Special values of L-functions and Ihara's Lemma for Quaternionic Shimura Varieties

Overview

K number field, E/K elliptic curve

$$L(E/K,s) = \prod_{\substack{v \text{ finite} \\ \text{place of } K}} L_v(E/K,s),$$

$$L_v(E/K,s) = \frac{1}{1 - a_v N(v)^{-s} + N(v)^{1-2s}}, \quad v \text{ good reduction place, } a_v = N(v)+1-\#E(\mathbb{F}_v)$$

Conj: (0) $L(E/K,s)$ has analytic continuation to $\mathbb{C}$

(1) $\mathrm{ord}_{s=1} L(E/K,s) = \mathrm{rk}\, E(K)$

(2)
$$\boxed{\underset{\substack{\text{first non-zero term} \\ \text{in Taylor expansion}}}{L^*(E/K,1)} = \frac{\overset{\text{real period}}{\Omega_{E/K}}\ \overset{\text{regulator}}{R_{E/K}}\ \#Ш(E/K) \prod_v \overset{\text{Tamagawa numbers}}{c_v(E/K)}}{\sqrt{|\mathrm{disc}\, K|}\ \#E(K)_{\mathrm{tors}}^2}}$$

Reformulations of (1) (Flach's lectures)

Bloch-Kato '90

$T_p(E) = \varprojlim_m E[p^m]$

$\cap$

$V_p(E) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow A_p(E) = \frac{V_p(E)}{T_p(E)}$

$0 \to E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to Sel(K, A_p(E)) \to Ш(E/K)[p^\infty] \to 0$

$$\boxed{\mathrm{ord}_{s=1} L(E/K,s) = \dim_{\mathbb{Q}_p} Sel(K, V_p(E))}$$

Tate '65

$\mathcal{E}/\mathcal{O}_K$ proper regular model of E, with zeta function $\zeta(\mathcal{E},s)$

$$\boxed{\mathrm{ord}_{s=1}\zeta(\mathcal{E},s) = \mathrm{rk}\, H^0(\mathcal{E},\mathcal{O}_{\mathcal{E}}^\times) - \mathrm{rk}(\underset{\overset{\|}{\mathrm{Pic}(\mathcal{E})}}{H^1(\mathcal{E},\mathcal{O}_{\mathcal{E}}^\times)})}$$

BLOCH-KATO: USUALLY APPROACHABLE ONLY IF:

- K IS (AN ABELIAN EXTENSION OF) A TOT. REAL OR CM FIELD
- $\mathrm{ord}_{s=1} L(E/K,s) \le 1$

NEED TWO INPUTS:

(i) E/K IS MODULAR, I.E. $\exists$ AUTOMORPHIC FORM f FOR $GL_{2,K}$ S.T. $\Rightarrow$ (0)

$$L(f,s) = L(E/K,s)$$ (KNOWN POTENTIALLY, [ACCGHLNSTT])

(ii) EXPLICIT "ZETA ELEMENT" RELATED TO $L^*(E/K,1)$

BEILINSON-KATO, HEEGNER, ... (LEI'S LECTURES)

Main Theorem: Statement

F TOT. REAL, K/F CM QUADRATIC EXTENSION.

ASSUME E/F MODULAR $\longrightarrow$ $f \in S_2(\Gamma_0(\mathfrak{m}))$ HILBERT CUSPIDAL EIGENFORM OF PARALLEL WEIGHT 2, TRIVIAL CENTRAL CHARACTER, INTEGRAL FOURIER COEFFS.

NOT KNOWN IF $[F:\mathbb{Q}]=2$, $\mathfrak{m}=1$.

$$\rho: \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Aut}(T(f)) \simeq GL_2(\mathbb{Z}_p) \subseteq \mathrm{Aut}(V(f)), \quad V(f) = T(f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$$\bar{\rho}: \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Aut}(T(f)/p) \simeq GL_2(\mathbb{F}_p)$$

Thm (T.): ASSUME:
- $\mathfrak{m}$ SQUAREFREE, ALL FACTORS INERT IN K
- f DOES NOT HAVE CM

THEN
$$\boxed{L(f_K,1) \neq 0 \Rightarrow \mathrm{length}_{\mathbb{Z}_p} \mathrm{Sel}(K, A(f)) \le v_p(L^{alg}(f_K,1)) \quad (\star)}$$

$A(f) = V(f)/T(f)$

FOR ALL BUT FINITELY MANY p.

RKS: IN ADDITION, CAN ESTABLISH:

(1) AN UPPER BOUND FOR $\mathrm{length}_{\mathbb{Z}_p} \mathrm{Sel}(K,A(f))/_{div}$ IN ANALYTIC RANK ONE

(2) A CRITERION IMPLYING THAT $(\star)$ IS AN EQUALITY (CF. BELOW)

- CAN GIVE EXPLICIT CONDITIONS ON p (E.G.: $\mathrm{Im}(\bar{\rho}) \supseteq SL_2(\mathbb{F}_p)$, $p>3$)

PREVIOUS WORKS W. SIMILAR METHOD (BERTOLINI-DARMON, LONGO, CHIDA-HSIEH, HOWARD, NEKOVÁŘ, ...)

PROVED:
- $L(f_K,1) \neq 0 \Rightarrow \mathrm{Sel}(K,A(f))$ FINITE
- $(\star)$ FOR p ORDINARY, VIA IWASAWA THEORY

· GOAL: PROVE (A) DIRECTLY; NO NEED TO DISTINGUISH ORDINARY/SUPERSINGULAR.

· THE RANK ONE CASE IS REDUCED TO THE RANK ZERO CASE (BASIC IDEA: W. ZHANG '14)

· OTHER RECENT RESULTS: FOUQUET-WAN, BURUNGALE-FLACH, ...

Main theorem: proof

REFINEMENT OF THE METHOD OF BERTOLINI-DARMON (2005), CF. ALSO BERTI-BERTOLINI-VENERUCCI 2015

A SIGN DICHOTOMY (CF. TALKS BY LIU, LAI, PAL, ZHANG); B/F QUAT. ALGEBRA, $K^\times \hookrightarrow B^\times$

	SIGN	SPACE	ZETA ELEMENT
DEFINITE CASE	$[F:\mathbb{Q}] \equiv \#\{q \mid m\}\ (2)$ $\varepsilon(f_K) = +1$	B RAMIFIED AT $m\infty$ $X(B^\times)$ (FINITE SET)	JL TRANSFER $L^{alg}(f_K,1) = \lvert z_K\rvert^2,\ z_K = \int_{[K^\times]} f^B$
INDEFINITE CASE	$[F:\mathbb{Q}] \not\equiv \#\{q \mid m\}\ (2)$ $\varepsilon(f_K) = -1$	B RAMIFIED AT m + ALL BUT ONE INFINITE PLACE $X(B^\times)$ (SHIMURA CURVE)	$L'(f_K,1) \doteq \langle z_K, z_K\rangle_{NT}$, z_K: HEEGNER DIVISOR ON $X(B^\times)$

WE NOW ASSUME $L(f_K,1) \neq 0$ AS IN THE THM ($\Rightarrow \varepsilon(f_K)=1$). FIX $m>>0$; $A_m(f) := A(f)[p^m]$

WANT TO SHOW:

$$\mathrm{length}_{\mathbb{Z}_p} \mathrm{Sel}(K, A_m(f)) \leq 2\, v_p(z_K(f)) \qquad (A)_m$$

DEF: A PRIME IDEAL $\ell \subseteq O_F$ IS CALLED ADMISSIBLE ($\ell \in \mathcal{L}$) IF:

- $\ell \nmid pm$
- ℓ INERT IN K
- $p \nmid N(\ell)^2 - 1$
- $(N(\ell)+1)^2 \equiv a_\ell(f)^2 \bmod p^m$ (LEVEL RAISING PRIME)

CONSEQUENCE: $H^1(K_\ell, A_m(f)) \simeq \underset{\simeq \mathbb{Z}/p^m\mathbb{Z}}{H^1_{ur}(K_\ell, A_m(f))} \oplus \underset{\simeq \mathbb{Z}/p^m\mathbb{Z}}{H^1_{sing}(K_\ell, A_m(f))}$

(a) <u>KEY CONSTRUCTION</u>: $\forall\, \ell_1 \neq \ell_2 \in A$, $c(\ell_1), c(\ell_2) \in H^1(K, A_m(f))$ S.T.

$$c(\ell_1) \xrightarrow{\mathrm{res}_{\ell_2}} H^1_{ur}(K_{\ell_2}, A_m(f)) \cong \mathbb{Z}/p^m\mathbb{Z} \ni z_K(g)$$

$$c(\ell_1) \xrightarrow{\mathrm{res}_{\ell_2}} H^1_{sing}(K_{\ell_1}, A_m(f)) \cong \mathbb{Z}/p^m\mathbb{Z} \ni z_K(f)$$

$$\mathbb{Z}/p^m\mathbb{Z} \cong H^1_{sing}(K_{\ell_2}, A_m(f)) \xleftarrow{\mathrm{res}_{\ell_2}} c(\ell_2) \xrightarrow{\mathrm{res}_{\ell_1}} H^1_{ur}(K_{\ell_2}, A_m(f)) \cong \mathbb{Z}/p^m\mathbb{Z}$$

$g \in S(B^\times_{\ell_1\ell_2}, \mathbb{Z}/p^m)$ CONGRUENT TO f

RAMIFIED AT $n\ell_1\ell_2\infty$

(RL1) $\mathrm{res}_{\ell_i}(c(\ell_i)) = z_K(f)$ (RL2) ; $\mathrm{res}_{\ell_j}(c(\ell_i)) = z_K(g)$, $i \neq j$

(b) THERE ARE "ENOUGH" ADMISSIBLE PRIMES: (CHEBOTAREV)

$\forall\, c \in H^1(K, A_1(f)) \setminus \{0\}$, $\exists$ INFINITELY MANY $\ell \in A$ S.T. $\mathrm{res}_\ell(c) \neq 0$

<u>RK</u>: THE ABOVE CLASSES ARE <u>NOT</u> AN EULER/KOLYVAGIN SYSTEM IN THE TRADITIONAL SENSE.

- IDEA OF CONSTRUCTION: $n \rightsquigarrow n\ell_i$ ALLOWS TO SWITCH FROM DEFINITE TO INDEFINITE $\Rightarrow$ REALISE $A_m(f)$ IN COHOMOLOGY OF SHIMURA CURVES $X(B^\times_{\ell_i})$ (<u>BAD</u> REDUCTION AT ℓ_i) ; USE CM POINTS ON $X(B^\times_{\ell_i})$ TO CONSTRUCT $c(\ell_i)$

1ST RECIPROCITY LAW (RL1) $\Leftarrow$ STUDY OF SPECIAL FIBRE OF $X(B^\times_{\ell_i})$ AT ℓ_i (LIU'S LECTURES)

2nd RECIPROCITY LAW (RL2) $\Leftarrow$ $X(B^\times_{\ell_i})_{\overline{\mathbb{F}}_{\ell_j}}$

$$X(B^\times_{\ell_i})_{\overline{\mathbb{F}}_{\ell_j}} \supseteq X(B^\times_{\ell_i})^{ss}_{\overline{\mathbb{F}}_{\ell_j}} \xrightarrow{\simeq} X(B^\times_{\ell_i\ell_j}) \xrightarrow{AJ} H^1(\mathbb{F}_{\ell_j}, A_m(f)) \cong \mathbb{Z}/p^m\mathbb{Z}$$

$$X(B^\times_{\ell_i\ell_j}) \overset{g}{\dashrightarrow} \mathbb{Z}/p^m\mathbb{Z}$$

GEOMETRIC JL, CF. PRASANNA'S TALK

GIVEN (a), (b), WE PROVE $(\#)_m$ BY INDUCTION ON $v_p(L^{alg}(f_K, 1)) = 2v_p(z_K(f))$

<u>RK</u>: SAME STRATEGY COULD BE USEFUL FOR

- TRIPLE PRODUCTS (WANG '20)
- UNITARY GROUPS (LTXZZ '19)

STEP I: $v_p(z_K(f))=0 \Rightarrow Sel(K, A_1(f))=0$

PROOF: ASSUME THERE IS $c \in Sel(K, A_1(f)) \setminus \{0\}$.

- BY (b), $\exists\, \ell \in \Lambda$ s.t. $res_\ell\, c \neq 0 \in H^1_{un}(K_\ell, A_1(f))$
- BY (RL1), $res_\ell(c(\ell)) \neq 0 \in H^1_{sing}(K_\ell, A_1(f))$
- BY GLOBAL CFT:

$$0 = \sum_v \langle res_v c, res_v c(\ell)\rangle = \langle res_\ell c, res_\ell c(\ell)\rangle$$

CONTRADICTION, SINCE $H^1_{sing}(K_\ell, A_1(f)) \times H^1_{un}(K_\ell, A_1(f)) \to \mathbb{Z}/p\mathbb{Z}$ IS PERFECT.

STEP II INDUCTIVE STEP. IF $v_p(z_K(f)) \geq 1$, THEN THERE ARE TWO POSSIBILITIES:

(i) EITHER $Sel(K, A_n(f))=0$

(ii) OR THERE EXIST $\ell_1 \neq \ell_2 \in \Lambda$, AND $g \in S(B^\times_{\ell_1\ell_2}, \mathbb{Z}/p^m)$ CONGR. TO f, S.T. $v_p(z_K(g)) < v_p(z_K(f))$ AND

(*) $\boxed{length_{\mathbb{Z}_p}(Sel(K, A_n(f))) - length_{\mathbb{Z}_p}(Sel(K, A_n(g))) = 2v_p(z_K(f)) - 2v_p(z_K(g))}$

(i), (ii) $\Rightarrow$ (A)$_n$ INDUCTIVELY

DEFINITION OF ℓ_1, ℓ_2: • $\ell_1 \in \Lambda$ CHOSEN S.T. $v_p(c(\ell_1))$ IS MINIMAL

SIMILARLY TO STEP I, $\boxed{Sel(K, A_n(f)) \neq 0 \Rightarrow v_p(c(\ell_1)) < v_p(z_K(f))}$

- CHOOSE $\ell_2 \neq \ell_1$ S.T. $res_{\ell_2}\left(\frac{c(\ell_1)}{p^{v_p(c(\ell_1))}}\right) \in \mathbb{Z}/p^n\mathbb{Z}$ IS A UNIT (USE (b))

SHOW, USING GLOBAL DUALITY + (RL2), THAT

$$v_p(z_K(g)) = v_p(c(\ell_1)) < v_p(z_K(f)) \text{ AND } (*) \text{ HOLDS}$$

RK: WE CANNOT EXCLUDE CASE (i) USING THE ABOVE METHOD. IN ORDER TO OBTAIN

EQUALITY IN OUR THM., WE NEED TO KNOW THAT

$$\boxed{v_p(z_K(f)) > 0 \Rightarrow \mathrm{Sel}(K, A_1(f)) \neq 0} \quad \left(\begin{array}{l}+\text{ SAME FOR LEVEL}\\ \text{RAISINGS OF } f\end{array}\right)$$

(GL_2-VERSION OF RIBET'S CONVERSE TO HERBRAND)

Second explicit reciprocity law and Ihara's lemma

$$\begin{tikzcd} X(B^\times_{\ell_i})_{\bar{\mathbb{F}}_{\ell_j}} \supseteq X(B^\times_{\ell_i})^{ss}_{\mathbb{F}_{\ell_j}} \arrow[r, "\sim"] & X(B^\times_{\ell_i \ell_j}) \arrow[r, "AJ"] \arrow[dr, dashed, "g"'] & H^1(\mathbb{F}_{\ell_j}, A_n(f)) \arrow[d, phantom, "\wr"] \\ & & \mathbb{Z}/p^n\mathbb{Z} \end{tikzcd}$$

KEY POINT: g MOD p IS NOT IDENTICALLY ZERO.

THIS IS REDUCED (LIU-TIAN '20, PROP. 4.8) TO THE FOLLOWING RESULT:

(IHARA'S LEMMA)

THM (MANNING-SHOTTON '21)

$$\begin{tikzcd} & X(B^\times_{\ell_i})_{\Gamma_0(\ell_j)} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\ X(B^\times_{\ell_i}) & X(B^\times_{\ell_i}) \end{tikzcd}$$

THE MAP $\pi_2^* \oplus \pi_2^* : H^1(X(B^\times_{\ell_i}), \mathbb{F}_p)^{\oplus 2}_f \to H^1(X(B^\times_{\ell_i})_{\Gamma_0(\ell_j)}, \mathbb{F}_p)_f$

IS INJECTIVE.

- IN JOINT WORK IN PROGRESS W. A. CARAIANI, WE GENERALISE THIS TO ARBITRARY QUATERNIONIC SHIM. VARS. THE PROOF (FOLLOWING MANNING-SHOTTON'S IDEA) USES:

(1) STRUCTURE OF SPECIAL FIBRE AT IWAHORI LEVEL

(2) VANISHING OF f-ISOTYPIC PART OF COHOMOLOGY OUTSIDE MIDDLE DEG. ((CARAIANI, T.)

(3) TAYLOR-WILES PATCHING.