Numerical Approach to Dissipative Weak Solutions to the Euler Equations

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Joint work with U. Frisch and L. Székelyhidi

Scaling in 'real' turbulent flow

(i) Kolmogorov scaling law (K41)

$$E(k) \propto \epsilon^{2/3} k^{-5/3} \qquad 1/L \ll k \ll 1/\eta,$$

$$u(x+r,t) - u(x,t) \propto \epsilon^{1/3} r^{1/3} \qquad \eta \ll r \ll L$$

(ii) Energy dissipation rate $\epsilon = \nu \sum_{\boldsymbol{k}} |\boldsymbol{k}|^2 \langle |\hat{\boldsymbol{u}}(\boldsymbol{k},t)|^2 \rangle = \nu \frac{1}{V} \int \left\langle \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right\rangle d\boldsymbol{x}$

$$\epsilon \to (\text{const.}) \quad \text{as} \quad \nu \to 0$$

(iii) Experiments and simulations lead Parisi & Frisch in 1980s to conjecture

$$u(x+r,t) - u(x,t) \propto r^h.$$

Exponent h is not limited to $1/3$ (multifractal model)

dissipation, $1/3$, multiple exponents h

Onsager's conjecture

- L. Onsager (1949)
  1. Turbulent dissipation could take place without viscosity.
  2. For this, the velocity field should not remain differentiable.
  3. "In fact, it is possible to show that the velocity field in such ideal turbulence cannot obey any LIPSCHITZ condition of the form

$$|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})| < (\text{const.})\, r^h,$$

     for any order h greater than $1/3$; otherwise the energy is conserved."
- $\nu = 0$, the energy conserves for $h > 1/3$ and dissipates for $h \leq 1/3$.

dissipation, $1/3$, multiple exponents h

Onsager's conjecture: afterwards

- Precise formulation with the weak form of the Euler eqs. (Eyink 1994)
- The energy conservation for $h > 1/3$: proven by Eyink 1994; Constantin, E, Titi 1994.

$$|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})| < (\text{const.})\ r^h$$

- The energy dissipation for $h \leq 1/3$: proven recently by De Lellis and Székelyhidi 2013; Daneri and Székelyhidi 2016; Isett 2018; Buckmaster *et al.* 2019.

  They constructed dynamic weak solutions with arbitrary (single) h in $0 < h < 1/3$.

  How? Can one do it numerically? (YES, BUT...)

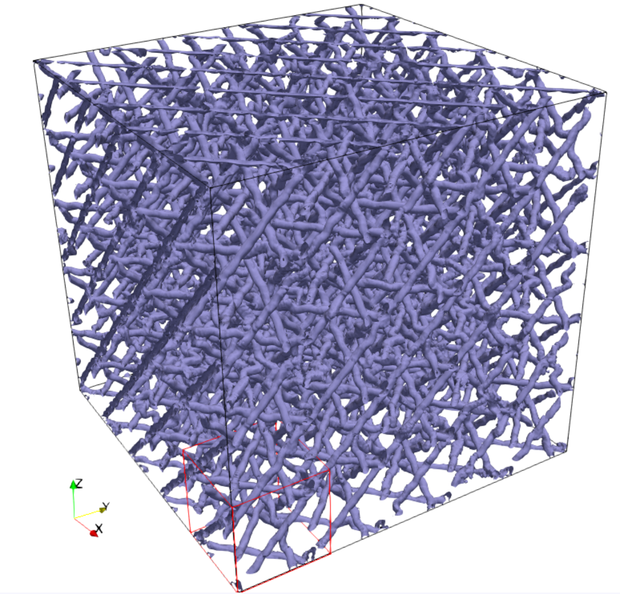
  Are the solutions relevant to real-world turbulence?

Why numerical simulation ?

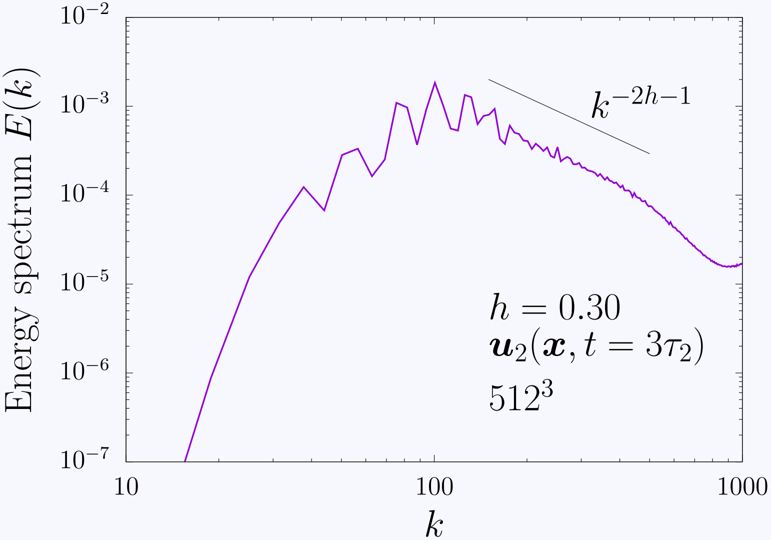
- How the dissipative weak solutions look like?
- Measure high-order structure functions, PDF, etc...
- Some aspects of the construction are similar to cascade models.
  It constructs velocity field $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)| \sim r^h \ (0 \le h \le 1/3)$
  The cascade model approved by the (weak) Euler eqs.

Outline

- Dissipative weak solution of the Euler equations.
- Mathematical construction by De Lellis, Székelyhidi and co-workers
- Numerical simulation (on-going)
- Summary and outlook



Typical appearance as iso-surface of $|\nabla \times \boldsymbol{u}_2(\boldsymbol{x}, t)|$
Simulation of the construction of Buckmaster *et al.* (2019)



Energy spectrum of $\boldsymbol{u}_2$ with $h = 0.30$
Simulation of the construction of Buckmaster *et al.* (2019)

Weak solution of the Euler eqs.

- Incompressible Euler eqs. with periodic boundary condition

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0$$

- test function $\varphi(\boldsymbol{x}, t)$ (smooth, finite support in $\boldsymbol{x}$ and t)

$$\overline{\boldsymbol{u}}(\boldsymbol{x}, t) = \int_0^T dt' \int d\boldsymbol{x}' \varphi(\boldsymbol{x} - \boldsymbol{x}', t - t')\boldsymbol{u}(\boldsymbol{x}', t')$$

- Weak solution to the Euler eqs. (this holds for all φ's)

$$\int_0^T dt \int d\boldsymbol{x}\, [(\partial_t \varphi)\boldsymbol{u} + \boldsymbol{u}(\boldsymbol{u} \cdot \nabla)\varphi + p\nabla\varphi] = \boldsymbol{0}, \quad \int_0^T dt \int d\boldsymbol{x}\, \boldsymbol{u} \cdot \nabla\varphi = 0$$

- There are famous examples of constructed weak solutions (Scheffer 1993 (2D); Schnirelman 1997 (2D), 2000 (3D)).

Construction (Buckmaster *et al.* 2019)

(Buckmaster, De Lellis, Székelyhidi, and Vicol, Comm. Pure Appl. Math. **72**, 229, 2019)

- time-dependent weak solution $\boldsymbol{u}(\boldsymbol{x}, t)$

  $\boldsymbol{x}$: unit periodic cube

  t: $0 \le t \le T$ (T some large time)

- Two inputs

  (1) Total energy $e(t) = \int |\boldsymbol{u}(\boldsymbol{x}, t)|^2 d\boldsymbol{x}$ in $0 \le t \le T$

  (2) Exponent h : $|\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t) - \boldsymbol{u}(\boldsymbol{x}, t)| \sim r^h$, $\quad 0 < h < 1/3$

- Iterative construction (space and time)

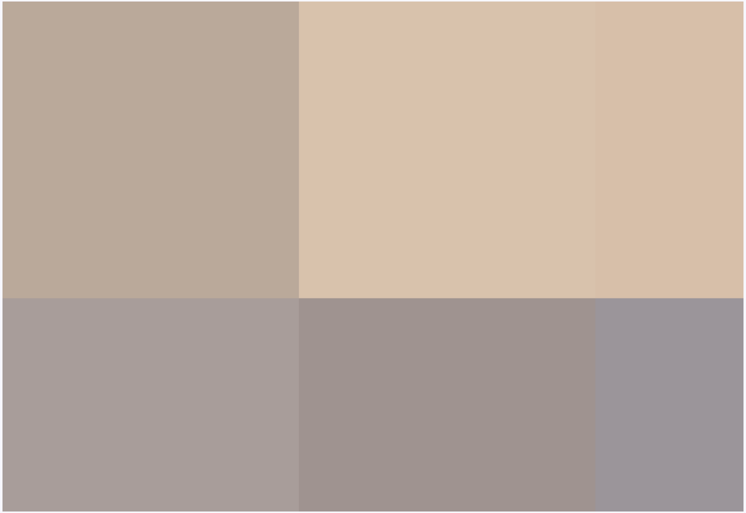
  Initial guess: $\boldsymbol{u}_0(\boldsymbol{x}, t)$: $\boldsymbol{x}$ in periodic cube, $0 \le t \le T$

  $\boldsymbol{u}_0(\boldsymbol{x}, t) \Rightarrow \boldsymbol{u}_1(\boldsymbol{x}, t) \Rightarrow \cdots \Rightarrow \boldsymbol{u}_n(\boldsymbol{x}, t) \to \boldsymbol{u}(\boldsymbol{x}, t)$ as $n \to \infty$

  $\boldsymbol{u}_n$ has characteristic wavenumber λ_n

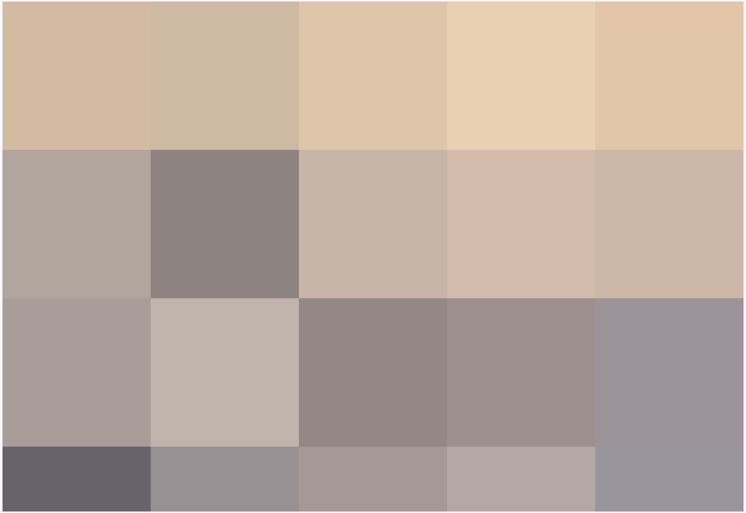
From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_0$: coarsest



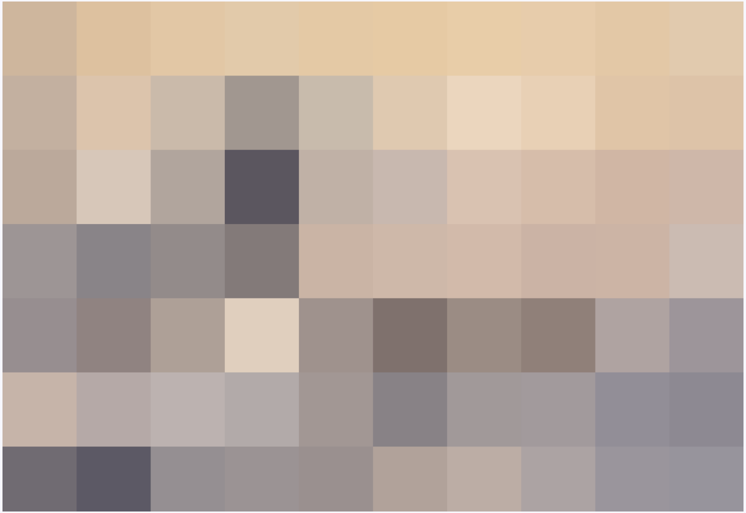
From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_1$: add details to $\boldsymbol{u}_0$



From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_2$: add further details to $\boldsymbol{u}_1$



From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_3$: add further details to $\boldsymbol{u}_2$

From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_4$: add further details to $\boldsymbol{u}_3$

From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_5$: add further details to $\boldsymbol{u}_4$

From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_6$: add further details to $\boldsymbol{u}_5$

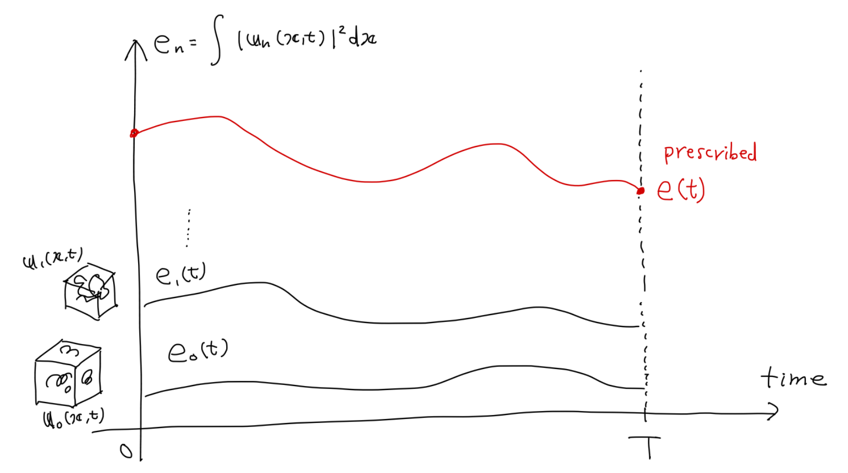
From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

$\boldsymbol{u}_\infty$:

(K. Hokusai, *Thirty-six views of Mt. Fuji – the great wave off the coast of Kanagawa* from Google Art Project)

Space-time construction

$\boldsymbol{u}_0(\boldsymbol{x}, t) \Rightarrow \boldsymbol{u}_1(\boldsymbol{x}, t) \Rightarrow \cdots \Rightarrow \boldsymbol{u}_n(\boldsymbol{x}, t) \Rightarrow \ldots \to \boldsymbol{u}(\boldsymbol{x}, t)$ as $n \to \infty$



$\dfrac{de(t)}{dt} < 0$: energy dissipation. $\boldsymbol{u}_n(\boldsymbol{x}, t)$ has maximum wavenumber λ_n

From $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$

- Classical Euler equation (periodic cube)

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0, \quad \nabla\cdot\boldsymbol{u} = 0$$

- Iterative construction from $\boldsymbol{u}_n$ to $\boldsymbol{u}_{n+1}$
  - n-th iteration: $(\boldsymbol{u}_n,\ p_n)$ for $0 \le t \le T$

$$\partial_t \boldsymbol{u}_n + (\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n + \nabla p_n = \boldsymbol{E}_n = -\operatorname{div}\mathsf{R}_n$$

    $\boldsymbol{E}_n$: error
    R_n : positive-definite symmetric 3×3 tensor

  - Add perturbation $(\boldsymbol{u}_{n+1}, p_{n+1}) = (\boldsymbol{u}_n + \boldsymbol{W}_n,\ p_n + Q_n)$

$$\partial_t \boldsymbol{u}_{n+1} + (\boldsymbol{u}_{n+1}\cdot\nabla)\boldsymbol{u}_{n+1} + \nabla p_{n+1} = \boldsymbol{E}_{n+1} = -\operatorname{div}\mathsf{R}_{n+1}$$

$$\boldsymbol{E}_{n+1} = \boldsymbol{E}_n + \partial_t \boldsymbol{W}_n + (\boldsymbol{u}_n\cdot\nabla)\boldsymbol{W}_n + (\boldsymbol{W}_n\cdot\nabla)\boldsymbol{u}_n + (\boldsymbol{W}_n\cdot)\nabla\boldsymbol{W}_n + \nabla Q_n$$

- Idea : add perturbation $\boldsymbol{W}_n(\boldsymbol{x}, t)$ to cancel error $\boldsymbol{E}_n(\boldsymbol{x}, t)$

Perturbation to cancel error

- $(n+1)$-th iteration: $(\boldsymbol{u}_{n+1}, p_{n+1}) = (\boldsymbol{u}_n + \boldsymbol{W}_n,\ p_n + Q_n)$

$$\partial_t \boldsymbol{u}_{n+1} + (\boldsymbol{u}_{n+1} \cdot \nabla)\boldsymbol{u}_{n+1} + \nabla p_{n+1} = \boldsymbol{E}_{n+1} = -\operatorname{div} \mathsf{R}_{n+1}$$

$$\boldsymbol{E}_{n+1} = \boldsymbol{E}_n + \partial_t \boldsymbol{W}_n + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{W}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{u}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n + \nabla Q_n$$

- Idea：add perturbation $\boldsymbol{W}_n(\boldsymbol{x}, t)$ to cancel error $\boldsymbol{E}_n(\boldsymbol{x}, t)$
  - If $\|\boldsymbol{E}_{n+1}\| < \|\boldsymbol{E}_n\|$, then $\boldsymbol{u}_n$ approaches to weak solution.
  - $\boldsymbol{W}_n(\boldsymbol{x}, t)$: stationary solution of the classical Euler equation
  - (scale of $\boldsymbol{W}_n$) $\ll$ (scale of $\boldsymbol{u}_n$)

$$\boldsymbol{E}_{n+1} \sim \boldsymbol{E}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = -\operatorname{div} \mathsf{R}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = \operatorname{div}[-\mathsf{R}_n + \boldsymbol{W}_n \otimes \boldsymbol{W}_n] \sim 0$$

Two questions

- Question (1) : How to calculate the error $\boldsymbol{E}_n$?

  n-th iteration: $(\boldsymbol{u}_n,\ p_n)$ for $0 \le t \le T$

$$\partial_t \boldsymbol{u}_n + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n + \nabla p_n = \boldsymbol{E}_n = -\operatorname{div} \mathsf{R}_n$$

(The Euler eqs. : $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0$)

- Question (2) : How to cancel the error $\boldsymbol{E}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n \sim 0$?

$$\boldsymbol{E}_{n+1} \sim \boldsymbol{E}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = -\operatorname{div} \mathsf{R}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = \operatorname{div}[-\mathsf{R}_n + \boldsymbol{W}_n \otimes \boldsymbol{W}_n] \sim 0$$

A way to cancel the error

- Question (2) : How to cancel the error $\boldsymbol{E}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n \sim 0$?

$$\boldsymbol{E}_{n+1} \sim \boldsymbol{E}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = -\operatorname{div} \mathsf{R}_n + (\boldsymbol{W}_n \cdot \nabla)\boldsymbol{W}_n = \operatorname{div}[-\mathsf{R}_n + \boldsymbol{W}_n \otimes \boldsymbol{W}_n] \sim 0$$

- In the sense of spatial average

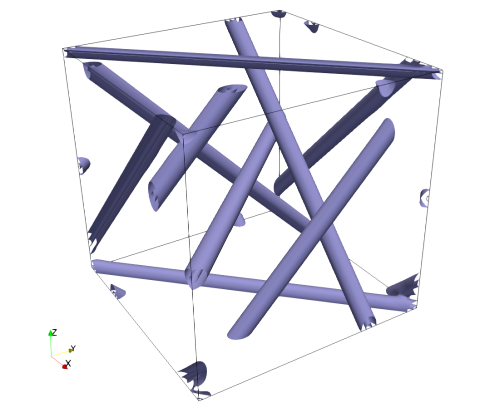
$$\overline{\mathsf{R}}_n = \overline{\boldsymbol{W}_n \otimes \boldsymbol{W}_n}$$

Mikado flow (six jets) as perturbation

- Cancel the error in the sense of spatial average

$$\overline{\mathsf{R}} = \overline{\boldsymbol{w} \otimes \boldsymbol{w}}$$

- Six axisymmetric jets can do this
  — Mikado flow
  (Daneri and Székelyhidi 2016).

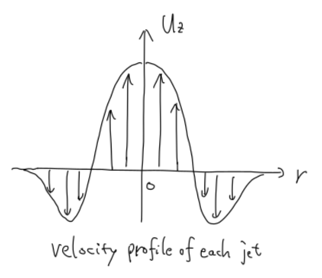


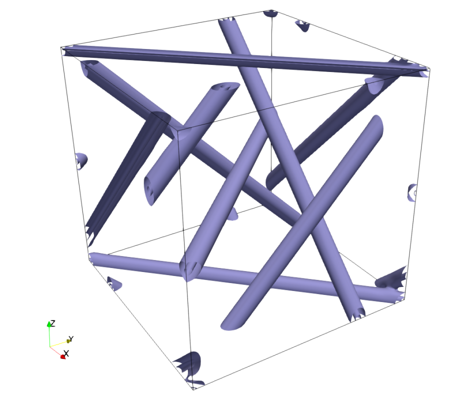
Mikado flow $|\nabla \times \boldsymbol{w}(\boldsymbol{x})| = \mathrm{const.}$

Mikado flow can cancel the error

- Given the error
  $\boldsymbol{E}_n = -\operatorname{div} \mathsf{R}_n \sim \operatorname{div} \boldsymbol{W}_n \otimes \boldsymbol{W}_n$,
  Spatial average: $\overline{R}_n = \overline{\boldsymbol{W}_n \otimes \boldsymbol{W}_n}$
- Mikado flow to cancel
  a constant, symmetric tensor R:

$$\mathsf{R} = \overline{\boldsymbol{w}(\boldsymbol{x}) \otimes \boldsymbol{w}(\boldsymbol{x})}$$





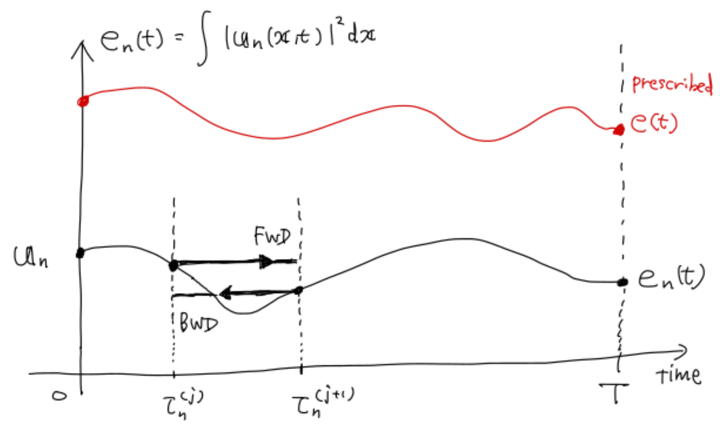
$$|\nabla \times \boldsymbol{w}(\boldsymbol{x})| = \text{const.}$$

How to calculate the error

- n-th iteration $(\boldsymbol{u}_n,\ p_n)$

$$\partial_t \boldsymbol{u}_n + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n + \nabla p_n = \boldsymbol{E}_n = -\operatorname{div} \mathsf{R}_n$$

- “Gluing” to obtain the error R_n (Isett 2018)



Superposition gives the error

- Classical solutions of the Euler eqs. : $\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0$

  Forward-in-time sol. from $\tau_n^{(j)}$: $\boldsymbol{u}^F_{n,\ j}$

  Backward-in-time sol. from $\tau_n^{(j+1)}$: $\boldsymbol{u}^B_{n,\ j+1}$

- Superposition ("gluing"): $\tilde{\boldsymbol{u}}_n = \chi_j(t)\boldsymbol{u}^F_{n,\ j} + (1-\chi_j(t))\boldsymbol{u}^B_{n,\ j+1}$

$$\begin{aligned}\partial_t \tilde{\boldsymbol{u}}_n + \mathrm{div}(\tilde{\boldsymbol{u}}_n \otimes \tilde{\boldsymbol{u}}_n) + \nabla \tilde{p}_n \quad &= \quad (\partial_t \chi_j)(\boldsymbol{u}^F_{n,\ j} - \boldsymbol{u}^B_{n,\ j+1}) \\ &\quad -\chi_j(1-\chi_j)\,\mathrm{div}[(\boldsymbol{u}^F_{n,\ j} - \boldsymbol{u}^B_{n,\ j+1}) \otimes (\boldsymbol{u}^F_{n,\ j} - \boldsymbol{u}^B_{n,\ j+1})].\end{aligned}$$

- The error in terms of the tensor:

$$\mathring{\tilde{\mathsf{R}}}_n \quad = \quad (\partial_t \chi_j)\mathcal{R}(\boldsymbol{u}^F_{n,j} - \boldsymbol{u}^B_{n,j+1}) - \chi_j(1-\chi_j)(\boldsymbol{u}^F_{n,j} - \boldsymbol{u}^B_{n,j+1})\mathring{\otimes}(\boldsymbol{u}^F_{n,j} - \boldsymbol{u}^B_{n,j+1})$$

$\mathcal{R}$: the inverse divergence operator $\mathrm{div}\,\mathcal{R}\boldsymbol{u} = \boldsymbol{u}$,

$$\begin{aligned}(\mathcal{R}\boldsymbol{u})^{ij} \quad &= \quad \mathcal{R}^{ijk} u^k, \\ \mathcal{R}^{ijk} \quad &= \quad -2\triangle^{-2}\partial_i\partial_j\partial_k - \triangle^{-1}\partial_k\delta_{ij} - \triangle^{-1}\partial_i\delta_{jk} - \triangle^{-1}\partial_j\delta_{ik}.\end{aligned}$$

- Equation for the glued velocity:

$$\partial_t \tilde{\boldsymbol{u}}_n + \mathrm{div}(\tilde{\boldsymbol{u}}_n \otimes \tilde{\boldsymbol{u}}_n) + \nabla \tilde{p}_n = \mathrm{div}\,\mathring{\tilde{\mathsf{R}}}_n, \quad \mathrm{div}\,\tilde{\boldsymbol{u}}_n = 0.$$

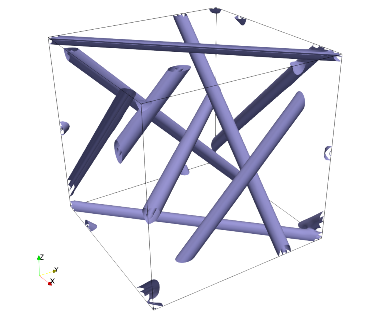
How to en-force $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x},t)| \sim r^h$

- Input of the construction: the exponent h ($0 \leq h < 1/3$)

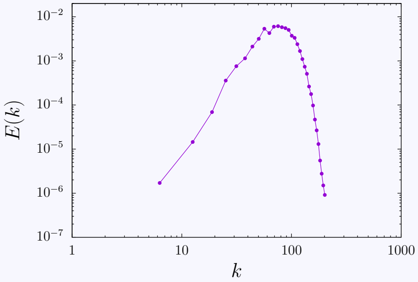
$$|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x},t)| \sim r^h$$

- Mikado flow to cancel a constant, symmetric tensor R:

$$\mathsf{R} = \overline{\boldsymbol{w}(\boldsymbol{x}) \otimes \boldsymbol{w}(\boldsymbol{x})}$$



Mikado ($|\nabla \times \boldsymbol{w}(\boldsymbol{x})| = \text{const.}$)

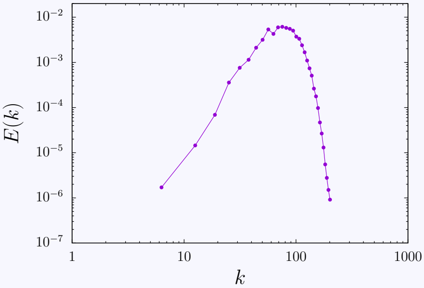


Energy spectrum of Mikado flow

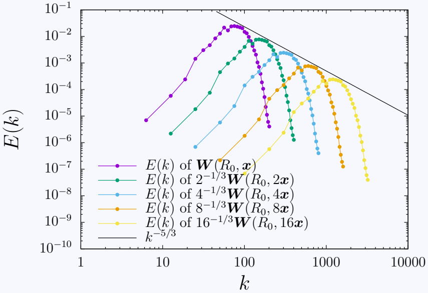
How to en-force the scaling r^h

- Perturbation $\boldsymbol{W}_n(\overline{\mathsf{R}}_n, \boldsymbol{x}) \sim \boldsymbol{w}(\overline{\mathsf{R}}_n, \lambda_n \boldsymbol{x})$

$$\begin{aligned}\boldsymbol{u}_{n+1}(\boldsymbol{x},t) &= \boldsymbol{u}_n(\boldsymbol{x},t) + \boldsymbol{W}_n(\overline{\mathsf{R}}_n, \boldsymbol{x}) \\ &= \boldsymbol{u}_0(\boldsymbol{x},t) + \sum_m A_m \boldsymbol{w}(\overline{\mathsf{R}}_m, \lambda_m \boldsymbol{x})\end{aligned}$$



$E(k)$ of $\boldsymbol{w}(\mathrm{Id}, \boldsymbol{x})$ (Mikado flow)



$E(k,t)$ of $\boldsymbol{u}_n(\boldsymbol{x},t) \sim \sum_m k_m^{-h} \boldsymbol{w}(\overline{\mathsf{R}}_m, \lambda_m \boldsymbol{x})$

$E(k,t) \propto k^{-2h-1}$ consistent to r^h

Formal statement

(Buckmaster, De Lellis, Székelyhidi and Vicol 2019)

- n-th step ($n = 0, 1, 2, \dots$) "Euler-Reynolds" equation $(\boldsymbol{u}_n, p_n, \mathring{\mathsf{R}}_n)$ in $0 \le t \le T$

$$\partial_t \boldsymbol{u}_n + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n + \nabla p_n = \mathrm{div}\mathring{\mathsf{R}}_n, \quad \nabla \cdot \boldsymbol{u}_n = 0$$

- $\boldsymbol{u}_n$: wavenumber $\lambda_n = 2\pi\lceil a^{b^q}\rceil$, squared amplitude $\delta_n = \lambda_n^{-2h}$ ($a \gg 1, b \simeq 1 (b > 1)$)
- Estimates

$$\begin{aligned}
\|\mathring{\mathsf{R}}_n\|_0 &\le \delta_{n+1}\lambda_n^{-3\alpha} = \lambda_{n+1}^{-2h}\lambda_n^{-3\alpha}, \\
\|\boldsymbol{u}_n\|_1 &\le M\delta_n^{1/2}\lambda_n, \\
\|\boldsymbol{u}_n\|_0 &\le 1 - \delta_n^{1/2}, \\
\delta_{n+1}\lambda_n^{-\alpha} &\le e(t) - \int |\boldsymbol{u}_n|^2 d\boldsymbol{x} \le \delta_{n+1}.
\end{aligned}$$

α suitable parameter ($0 < \alpha < 1$)、constant M depends on h and b.

Spatial Hölder norm: $\|f\|_0 = \sup_{\boldsymbol{x}\times[0,1]}|f|$, $[f]_m = \max_{|\theta|=m}\|D^\theta f\|_0$, $\|f\|_m = \sum_{j=0}^m [f]_j$

Three stages in the construction

Procedure to make $\boldsymbol{u}_{q+1}$ from $\boldsymbol{u}_q$ (3 stages)

(1) mollification stage:

low-pass filter larger than length $\ell_n = \lambda_n^{-1-3\alpha/2}$

(2) gluing stage

- $0 \leq t \leq T$ is divided with small interval τ_n.
- Solve strong Euler eqs. with $\boldsymbol{u}_n^{(\ell_n)}(\boldsymbol{x}, t = j\tau_n)$ as initial data
- Superpose the forward-in-time and backward-in-time solutions.
- Calculate the Reynolds stress R from the superposed solution.

(3) perturbation stage

- calculate the Mikado flow to cancel the Reynolds stress R.
- $\boldsymbol{u}_{n+1}$ = (superposed solutions) + mikado flow

Precise form of the perturbation

- Difference between the glued-velocity energy and the prescribed energy $e(t)$

$$\rho_n(t) = \frac{1}{3}\left(e(t) - \frac{\delta_{n+2}}{2} - \int |\tilde{\boldsymbol{u}}_n|^2 d\boldsymbol{x}\right), \quad \rho_{n,i}(\boldsymbol{x},t) = \frac{\eta_i^2(\boldsymbol{x},t)}{\sum_j \int \eta_j^2(\boldsymbol{y},t)d\boldsymbol{y}}\rho_n(t)$$

  $\eta_i(\boldsymbol{x},t)$: cut-off function
- Inverse Lagrangian map of the glued velocity $\tilde{\boldsymbol{u}}_n$

$$(\partial_t + \tilde{\boldsymbol{u}}_n \cdot \nabla)\boldsymbol{a}_j = \boldsymbol{0}, \quad \boldsymbol{a}_j(\boldsymbol{x}, j\tau_n) = \boldsymbol{x}$$

- Transform of the Reynolds stress

$$\mathsf{R}_{n,j} = \rho_{n,j}\mathrm{Id} - \eta_j^2 \mathring{\tilde{\mathsf{R}}}_n, \quad \tilde{\mathsf{R}}_{n,j} = \frac{\nabla\boldsymbol{a}_j \mathsf{R}_{n,j}(\nabla\boldsymbol{a}_j)^T}{\rho_{n,j}}$$

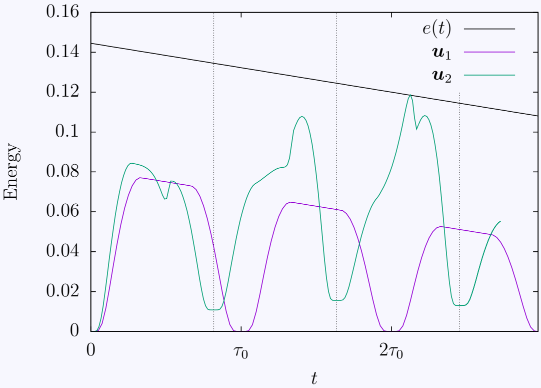
- Main part of the perturbation ($\boldsymbol{w}(\mathsf{R},\boldsymbol{x})$: six jets)

$$\boldsymbol{W}_{o,n} = \sum_j [\rho_{n,j}(\boldsymbol{x},t)]^{1/2}(\nabla\boldsymbol{a}_j(\boldsymbol{x},t))^{-1}\boldsymbol{w}\left(\bar{\tilde{\mathsf{R}}}_{n,j}, \frac{\lambda_n}{2\pi}\boldsymbol{a}_j(\boldsymbol{x},t)\right)$$

- $(n+1)$-th step velocity $\boldsymbol{u}_{n+1} = \tilde{\boldsymbol{u}}_n + \boldsymbol{W}_{o,n} + \boldsymbol{W}_{c,n}$
  ($\boldsymbol{W}_{c,n}$ guarantees divergence-free condition)

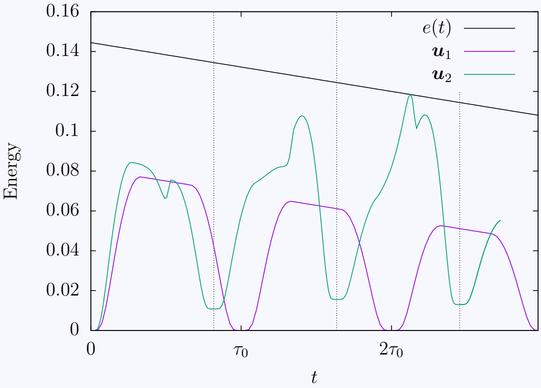
Simulation example: parameters

- Difficulty : n-th step wavenumber $\lambda_n = 2\pi\lceil a^{b^n}\rceil$ (periodic unit cube)
- $e(t) = e_0 - At$ (linear decrease), exponent $h = 0.30$ ($a = 3.0, b = 1.15, \alpha = 0.1$)
- Initial guess : $\boldsymbol{u}_0(\boldsymbol{x}, t) = \boldsymbol{0}$



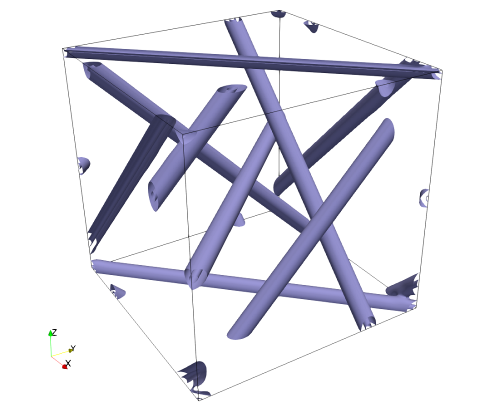
Simulation example: energy

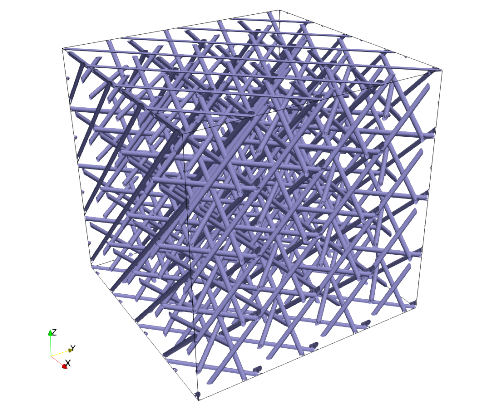
- $n = 0$: $\boldsymbol{u}_0(\boldsymbol{x}, t) = \boldsymbol{0}$
- $n = 1$: $\boldsymbol{u}_1(\boldsymbol{x}, t) = \eta_1(t)\boldsymbol{w}(\mathrm{Id}, \frac{\lambda_1}{2\pi}\boldsymbol{x}) = \eta_1(t)\boldsymbol{w}(\mathrm{Id}, 4\boldsymbol{x})$



Simulation example: $n = 1$

- $n = 1$: $\boldsymbol{u}_1(\boldsymbol{x}, t) = \eta_1(t)\boldsymbol{w}(\mathrm{Id}, \frac{\lambda_1}{2\pi}\boldsymbol{x}) = \eta_1(t)\boldsymbol{w}(\mathrm{Id}, 4\boldsymbol{x})$



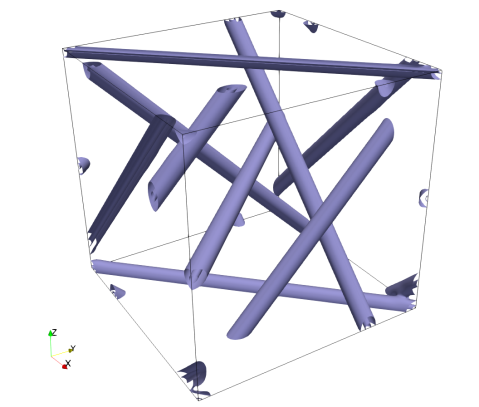


$|\nabla \times \boldsymbol{w}(\mathrm{Id}, \boldsymbol{x})| = \mathrm{const.}$

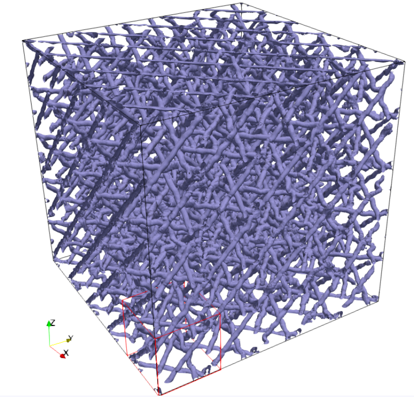
$|\nabla \times \boldsymbol{u}_1(\boldsymbol{x}, t)| = \mathrm{const.}$

Simulation example: $n = 2$

- $n = 2$: $\boldsymbol{u}_2(\boldsymbol{x}, t) \simeq \boldsymbol{w}(\mathsf{R}, \frac{\lambda_2}{2\pi}\boldsymbol{x}) = \boldsymbol{w}(\mathsf{R}, 5\boldsymbol{x})$

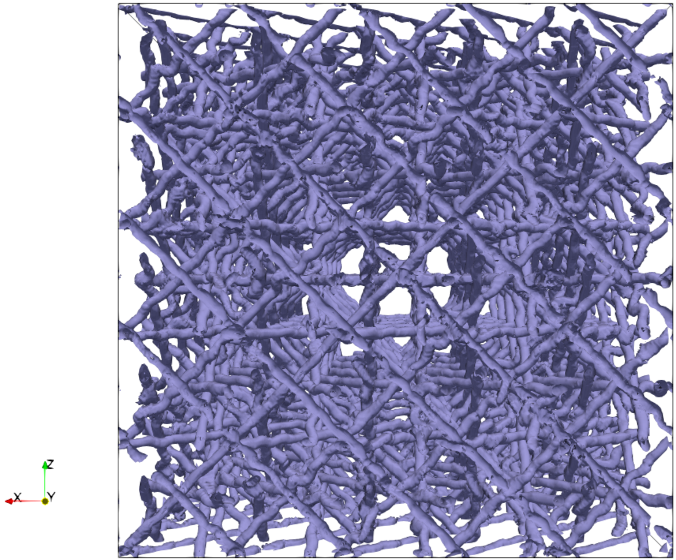


$|\nabla \times \boldsymbol{w}(\mathrm{Id}, \boldsymbol{x})| = \mathrm{const.}$



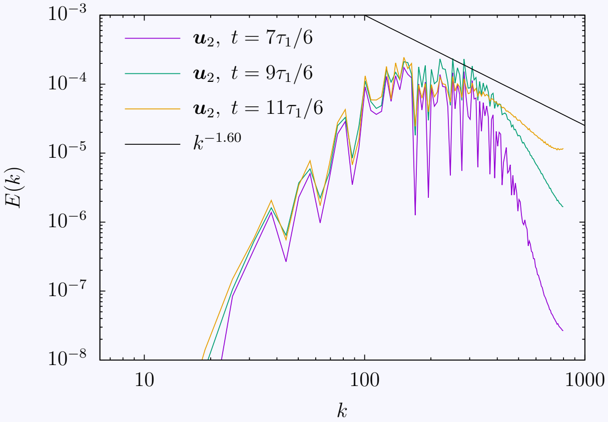
$|\nabla \times \boldsymbol{u}_2(\boldsymbol{x}, t = 3\tau_1/2)| = \mathrm{const.}$

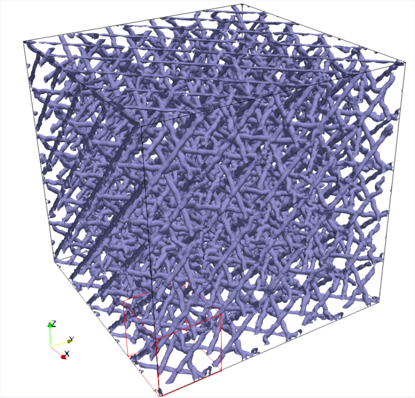
Simulation example: $n = 2$



Difficulty to obtain scaling of $\boldsymbol{u}_2$

- Prescribed exponent $h = 0.30$
- $n = 2$: $\boldsymbol{u}_2(\boldsymbol{x}, t)$ $\quad E_n(k) \propto k^{-2h-1} = k^{-1.60}$ **???**

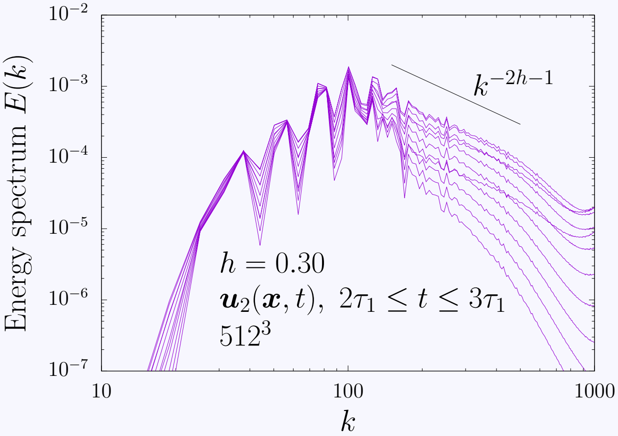




- To get scaling at $\boldsymbol{u}_2$, we use thicker Mikado jets.

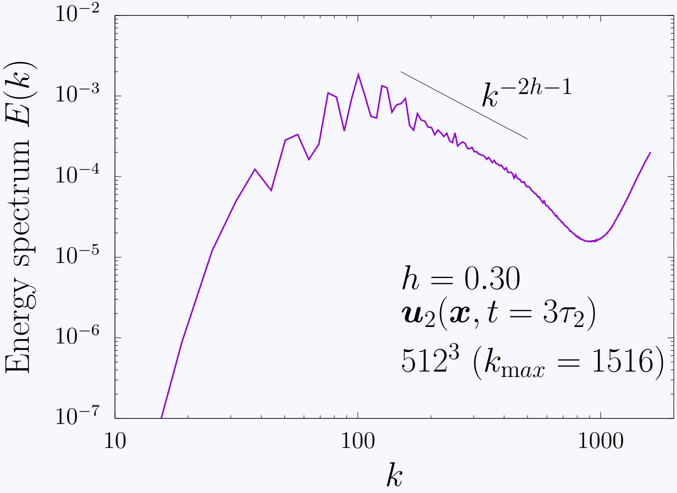
Scaling of $\boldsymbol{u}_2$ with thicker Mikados

- Prescribed exponent $h = 0.30$ ($a = 3, b = 1.15, \alpha = 0$)
- $n = 2$: $\boldsymbol{u}_2(\boldsymbol{x}, t)$ $\quad E_n(k) \propto k^{-2h-1} = k^{-1.60}$ **???**



- Mikado radius doubled — a dirty trick to obtain scaling at $n = 2$

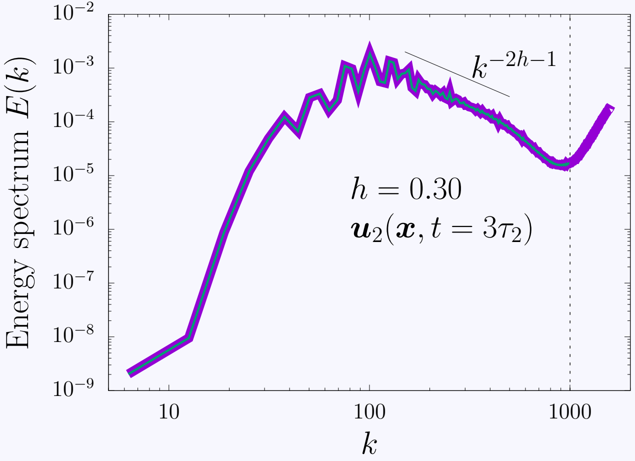
Scaling of $\boldsymbol{u}_2$ with thicker Mikados



- Numerical problem: noise grows at high k.
  - Using thinner Mikados can cure this problem.

Longitudinal structure function of $\boldsymbol{u}_2$

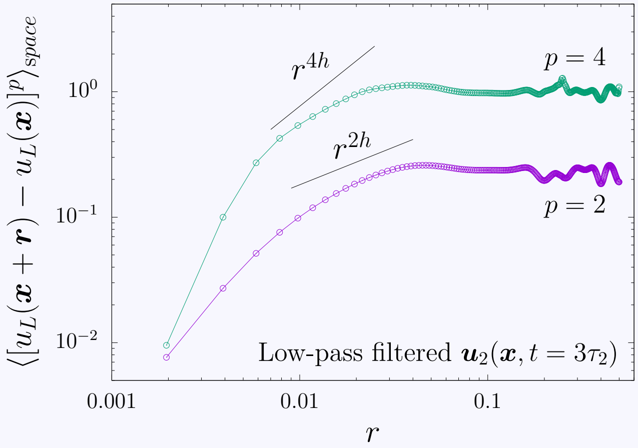
- Prescribed exponent $h = 0.30$ ($a = 3, b = 1.15, \alpha = 0$)
- $|\boldsymbol{u}_2(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}_2(\boldsymbol{x},t)| \sim r^h$



- $k \geq 1000$ part is removed (low-pass filter).

Longitudinal structure function of $\boldsymbol{u}_2$

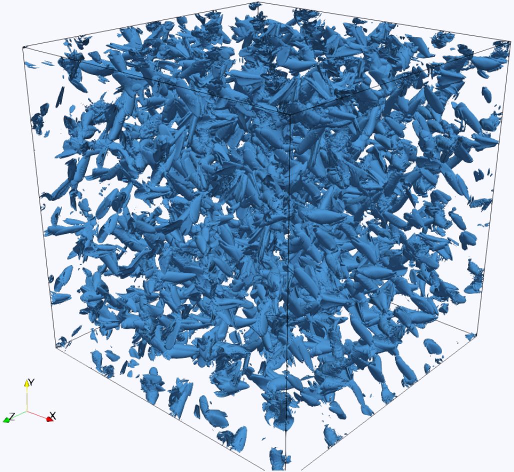
- Prescribed exponent $h = 0.30$ $(a = 3, b = 1.15, \alpha = 0)$
- $|\boldsymbol{u}_2(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}_2(\boldsymbol{x},t)| \sim r^h$



- Order $p = 2$ and 4: consistent with r^{2h} and r^{4h}?

Vorticity iso-surface of $\boldsymbol{u}_2$

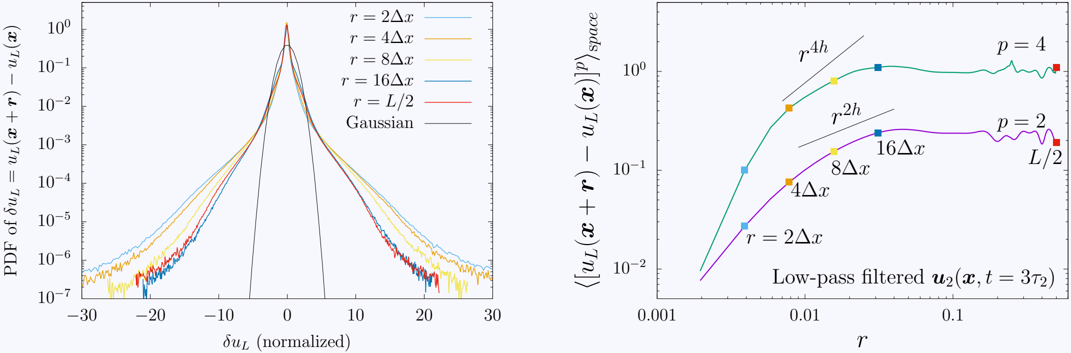
- Prescribed exponent $h = 0.30$ $(a = 3, b = 1.15, \alpha = 0)$



- Low-pass filtered $\boldsymbol{u}_2(\boldsymbol{x}, 3\tau_2)$

PDF: longitudinal velocity increment $\boldsymbol{u}_2$

- Prescribed exponent $h = 0.30$ $(a = 3, b = 1.15, \alpha = 0)$, $\boldsymbol{u}_2(\boldsymbol{x}, 3\tau_2)$



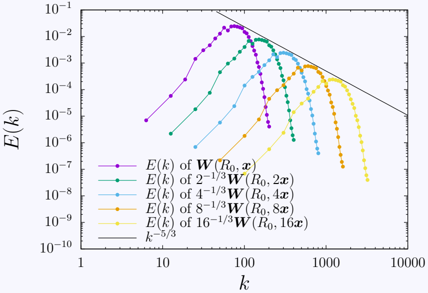
- Normalized PDFs in the “scaling range” do not collapse?

Discussion

- Insight from the mathematical construction?

  Prescribed (single) exponent h : $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)| \propto r^h$

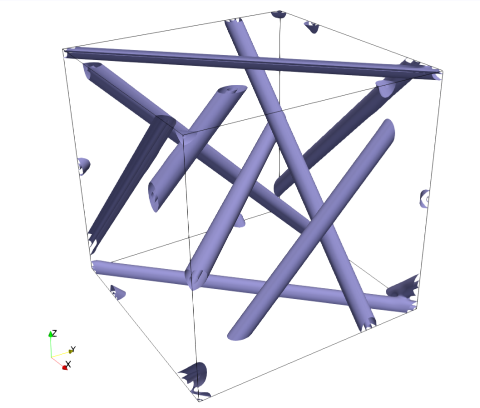
  — how we tile the Mikado flows, not the profile of each jet



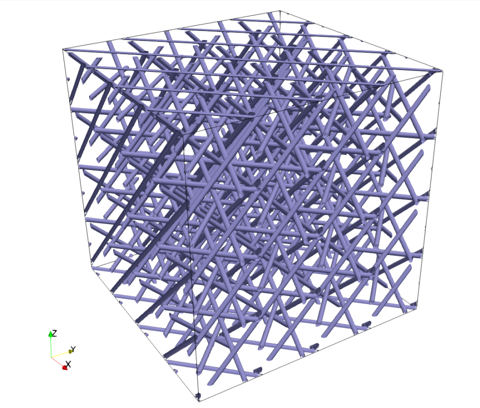
- Question: is the constructed solution multi-fractal or not?

  $\langle [u_L(\boldsymbol{x}+\boldsymbol{r},t)-u_L(\boldsymbol{x},t)]^p\rangle \not\propto r^{ph}$?

Tiling of the Mikado flows



fundamental Mikado $\boldsymbol{w}(\mathrm{Id}, \boldsymbol{x})$



$4 \times 4 \times 4$ tiling: $\boldsymbol{w}(\mathrm{Id}, 4\boldsymbol{x})$

Can we introduce an in-homogeneous factor for each sub-cube?
— reminiscent of cascade models of turbulence

Summary and outlook

- Summary
  - Mathematical construction of dissipative Euler solutions by De Lellis, Székelyhidi and co-workers
  - Numerical simulation of the construction hopefully with scaling (on-going)
- Outlook
  - Optimize Mikado thickness to obtain scaling with $n = 2, 3, 4$.
  - Increase n (iteration) with larger resolution.
  - Check the scaling $\langle \{[\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)] \cdot \hat{\boldsymbol{r}}\}^p \rangle \propto r^{\xi_p}$
    In particular, the 3rd order? (Duchon-Robert?)
    Are the constructed solutions multi-scaling (intermittent)?
  - Can we relate the construction to cascade models?
  - Adapt to lower dimensional flows (e.g., surface quasi-geostrophic (SQG) model).