

# Brill-Noether loci inside moduli space of stable vector bundles over curves

Pritthijit Biswas

Indian Institute of Technology Bombay

February 25, 2026

Joint with Jaya N. Iyer  
Geometric Structures and Stability  
ICTS, Bangalore

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .
- ▶  $X$  - smooth projective curve of genus  $g \geq 2$ .

$L$  - Line bundle on  $X$  of degree  $2g - 1$ .

$M_X(2, L)$  - Moduli space of rank 2 stable vector bundles on  $X$  with fixed determinant  $L$ .

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .
- ▶  $X$  - smooth projective curve of genus  $g \geq 2$ .

$L$  - Line bundle on  $X$  of degree  $2g - 1$ .

$M_X(2, L)$  - Moduli space of rank 2 stable vector bundles on  $X$  with fixed determinant  $L$ .

- ▶  $W_X^1(2, L) := \{E \in M_X(2, L) : h^0(E) \geq 2\} \subseteq M_X(2, L)$  is the Brill-Noether subvariety.

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .
- ▶  $X$  - smooth projective curve of genus  $g \geq 2$ .

$L$  - Line bundle on  $X$  of degree  $2g - 1$ .

$M_X(2, L)$  - Moduli space of rank 2 stable vector bundles on  $X$  with fixed determinant  $L$ .

- ▶  $W_X^1(2, L) := \{E \in M_X(2, L) : h^0(E) \geq 2\} \subseteq M_X(2, L)$  is the Brill-Noether subvariety.
- ▶ The expected dimension of  $W_X^1(2, L)$  is  $3g - 5$ .

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .
- ▶  $X$  - smooth projective curve of genus  $g \geq 2$ .  
 $L$  - Line bundle on  $X$  of degree  $2g - 1$ .  
 $M_X(2, L)$  - Moduli space of rank 2 stable vector bundles on  $X$  with fixed determinant  $L$ .
- ▶  $W_X^1(2, L) := \{E \in M_X(2, L) : h^0(E) \geq 2\} \subseteq M_X(2, L)$  is the Brill-Noether subvariety.
- ▶ The expected dimension of  $W_X^1(2, L)$  is  $3g - 5$ .
- ▶ The projective space  $\mathbb{P}(\text{Ext}^1(L, \mathcal{O})^*)$  will be denoted by  $\mathbb{P}_L$ .

# Notations

- ▶ Will work over the field  $\mathbb{C}$ .
- ▶  $X$  - smooth projective curve of genus  $g \geq 2$ .  
 $L$  - Line bundle on  $X$  of degree  $2g - 1$ .  
 $M_X(2, L)$  - Moduli space of rank 2 stable vector bundles on  $X$  with fixed determinant  $L$ .
- ▶  $W_X^1(2, L) := \{E \in M_X(2, L) : h^0(E) \geq 2\} \subseteq M_X(2, L)$  is the Brill-Noether subvariety.
- ▶ The expected dimension of  $W_X^1(2, L)$  is  $3g - 5$ .
- ▶ The projective space  $\mathbb{P}(\text{Ext}^1(L, \mathcal{O})^*)$  will be denoted by  $\mathbb{P}_L$ .
- ▶ The aim of this talk is to study the rationality properties of the variety  $W_X^1(2, L)$ .

# Rank 2 special vector bundles and the extension map

## Rank 2 special vector bundles and the extension map

- ▶ Let  $\pi_X$  and  $\pi_L$  denote the projections from  $X \times \mathbb{P}_L$  to  $X$  and  $\mathbb{P}_L$  respectively. Then there is a natural Poincaré extension on  $X \times \mathbb{P}_L$ , given by:

$$0 \rightarrow \pi_L^* \mathcal{O}_{\mathbb{P}_L}(1) \rightarrow \mathcal{E}_L \rightarrow \pi_X^* L \rightarrow 0$$

which satisfies the property that for any closed point  $x \in \mathbb{P}_L$ , the restriction of the above to  $X \cong X \times x$  corresponds to  $x$ .

## Rank 2 special vector bundles and the extension map

- ▶ Let  $\pi_X$  and  $\pi_L$  denote the projections from  $X \times \mathbb{P}_L$  to  $X$  and  $\mathbb{P}_L$  respectively. Then there is a natural Poincaré extension on  $X \times \mathbb{P}_L$ , given by:

$$0 \rightarrow \pi_L^* \mathcal{O}_{\mathbb{P}_L}(1) \rightarrow \mathcal{E}_L \rightarrow \pi_X^* L \rightarrow 0$$

which satisfies the property that for any closed point  $x \in \mathbb{P}_L$ , the restriction of the above to  $X \cong X \times x$  corresponds to  $x$ .

- ▶ The existence of the Poincaré extension is given by the image of  $id_{H^1(X, L^*)}$  under the Künneth isomorphism:

$$H^1(X \times \mathbb{P}_L, \pi_X^* L^* \otimes \pi_L^* \mathcal{O}_{\mathbb{P}_L}(1)) \cong H^1(X, L^*) \otimes H^1(X, L^*)^*$$

- ▶ This gives a rational map

$$\phi_L : \mathbb{P}_L \dashrightarrow M_X(2, L)$$

- ▶ This gives a rational map

$$\phi_L : \mathbb{P}_L \dashrightarrow M_X(2, L)$$

- ▶ if  $\deg(L) < 0$ , then the domain of the map  $\phi_L$  is empty, whereas if  $\deg(L) = 0, 1$  or  $2$ , then domain of  $\phi_L$  is whole of  $\mathbb{P}_L$ .

- ▶ This gives a rational map

$$\phi_L : \mathbb{P}_L \dashrightarrow M_X(2, L)$$

- ▶ if  $\deg(L) < 0$ , then the domain of the map  $\phi_L$  is empty, whereas if  $\deg(L) = 0, 1$  or  $2$ , then domain of  $\phi_L$  is whole of  $\mathbb{P}_L$ .
- ▶ If  $D$  is an effective divisor on  $X$ , then there is a natural map  $h : L^* \rightarrow L^*(D)$ .

- ▶ This gives a rational map

$$\phi_L : \mathbb{P}_L \dashrightarrow M_X(2, L)$$

- ▶ if  $\deg(L) < 0$ , then the domain of the map  $\phi_L$  is empty, whereas if  $\deg(L) = 0, 1$  or  $2$ , then domain of  $\phi_L$  is whole of  $\mathbb{P}_L$ .
- ▶ If  $D$  is an effective divisor on  $X$ , then there is a natural map  $h : L^* \rightarrow L^*(D)$ .
- ▶ We denote the *linear span* of an effective divisor  $D$  by  $\overline{D}$  and define it by  $\mathbb{P}(\text{Ker}(h_* : H^1(X, L^*) \rightarrow H^1(X, L^*(D))))^* \subseteq \mathbb{P}_L$ .

- ▶ This gives a rational map

$$\phi_L : \mathbb{P}_L \dashrightarrow M_X(2, L)$$

- ▶ if  $\deg(L) < 0$ , then the domain of the map  $\phi_L$  is empty, whereas if  $\deg(L) = 0, 1$  or  $2$ , then domain of  $\phi_L$  is whole of  $\mathbb{P}_L$ .
- ▶ If  $D$  is an effective divisor on  $X$ , then there is a natural map  $h : L^* \rightarrow L^*(D)$ .
- ▶ We denote the *linear span* of an effective divisor  $D$  by  $\overline{D}$  and define it by  $\mathbb{P}(\text{Ker}(h_* : H^1(X, L^*) \rightarrow H^1(X, L^*(D))))^* \subseteq \mathbb{P}_L$ .
- ▶ As  $h_* : H^1(X, L^*) \rightarrow H^1(X, L^*(D))$  is a surjection,

$$\dim(\overline{D}) = h^1(L^*) - h^1(L^*(D)) - 1$$

# Stability in terms of linear span of effective divisors

# Stability in terms of linear span of effective divisors

## Lemma

Let  $e = 0 \rightarrow \mathcal{O} \xrightarrow{i} E \xrightarrow{\pi'} L \rightarrow 0$  be non split,  $L$  be of any degree and denote by  $x$ , the element in  $\mathbb{P}_L$  determined by  $e$ . Then the following are true:

- (i)  $E$  is not stable if and only if  $x \in \overline{D}$  for some effective divisor  $D$  of degree at most  $\frac{\deg(E)}{2}$
- (ii)  $E$  is not semistable if and only if  $x \in \overline{D}$  for some effective divisor  $D$  of degree strictly less than  $\frac{\deg(E)}{2}$ .
- (iii) If  $E$  is semistable, then  $E$  is not stable if and only if  $x \in \overline{D}$  for some effective divisor  $D$  of degree  $\frac{\deg(E)}{2}$ .

- ▶  $X$  is embedded in  $\mathbb{P}_L$  since  $g \geq 2$  and  $\deg(L) = 2g - 1$  so that  $\deg(K \otimes L) \geq 2g + 1$ .

- ▶  $X$  is embedded in  $\mathbb{P}_L$  since  $g \geq 2$  and  $\deg(L) = 2g - 1$  so that  $\deg(K \otimes L) \geq 2g + 1$ .
- ▶ For any  $r \geq 1$ ,  $\text{Sec}^r(X)$  will denote the *secant variety* of  $(r - 1)$ -planes through  $r$  points of  $X$  inside  $\mathbb{P}_L$ , and is defined by

$$\text{Sec}^r(X) := \bigcup_{D \geq 0, \deg D=r} \overline{D}.$$

- ▶  $X$  is embedded in  $\mathbb{P}_L$  since  $g \geq 2$  and  $\deg(L) = 2g - 1$  so that  $\deg(K \otimes L) \geq 2g + 1$ .
- ▶ For any  $r \geq 1$ ,  $\text{Sec}^r(X)$  will denote the *secant variety* of  $(r - 1)$ -planes through  $r$  points of  $X$  inside  $\mathbb{P}_L$ , and is defined by

$$\text{Sec}^r(X) := \bigcup_{D \geq 0, \deg D=r} \overline{D}.$$

- ▶  $\text{Sec}^1(X) = X$ , for any  $r \geq 1$ ,  $\text{Sec}^r(X)$  is at most of dimension  $2r - 1$  and there is an inclusion  $\text{Sec}^r(X) \subseteq \text{Sec}^{r+1}(X)$ .

- ▶  $X$  is embedded in  $\mathbb{P}_L$  since  $g \geq 2$  and  $\deg(L) = 2g - 1$  so that  $\deg(K \otimes L) \geq 2g + 1$ .
- ▶ For any  $r \geq 1$ ,  $\text{Sec}^r(X)$  will denote the *secant variety* of  $(r - 1)$ -planes through  $r$  points of  $X$  inside  $\mathbb{P}_L$ , and is defined by

$$\text{Sec}^r(X) := \bigcup_{D \geq 0, \deg D=r} \overline{D}.$$

- ▶  $\text{Sec}^1(X) = X$ , for any  $r \geq 1$ ,  $\text{Sec}^r(X)$  is at most of dimension  $2r - 1$  and there is an inclusion  $\text{Sec}^r(X) \subseteq \text{Sec}^{r+1}(X)$ .
- ▶ The closed subvariety  $\text{Sec}^{g-1}(X)$  of  $\mathbb{P}_L$  is precisely the locus of those extensions with middle term being a rank 2 vector bundle with determinant  $L$  and is not stable.

- ▶ Thus the rational map  $\phi_L$  restricts to a morphism:

$$\phi_L : \mathbb{P}_L \setminus \text{Sec}^{k(L)}(X) \rightarrow M_X(2, L)$$

- ▶ Thus the rational map  $\phi_L$  restricts to a morphism:

$$\phi_L : \mathbb{P}_L \setminus \text{Sec}^{k(L)}(X) \rightarrow M_X(2, L)$$

- ▶ The morphism  $\phi_L$  is birational.

[Bertram, A. *Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective spaces*. J. Differential Geom. **35**, pp. 429-469,1992.]

- ▶ Thus the rational map  $\phi_L$  restricts to a morphism:

$$\phi_L : \mathbb{P}_L \setminus \text{Sec}^{k(L)}(X) \rightarrow M_X(2, L)$$

- ▶ The morphism  $\phi_L$  is birational.

[Bertram, A. *Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective spaces*. J. Differential Geom. **35**, pp. 429-469,1992.]

- ▶ We will use the extension map  $\phi_L$  to study the rationality properties of the subvariety  $W_X^1(2, L)$

# Brill-Noether hypersurface in $\mathbb{P}_L$

## Brill-Noether hypersurface in $\mathbb{P}_L$

- ▶ On pushing down the Poincaré extension to  $\mathbb{P}_L$ , we get the following exact sequence of sheaves on  $\mathbb{P}_L$ .

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}_L}(1) \rightarrow \pi_{L*}\mathcal{E}_L \rightarrow H^0(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \xrightarrow{\lambda} H^1(X, \mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}_L}(1) \\ \rightarrow R^1\pi_{L*}\mathcal{E}_L \rightarrow H^1(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \rightarrow 0 \end{aligned}$$

## Brill-Noether hypersurface in $\mathbb{P}_L$

- ▶ On pushing down the Poincaré extension to  $\mathbb{P}_L$ , we get the following exact sequence of sheaves on  $\mathbb{P}_L$ .

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}_L}(1) \rightarrow \pi_{L*}\mathcal{E}_L \rightarrow H^0(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \xrightarrow{\lambda} H^1(X, \mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}_L}(1) \\ \rightarrow R^1\pi_{L*}\mathcal{E}_L \rightarrow H^1(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \rightarrow 0 \end{aligned}$$

- ▶ As  $\deg(L) = 2g - 1$ , the map  $\lambda$  is a  $g \times g$  matrix of linear forms. Let  $\mathcal{H}$  be the closed subscheme of  $\mathbb{P}_L$  corresponding to the ideal sheaf generated locally by  $\det(\lambda)$ .

## Brill-Noether hypersurface in $\mathbb{P}_L$

- ▶ On pushing down the Poincaré extension to  $\mathbb{P}_L$ , we get the following exact sequence of sheaves on  $\mathbb{P}_L$ .

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}_L}(1) \rightarrow \pi_{L*}\mathcal{E}_L \rightarrow H^0(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \xrightarrow{\lambda} H^1(X, \mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}_L}(1) \\ \rightarrow R^1\pi_{L*}\mathcal{E}_L \rightarrow H^1(X, L) \otimes \mathcal{O}_{\mathbb{P}_L} \rightarrow 0 \end{aligned}$$

- ▶ As  $\deg(L) = 2g - 1$ , the map  $\lambda$  is a  $g \times g$  matrix of linear forms. Let  $\mathcal{H}$  be the closed subscheme of  $\mathbb{P}_L$  corresponding to the ideal sheaf generated locally by  $\det(\lambda)$ .
- ▶ The set of closed points of  $\mathcal{H}$  is given by the set

$$\{x = [[0 \rightarrow \mathcal{O} \rightarrow E \rightarrow L \rightarrow 0]] \in \mathbb{P}_L \mid h^0(E) \geq 2\}.$$

## Proposition (\_\_\_, Iyer, 2025)

*If  $g \geq 2$  and  $\deg(L) = 2g - 1$ , then there is a stable rank 2 vector bundle  $E$  on  $X$  satisfying a non-split short exact sequence*

$$e := 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow L \rightarrow 0$$

*of vector bundles on  $X$  such that  $e$  lies in the hypersurface  $\mathcal{H}$  inside  $\mathbb{P}_L$ . Moreover one can choose the above  $E$  to satisfy  $h^0(E) = 2$ .*

## Proposition (\_\_, Iyer, 2025)

If  $g \geq 2$  and  $\deg(L) = 2g - 1$ , then there is a stable rank 2 vector bundle  $E$  on  $X$  satisfying a non-split short exact sequence

$$e := 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow L \rightarrow 0$$

of vector bundles on  $X$  such that  $e$  lies in the hypersurface  $\mathcal{H}$  inside  $\mathbb{P}_L$ . Moreover one can choose the above  $E$  to satisfy  $h^0(E) = 2$ .

- ▶ If  $\mathcal{H}$  does not contain an extension with stable middle term, then

$$\mathcal{H} \subseteq \bigcup_{0 \leq r \leq g-1} \text{Sec}^r(X).$$

As  $g \geq 2$ , a dimension count gives a contradiction.

# Non-emptiness results for Brill-Noether subvarieties

# Non-emptiness results for Brill-Noether subvarieties

Theorem (M. Teixidor. I. Bigas, 1991)

*$W_X^1(2, 2g - 1)$  has at least one component of the expected dimension  $4g - 5$  and a generic  $E$  in  $W_X^1(2, 2g - 1)$  satisfies  $h^0(E) = 2$ . If  $X$  is generic, then this is the only component of  $W_X^1(2, 2g - 1)$ .*

# Non-emptiness results for Brill-Noether subvarieties

Theorem (M. Teixidor. I. Bigas, 1991)

*$W_X^1(2, 2g - 1)$  has at least one component of the expected dimension  $4g - 5$  and a generic  $E$  in  $W_X^1(2, 2g - 1)$  satisfies  $h^0(E) = 2$ . If  $X$  is generic, then this is the only component of  $W_X^1(2, 2g - 1)$ .*

Theorem (M. Teixidor. I. Bigas, 2010)

*If  $X$  and  $L$  are both generic with  $\deg(L) = 2g - 1$ , then  $W_X^1(2, L)$  is non empty and has at least one irreducible component of the expected dimension  $3g - 5$ .*

## Proposition (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$  and  $L$  be a line bundle on  $X$  with  $\deg(L) = 2g - 1$ . If  $X$  and  $L$  are both generic, then the scheme  $W_X^1(2, L)$  is irreducible of dimension  $3g - 5$ .*

## Proposition (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$  and  $L$  be a line bundle on  $X$  with  $\deg(L) = 2g - 1$ . If  $X$  and  $L$  are both generic, then the scheme  $W_X^1(2, L)$  is irreducible of dimension  $3g - 5$ .*

- ▶  $V := \mathcal{H} \cap (\mathbb{P}_L \setminus \text{Sec}^{k(L)}(X))$  is a non empty open subset of  $\mathcal{H}$ .

## Proposition (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$  and  $L$  be a line bundle on  $X$  with  $\deg(L) = 2g - 1$ . If  $X$  and  $L$  are both generic, then the scheme  $W_X^1(2, L)$  is irreducible of dimension  $3g - 5$ .*

- ▶  $V := \mathcal{H} \cap (\mathbb{P}_L \setminus \text{Sec}^{k(L)}(X))$  is a non empty open subset of  $\mathcal{H}$ .
- ▶  $\phi_L$  restricts to a morphism

$$\phi_L : V \rightarrow W_X^1(2, L)$$

## Proposition (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$  and  $L$  be a line bundle on  $X$  with  $\deg(L) = 2g - 1$ . If  $X$  and  $L$  are both generic, then the scheme  $W_X^1(2, L)$  is irreducible of dimension  $3g - 5$ .*

- ▶  $V := \mathcal{H} \cap (\mathbb{P}_L \setminus \text{Sec}^{k(L)}(X))$  is a non empty open subset of  $\mathcal{H}$ .
- ▶  $\phi_L$  restricts to a morphism

$$\phi_L : V \rightarrow W_X^1(2, L)$$

## Proposition (\_\_\_, Iyer, 2025)

*If  $X$  and  $L$  are generic with  $\deg(L) = 2g - 1$ , then  $\phi_L : V \rightarrow W_X^1(2, L)$  is dominant*

# Stably-rationality of $W_X^1(2, L)$ when $g = 3$

# Stably-rationality of $W_X^1(2, L)$ when $g = 3$

Proposition (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$ ,  $L$  be a line bundle on  $X$  of degree  $2g - 1$  and  $U$  be the open subscheme of  $W_X^1(2, L)$  parameterised by rank 2 stable vector bundles  $E$  with determinant  $L$  and  $h^0(E) = 2$ . Assume that  $X$  and  $L$  are generic.*

*Then the morphism  $\phi_L : \phi_L^{-1}(U) \rightarrow U$  is birationally equivalent to a  $\mathbb{P}^1$ -bundle.*

## Theorem (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus 3 and  $L$  be a line bundle on  $X$  of degree 5. Assume that  $X$  and  $L$  are generic. Then  $W_X^1(2, L)$  is stably-rational.*

# Unirationality of $W_X^1(2, L)$ when $g = 4$

## Unirationality of $W_X^1(2, L)$ when $g = 4$

- ▶  $\mathbb{P} := \mathbb{P}(S^d(V^*))$ . Then  $\mathbb{P}$  parametrises all the degree  $d$  hypersurfaces of  $\mathbb{P}^n$ .

## Unirationality of $W_X^1(2, L)$ when $g = 4$

- ▶  $\mathbb{P} := \mathbb{P}(S^d(V^*))$ . Then  $\mathbb{P}$  parametrises all the degree  $d$  hypersurfaces of  $\mathbb{P}^n$ .
- ▶  $\mathbb{G}(r, n)$  denote the Grassmann variety parametrising the  $r$ -dimensional projective linear subspaces of  $\mathbb{P}^n$ . Consider the incidence variety

$$I(r, n, d) := \{(X, \Lambda) \in \mathbb{P} \times \mathbb{G}(r, n) : \Lambda \subseteq X\}.$$

## Unirationality of $W_X^1(2, L)$ when $g = 4$

- ▶  $\mathbb{P} := \mathbb{P}(S^d(V^*))$ . Then  $\mathbb{P}$  parametrises all the degree  $d$  hypersurfaces of  $\mathbb{P}^n$ .
- ▶  $\mathbb{G}(r, n)$  denote the Grassmann variety parametrising the  $r$ -dimensional projective linear subspaces of  $\mathbb{P}^n$ . Consider the incidence variety

$$I(r, n, d) := \{(X, \Lambda) \in \mathbb{P} \times \mathbb{G}(r, n) : \Lambda \subseteq X\}.$$

- ▶  $I(r, n, d)$  is non empty and has dimension  $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$ .

## Unirationality of $W_X^1(2, L)$ when $g = 4$

- ▶  $\mathbb{P} := \mathbb{P}(S^d(V^*))$ . Then  $\mathbb{P}$  parametrises all the degree  $d$  hypersurfaces of  $\mathbb{P}^n$ .
- ▶  $\mathbb{G}(r, n)$  denote the Grassmann variety parametrising the  $r$ -dimensional projective linear subspaces of  $\mathbb{P}^n$ . Consider the incidence variety

$$I(r, n, d) := \{(X, \Lambda) \in \mathbb{P} \times \mathbb{G}(r, n) : \Lambda \subseteq X\}.$$

- ▶  $I(r, n, d)$  is non empty and has dimension  $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$ .
- ▶  $p_1, p_2$  denote the natural projections of  $I(r, n, d)$  to  $\mathbb{P}$  and  $\mathbb{G}(r, n)$  respectively.

## Unirationality of $W_X^1(2, L)$ when $g = 4$

- ▶  $\mathbb{P} := \mathbb{P}(S^d(V^*))$ . Then  $\mathbb{P}$  parametrises all the degree  $d$  hypersurfaces of  $\mathbb{P}^n$ .
- ▶  $\mathbb{G}(r, n)$  denote the Grassmann variety parametrising the  $r$ -dimensional projective linear subspaces of  $\mathbb{P}^n$ . Consider the incidence variety

$$I(r, n, d) := \{(X, \Lambda) \in \mathbb{P} \times \mathbb{G}(r, n) : \Lambda \subseteq X\}.$$

- ▶  $I(r, n, d)$  is non empty and has dimension  $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$ .
- ▶  $p_1, p_2$  denote the natural projections of  $I(r, n, d)$  to  $\mathbb{P}$  and  $\mathbb{G}(r, n)$  respectively.
- ▶ if  $d \geq 2$ , then  $p_1$  is surjective if and only if the following two conditions hold:
  - (i) If  $d \geq 3$ , then  $(r+1)(n-r) \geq \binom{r+d}{d}$
  - (ii) If  $d = 2$ , then  $n \geq 2r + 1$ .

## Theorem (Marchisio, 2000)

*Any degree  $d$  projective hypersurface which contains a projective linear subspace of dimension  $d - 2$  is unirational for  $d = 4, 5$ .*

## Theorem (Marchisio, 2000)

*Any degree  $d$  projective hypersurface which contains a projective linear subspace of dimension  $d - 2$  is unirational for  $d = 4, 5$ .*

## Proposition (\_\_\_, Iyer, 2025)

*When the genus  $g = 4$ , the quartic hypersurface  $\mathcal{H}$  is unirational.*

## Theorem (Marchisio, 2000)

*Any degree  $d$  projective hypersurface which contains a projective linear subspace of dimension  $d - 2$  is unirational for  $d = 4, 5$ .*

## Proposition (\_\_\_, Iyer, 2025)

*When the genus  $g = 4$ , the quartic hypersurface  $\mathcal{H}$  is unirational.*

## Theorem (\_\_\_, Iyer, 2025)

*Assume  $X$  and  $L$  are generic,  $g = 4$ , then  $W_X^1(2, L)$  is unirational.*

# Rational chain-connectedness for $g \geq 5$

Theorem (\_\_\_, Iyer, 2025)

*When  $g \geq 5$  the Brill Noether loci  $W_X^1(2, L)$  is chain connected by Hecke curves of length 2.*

# Rational Chow groups of the Brill-Noether hypersurface $\mathcal{H}$

Theorem (\_\_\_, Iyer, 2025)

*Let  $X$  be a smooth projective curve of genus  $g \geq 2$ ,  $L$  be a line bundle on  $X$  of degree  $2g - 1$  and  $\mathcal{H}$  be the degree  $g$  hypersurface in  $\mathbb{P}_L$ . Then*

$$CH_0(\mathcal{H})_{\mathbb{Q}} \cong \mathbb{Q}.$$

*Furthermore If  $g$  is either 3,4,5 or 6, then*

$$CH_1(\mathcal{H})_{\mathbb{Q}} \cong \mathbb{Q}.$$

## References

- ▶ Bertram, A. *Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space*. J. Differential Geom. **35**, pp. 429-469, 1992.
- ▶ Lange, H.; Narasimhan, M.S. *Maximal Subbundles of Rank Two Vector Bundles on Curves*. Math. Ann. **266**, pp. 55-72, 1983.
- ▶ Marchisio, M. R. *Unirational quartic hypersurfaces*. Bolletino dell'Unione Matematica Italiana, Serie 8, **3-B**. No. 2, pp. 301-314. 2000.

- ▶ Otwinowska, A. *Remarques sur les groupes de Chow des hypersurfaces de petit degré*. C.R. Acad. Sci. Paris Sér. I. Math. **329**, no.1, pp. 51-56, 1999.
- ▶ Teixidor. I. Bigas, M. *Brill-Noether theory for vector bundles of rank 2*. Tôhoku Math. J. **43**, pp. 123-126, 1991.
- ▶ Teixidor. I. Bigas, M. *Existence of vector bundles of rank two with fixed determinant and sections*. Proc. Japan Acad. Ser. A Math. Sci. **86**, no. 7, pp. 113–118, 2010.

# Thank You