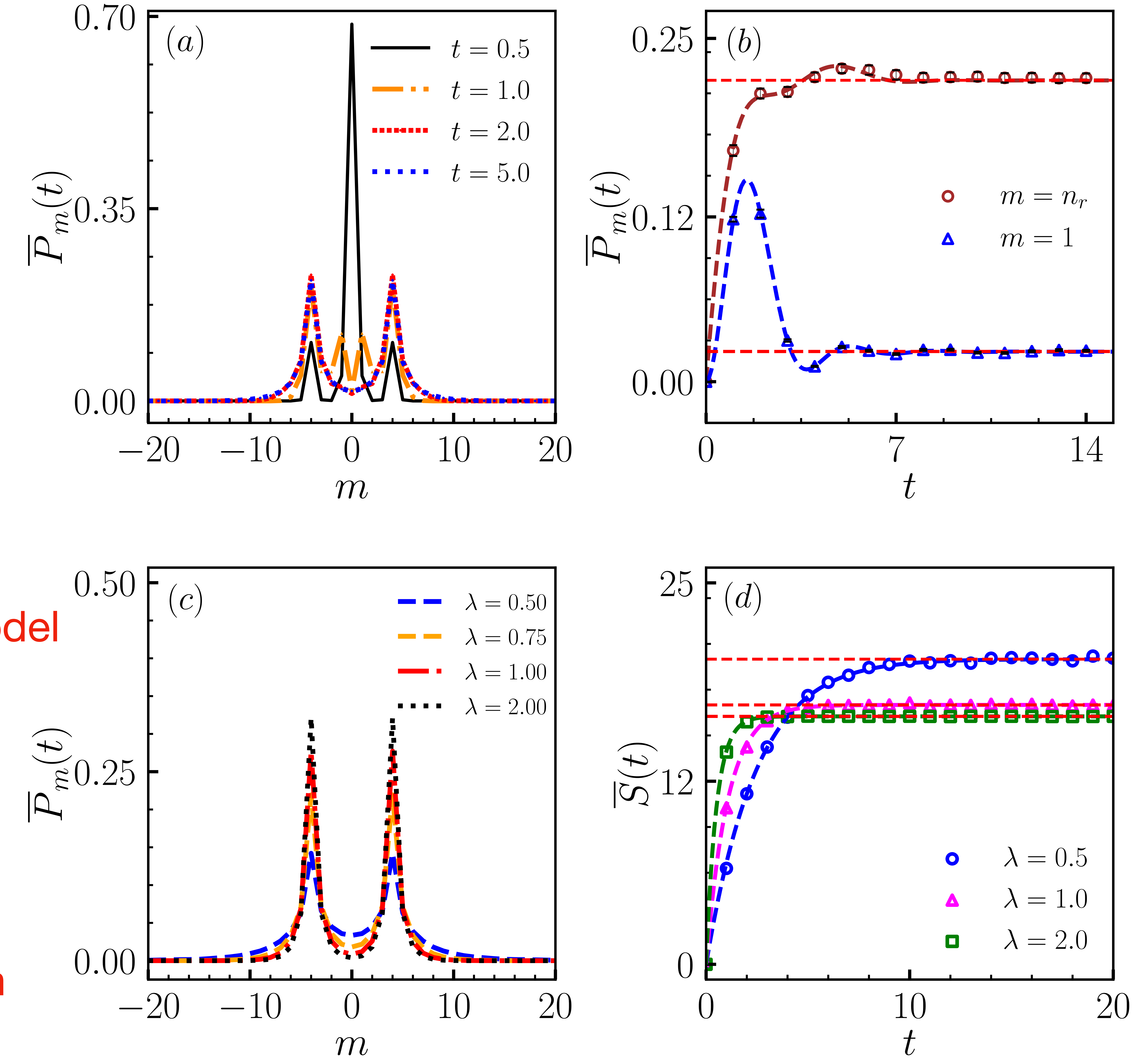
$$n_l = -4,\ n_r = 4$$

1. Stationary state, unlike the bare model

2. Localization around reset locations

3. Enhanced localisation with increase of reset rate

4. Particle most likely to be found with equal prob. at two reset sites



Resetting locations symmetric with respect to origin; Initial location NOT at origin

$$n_0 = 1$$

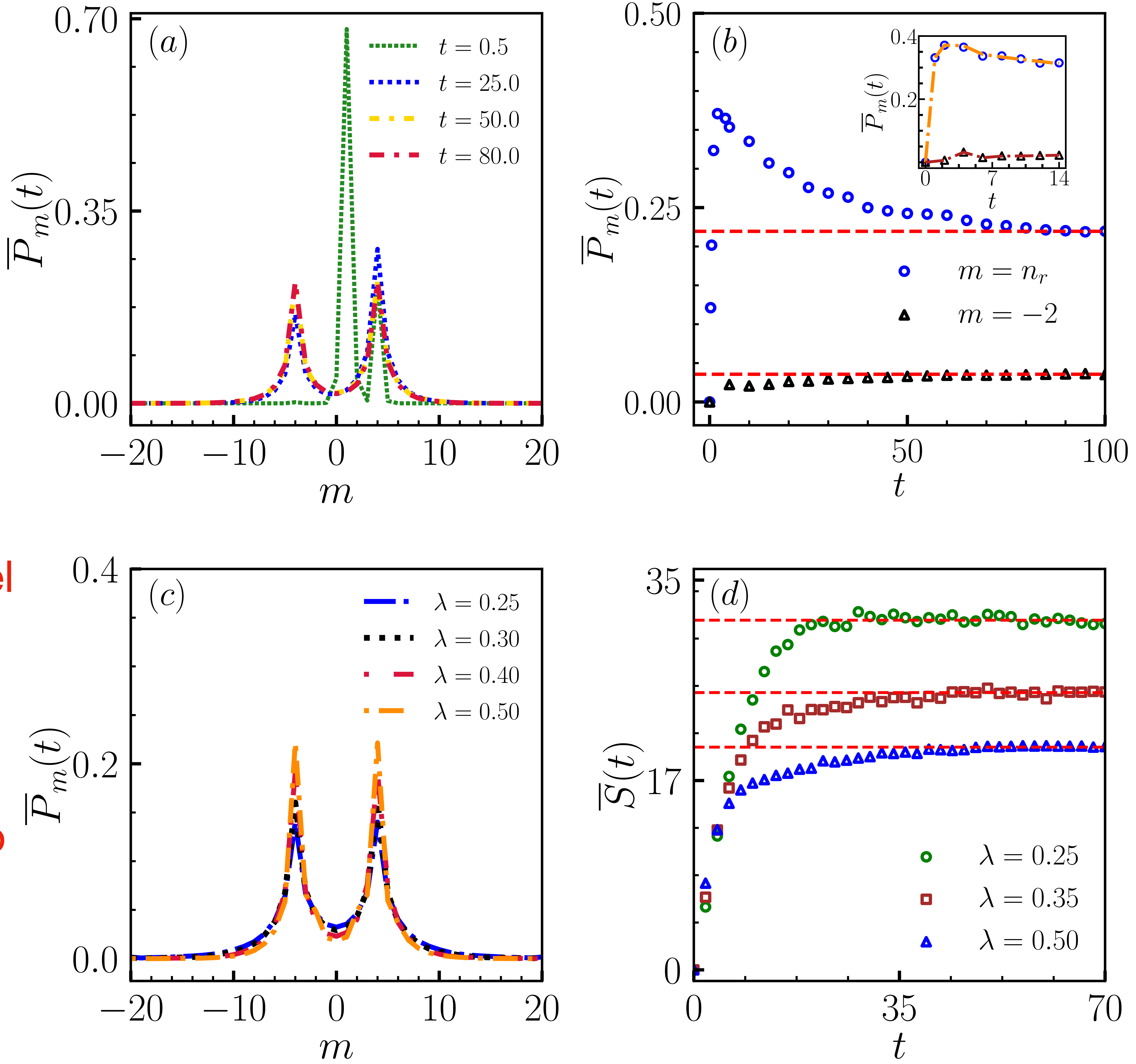
$$n_l = -4,\ n_r = 4$$

1. Stationary state, unlike the bare model

2. Localization around reset locations

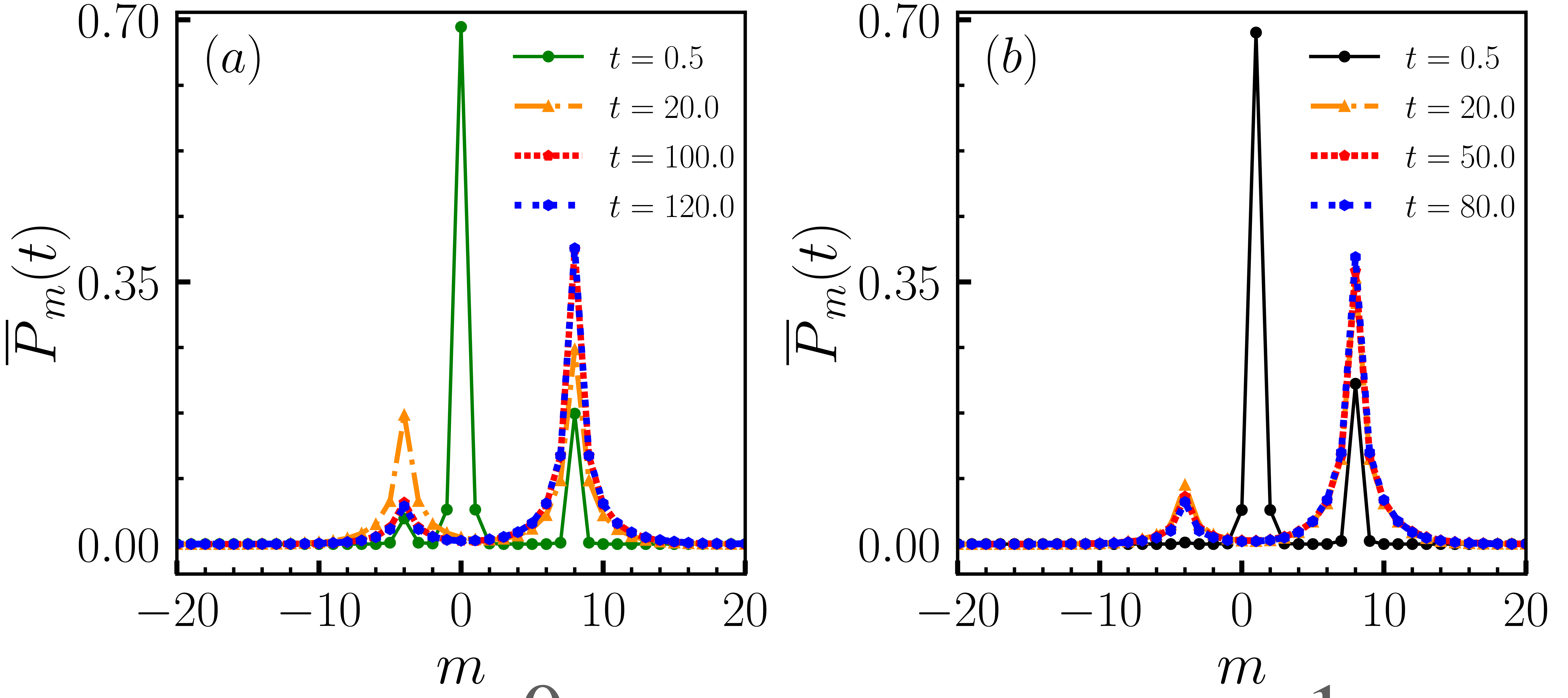
3. Transient state: particle more likely to be found to the right of the origin than to the left

4. Stationary state independent of initial location



Resetting locations NOT symmetric with respect to origin;
Initial location MAY or MAY NOT be at origin

$n_l = -4,\ n_r = 8$



$n_0 = 0$ $n_0 = 1$

1. Stationary state

2. Localization around reset locations

3. Particle more likely to be found at the reset location to the right than to the left

4. No bias in bare TBM dynamics, yet an effective bias towards one of the reset sites due to conditional resetting

Power-law resetting:

Time interval between reset satisfies

$$p(\tau) = \frac{\alpha}{\tau_0 (\tau/\tau_0)^{1+\alpha}}; \quad \alpha > 0; \quad \tau \in [\tau_0, \infty)$$

Resetting locations symmetric with respect to initial location; Initial location at origin

$$n_0 = 0$$
$$n_l = -4, \; n_r = 4$$

1. Stationary state for $\alpha > 2$

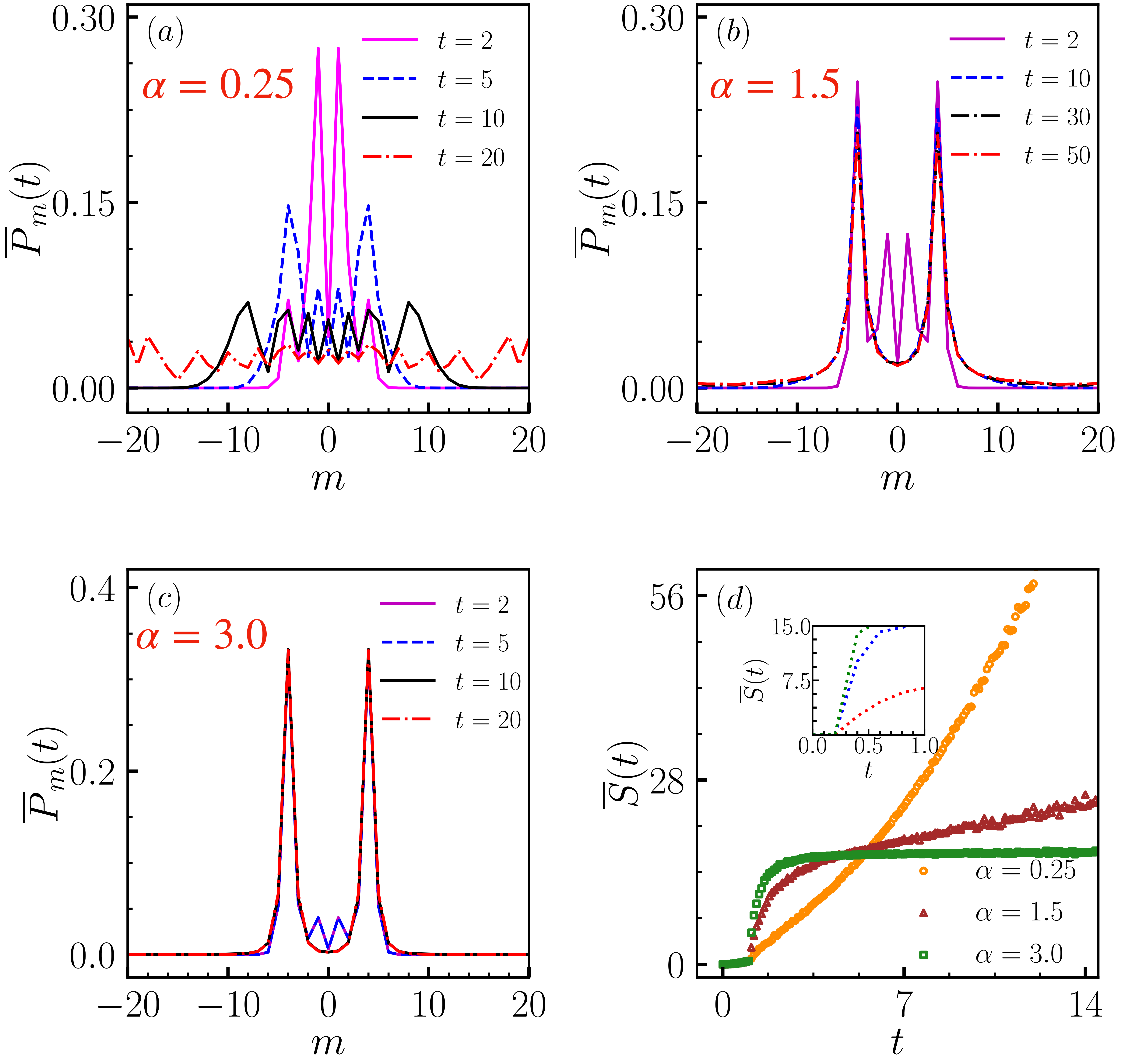
2. No stationary state for $\alpha < 1$

3. Stationary state and yet unbounded MSD for $1 < \alpha < 2$

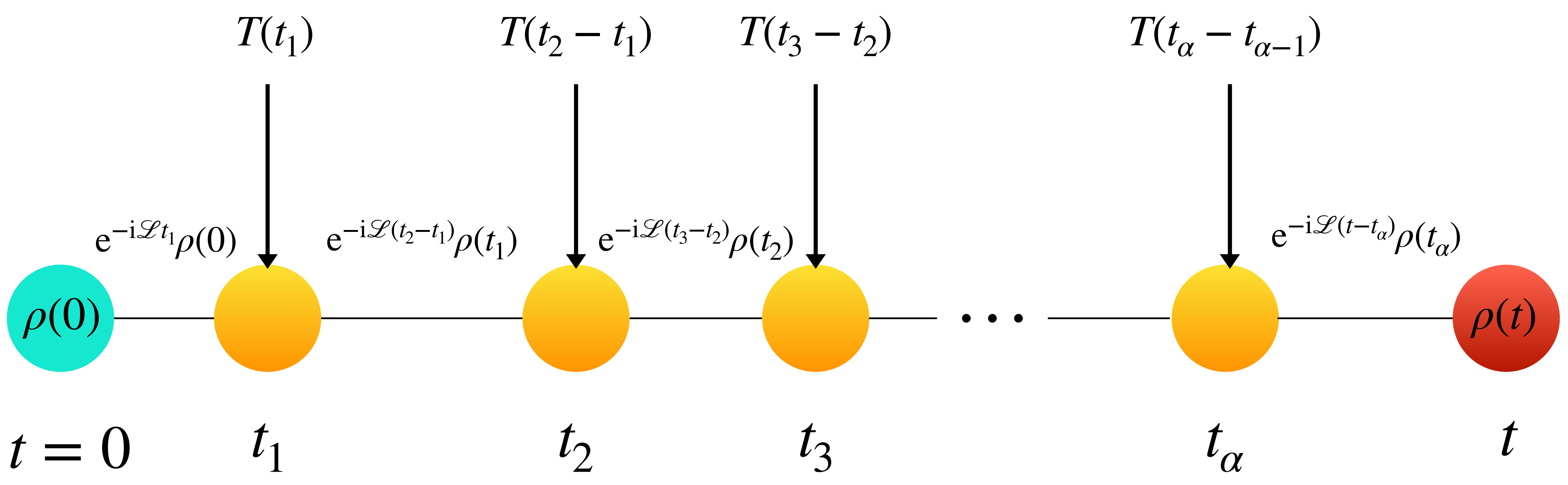
4. ***Lesson:*** Conditional reset does not ALWAYS lead to a stationary state

   $\langle \tau \rangle$ finite: Stationary State

   $\langle \tau \rangle$ infinite: NO Stationary State



Analytical calculation: A glimpse



Averaged density operator: $\overline{\rho}(t) = U(t)\rho(0)$

$$U(t) \equiv \sum_{\alpha=0}^{\infty} \int_0^t \mathrm{d}t_\alpha \int_0^{t_\alpha} \mathrm{d}t_{\alpha-1} \ldots \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 F(t-t_\alpha)\, \mathrm{e}_+^{-\mathrm{i}\int_{t_\alpha}^{t} \mathrm{d}t' \mathscr{L}(t')}\, T(t_\alpha - t_{\alpha-1})\, p(t_\alpha - t_{\alpha-1}) \mathrm{e}_+^{-\mathrm{i}\int_{t_{\alpha-1}}^{t_\alpha} \mathrm{d}t' \mathscr{L}(t')}$$
$$\ldots T(t_2 - t_1)\, p(t_2 - t_1)\, \mathrm{e}_+^{-\mathrm{i}\int_{t_1}^{t_2} \mathrm{d}t' \mathscr{L}(t')}\, T(t_1)\, p(t_1)\, \mathrm{e}_+^{-\mathrm{i}\int_0^{t_1} \mathrm{d}t' \mathscr{L}(t')}$$

Laplace space and exponential $p(\tau) = \lambda e^{-\lambda\tau}$ **:** $\widetilde{\overline{\rho}}(s) = \widetilde{U}(s)\rho(0)$

$$\widetilde{U}(s) = \widetilde{U}_0(s) \sum_{\alpha=0}^{\infty} \lambda^\alpha (\widetilde{T}'(s'))^\alpha$$

Analytical calculation: A glimpse (contd.)

$$\overline{P}_m(t) = \langle m \,|\, \overline{\rho}(t) \,|\, m \rangle \qquad \widetilde{\overline{P}}_m(s) = \langle m \,|\, \widetilde{U}_0(s) \sum_{\alpha=0}^{\infty} \lambda^\alpha (\widetilde{T}'(s'))^\alpha \rho(0) \,|\, m \rangle = \sum_{\alpha=0}^{\infty} \widetilde{\overline{P}}{}_m^{(\alpha)}(s)$$

$$\Omega_{n_r}^{n_0}(\gamma\tau_1) = \sum_{j=1}^{\infty} J^2_{|j-n_0|}(\gamma\tau_1) + \frac{1}{2} J^2_{|n_0|}(\gamma\tau_1); \;\; \Omega_{n_l}^{n_0}(\gamma\tau_1) = \sum_{j=-1}^{-\infty} J^2_{|j-n_0|}(\gamma\tau_1) + \frac{1}{2} J^2_{|n_0|}(\gamma\tau_1)$$

$$\overline{P}_m^{(1)}(t) = \lambda e^{-\lambda t} \int_0^t dt_1 (J^2_{|m-n_r|}(\gamma(t-t_1)) \Omega_{n_r}^{n_0}(\gamma t_1) + J^2_{|m-n_l|}(\gamma(t-t_1)) \Omega_{n_l}^{n_0}(\gamma t_1))$$

$$\overline{P}_m^{(2)}(t) = \lambda^2 e^{-\lambda t} \int_0^t dt_2 \int_0^{t_2} dt_1 \Big[J^2_{|m-n_r|}(\gamma(t-t_2)) \Big(\Omega_{n_r}^{n_r}(\gamma(t_2-t_1)) \Omega_{n_r}^{n_0}(\gamma t_1) + \Omega_{n_r}^{n_l}(\gamma(t_2-t_1)) \Omega_{n_l}^{n_0}(\gamma t_1) \Big)$$
$$+ J^2_{|m-n_l|}(\gamma(t-t_2)) \Big(\Omega_{n_l}^{n_r}(\gamma(t_2-t_1)) \Omega_{n_r}^{n_0}(\gamma t_1) + \Omega_{n_l}^{n_l}(\gamma(t_2-t_1)) \Omega_{n_l}^{n_0}(\gamma t_1) \Big) \Big]$$

$$\overline{P}_m^{(\alpha \geq 1)}(t) = \lambda^\alpha e^{-\lambda t} \Big[\int_0^t dt_\alpha \int_0^{t_\alpha} dt_{\alpha-1} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \Big[J^2_{|m-n_r|}(\gamma(t-t_\alpha)) \sum_{\mu_1,\mu_2,\ldots,\mu_\alpha} \Omega_{n_r}^{\mu_\alpha}(\gamma(t_\alpha - t_{\alpha-1})) \ldots \Omega_{\mu_2}^{\mu_1}(\gamma(t_2-t_1)) \Omega_{\mu_1}^{n_0}(\gamma t_1)$$
$$+ J^2_{|m-n_l|}(\gamma(t-t_\alpha)) \sum_{\mu_1,\mu_2,\ldots,\mu_\alpha} \Omega_{n_l}^{\mu_\alpha}(\gamma(t_\alpha - t_{\alpha-1})) \ldots \Omega_{\mu_2}^{\mu_1}(\gamma(t_2-t_1)) \Omega_{\mu_1}^{n_0}(\gamma t_1) \Big]$$

Take-home message(s)

1. Studied conditional reset of tight-binding (TBM) quantum particle to two reset locations, conditioned on the location of the particle at the time instant of reset; reset rates symmetric with respect to origin
2. Choice of reset locations with respect to origin crucial in dictating the stationary state
3. Considering reset locations asymmetrically disposed around the origin leads to enhanced localization around one of the reset sites, despite no explicit bias in the dynamics
4. Future scope: TBM with an inherent bias and its interplay with reset-induced bias; More general (?) protocol for conditional resetting, ….

Thank You for your attention !