Macroscopic fluctuation theory for integrable systems

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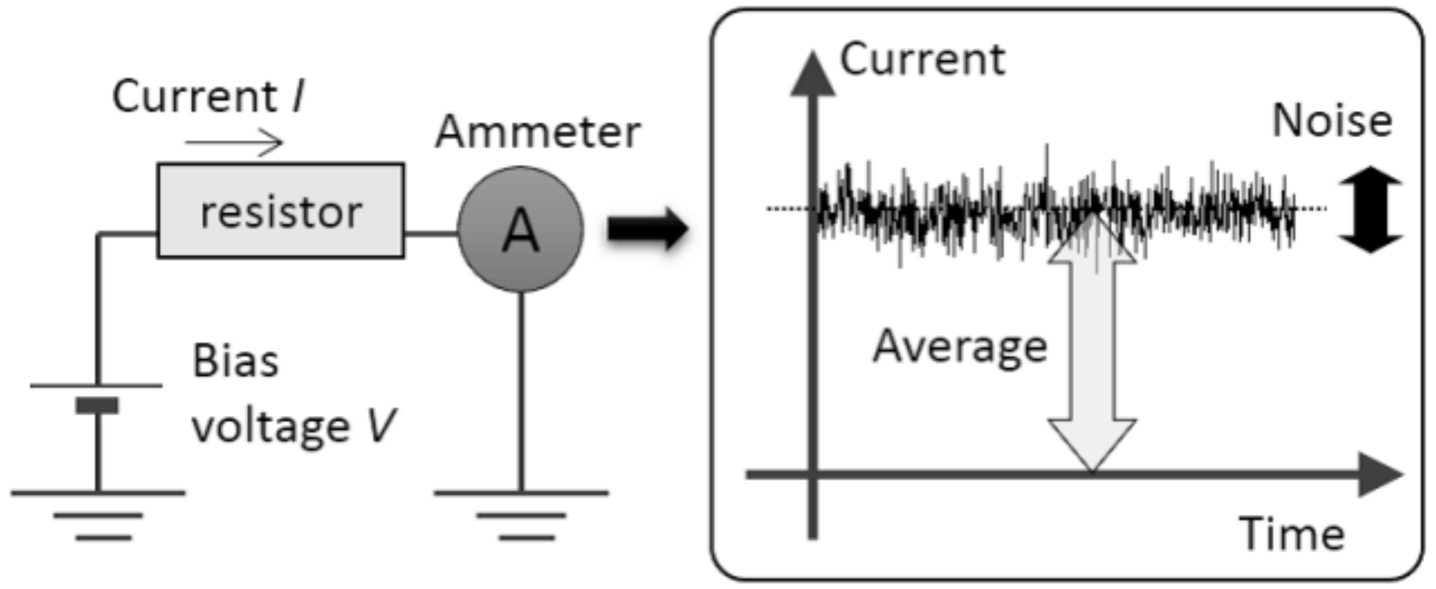
Large deviation in experiments

- Traditionally, rare fluctuations, i.e. large deviation, of transport in quantum many-body systems have been studied in the context of full counting statistics (FCS) of electron transport.

- In particular people have focused on observing the average current $I = \langle I(t) \rangle$ as well as the current noise $\langle (I(t) - I)^2 \rangle$ in mesoscopic systems.



[Kobayashi, 2016]

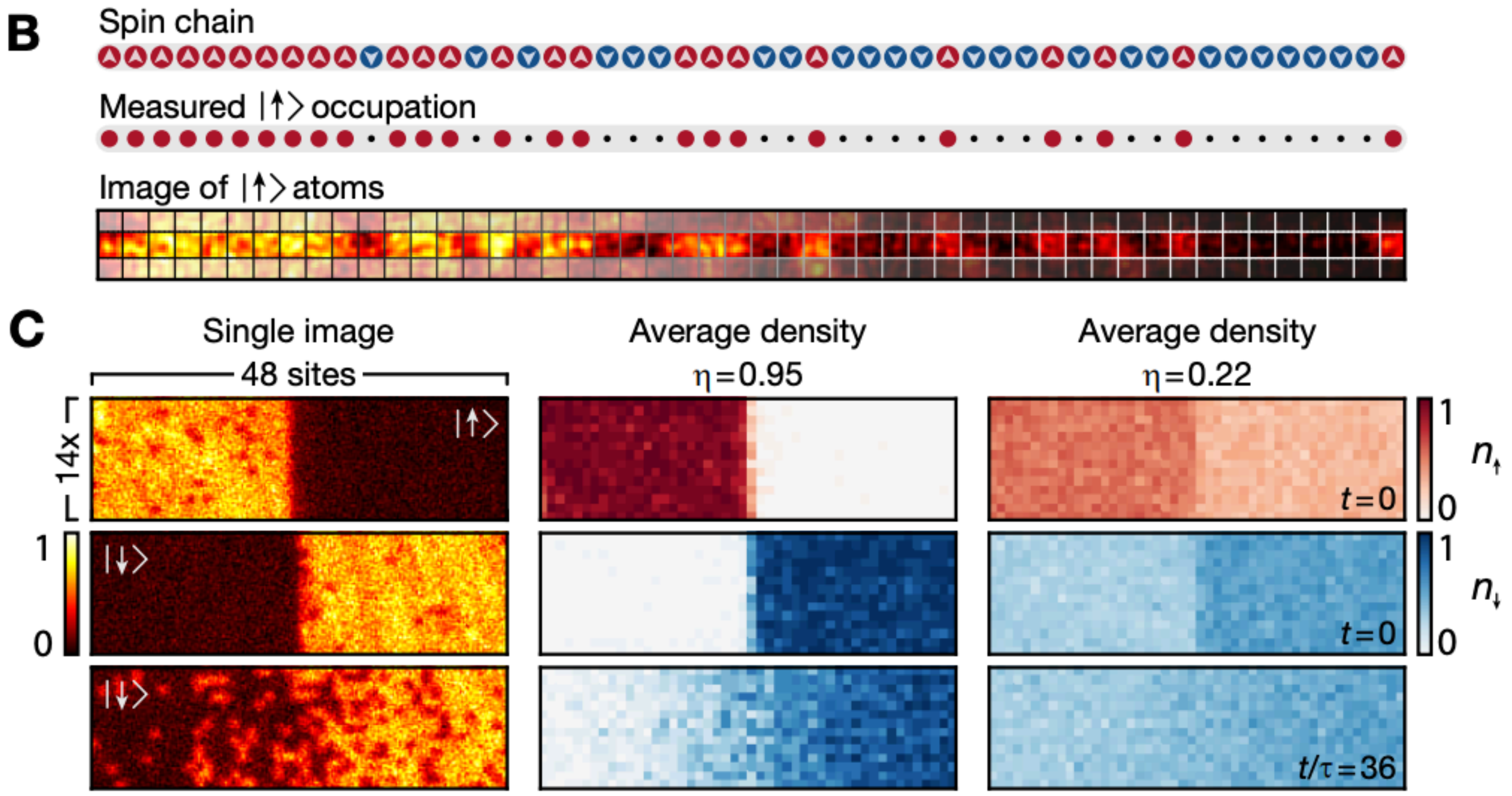
- One can also perform a more elaborate experiment and measure the skewness $\langle (I(t) - I)^3 \rangle$ ([Reulet, Senzier, and Prober, 2003]) as well as the fluctuation theorem ([Küng et al., 2012]).

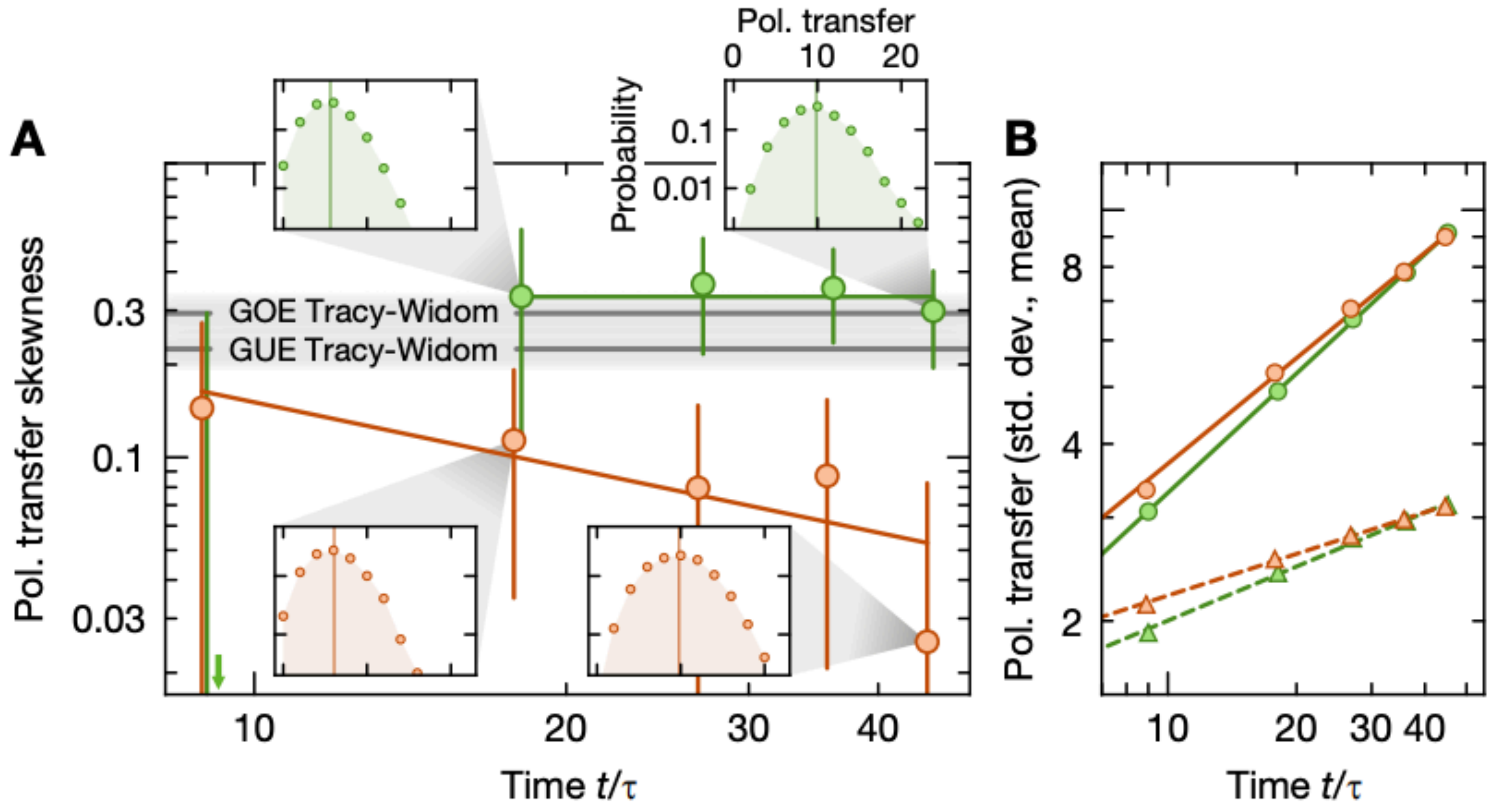
- A measurement of the (shot) noise also has also confirmed the existence of fractional charges. [de-Picciotto et al., 1997]

Large deviation in ultra-cold atom experiments

- In recent years it has become possible to experimentally study FCS using ultra-cold atoms.

- For instance very recently FCS of spin transport in the isotropic XXZ chain was investigated by probing the bosonic $^{87}\mathrm{Rb}$ atoms trapped in an optical lattice. [D. Wei et al., 2021]





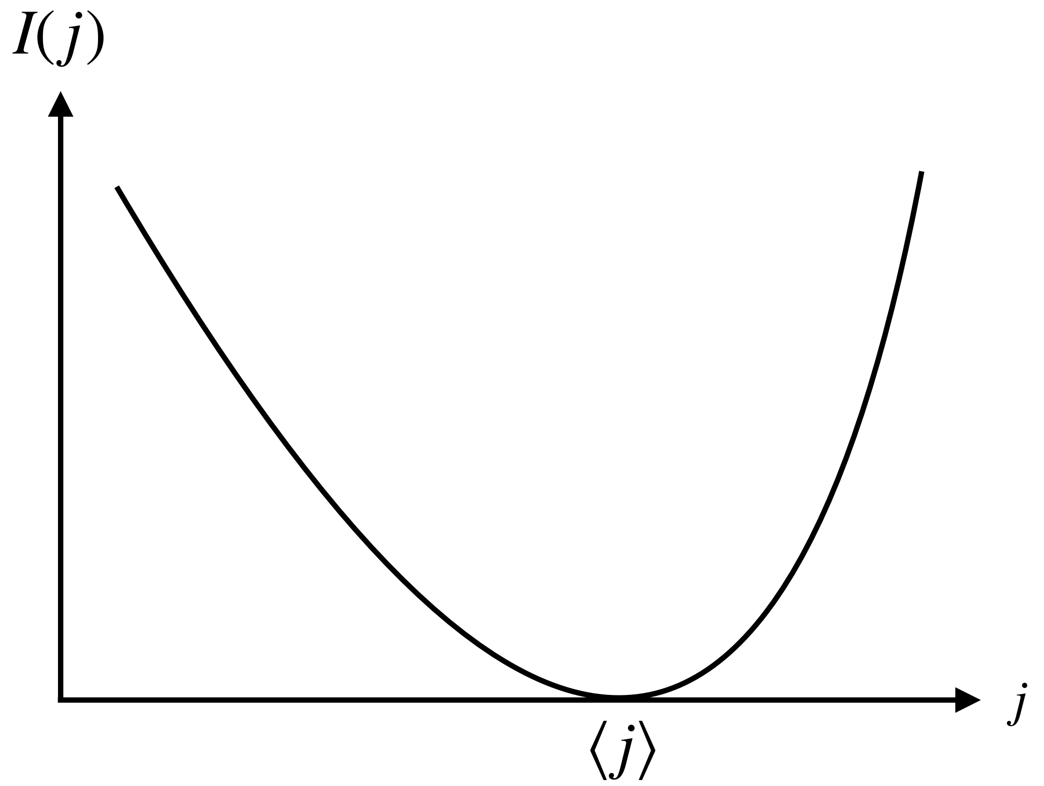
- A measurement of the skewness allowed the authors to conclude that the fluctuation is governed by Gaussian Orthogonal Ensemble (GOE) if the system is initially in a weakly-polarised state. [D. Wei et al., 2021]

- A versatile tool to study FCS in quantum many-body systems, in particular in integrable systems, is highly desired.

Large deviation theory in many-body systems

- In large deviation theory (LDT), one is primarily concerned with the rare fluctuation of some fluctuating quantity A_T that is extensive in T.
- In particular, LDT tells us about the probability distribution of A_T that has a peak at the most likely value $\langle a \rangle := \lim_{T\to\infty} A_T/T$.
- Of prime interest is the probability distribution of the time-integrated current (or transferred charge) associated to some charge.

$$J_T := \int_0^T \mathrm{d}t\, j(t) = N_T - N_0, \quad N_t := \int_0^\infty \mathrm{d}x\, \rho(x,t)$$



- LDT asserts that for large T the probability goes as

$$\mathrm{Prob}(J_T = Tj) \sim e^{-TI(j)}, \quad I(\langle j \rangle) = 0$$

- An often more convenient object to work with is the generating function $\langle e^{\lambda J_T} \rangle \sim e^{TF(\lambda)}$. The scaled cumulant generating function (SCGF) $F(\lambda)$ is related to the large deviation function $I(j)$ via the Legendre transformation

$$F(\lambda) = \max_j [\lambda j - I(j)]$$

Large deviation theory in quantum many-body systems

- In quantum systems the notion of fluctuating variable becomes unclear. J_T is not a natural observable either.

- Instead we can consider the probability of measuring the total charge q_0 at time 0 and q_T at time T in the right half of the system.

- The knowledge of the SCGF $F(\lambda)$ is equivalent to that of all the cumulants c_n:

$$c_n := \lim_{T\to\infty} \frac{\langle J_T^n \rangle^{\mathrm{c}}}{T} = \frac{\mathrm{d}^n F(\lambda)}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0}$$

- Clearly $c_1 = \langle j \rangle$. The variance $c_2 = \int_{\mathbb{R}} \mathrm{d}t \, \langle j(0,t) j(0,0) \rangle$ is sometimes also called Drude self-weight.

- There are a number of spectacular theoretical results in the study of FCS in quantum many-body systems. The most prominent one is the celebrated Levitov-Lesovik formula, which provides an exact SCGF for charge transport. [Levitov and Lesovik, 1993]

- Some exact results are also available for integrable quantum impurity systems [e.g. Saleur and Weiss, 2001; Komnik and Saleur, 2011], conformal field theories [Bernard and Doyon; 2015; Doyon and Myers, 2019], quantum harmonic chains [Saito and Dhar; 2008], and integrable systems [Myers, Bhaseen, Harris, and Doyon, 2018]

Conventional macroscopic fluctuation theory

- MFT is also known as the hydrodynamic large deviation theory, and provides a universal framework to understand large deviation in many-body systems. [Bertini et al., 2002 and 2014; Bodineau and Derrida, 2006; Derrida and Gerschenfeld, 2009; Krapivski, Mallick, Sadhu, 2015, etc]

- It has been primarily developed for classical driven diffusive systems, but the idea is supposed to be applicable to any systems where large deviation principle holds.

- Consider the probability of observing a hydrodynamic density and current profile, $\rho(x,t)$ and $j(x,t)$ during a time T. MFT then claims, for diffusive systems in the infinite volume, that $\mathrm{Prob}(\{\rho(x,t), j(x,t)\}) \asymp \exp\left[-I_{[0,T]}(\rho, j)\right]$ with

$$I_{[0,T]}(\rho, j) = \int_0^T \mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}x \frac{[j(x,\tau) - j_{\mathrm{diff}}(x,\tau)]^2}{2\sigma(\rho(x,\tau))}, \quad \begin{cases} j_{\mathrm{diff}}(x,t) := -\mathfrak{D}(\rho(x,t))\partial_x \rho(x,t) & \text{Fick's law} \\ \sigma(\rho) & \text{Mobility} \end{cases}$$

- There are a few ways of justifying this assertion. With this probability the SCGF reads

$$\langle e^{\lambda J_T} \rangle \asymp \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathscr{D}\rho(x,t)\mathscr{D}j(x,t) e^{\lambda J_T} \mathrm{Prob}[\rho(x,0)] \delta(\partial_t \rho + \partial_x j) \mathrm{Prob}[\{\rho(x,t), j(x,t)\}]$$

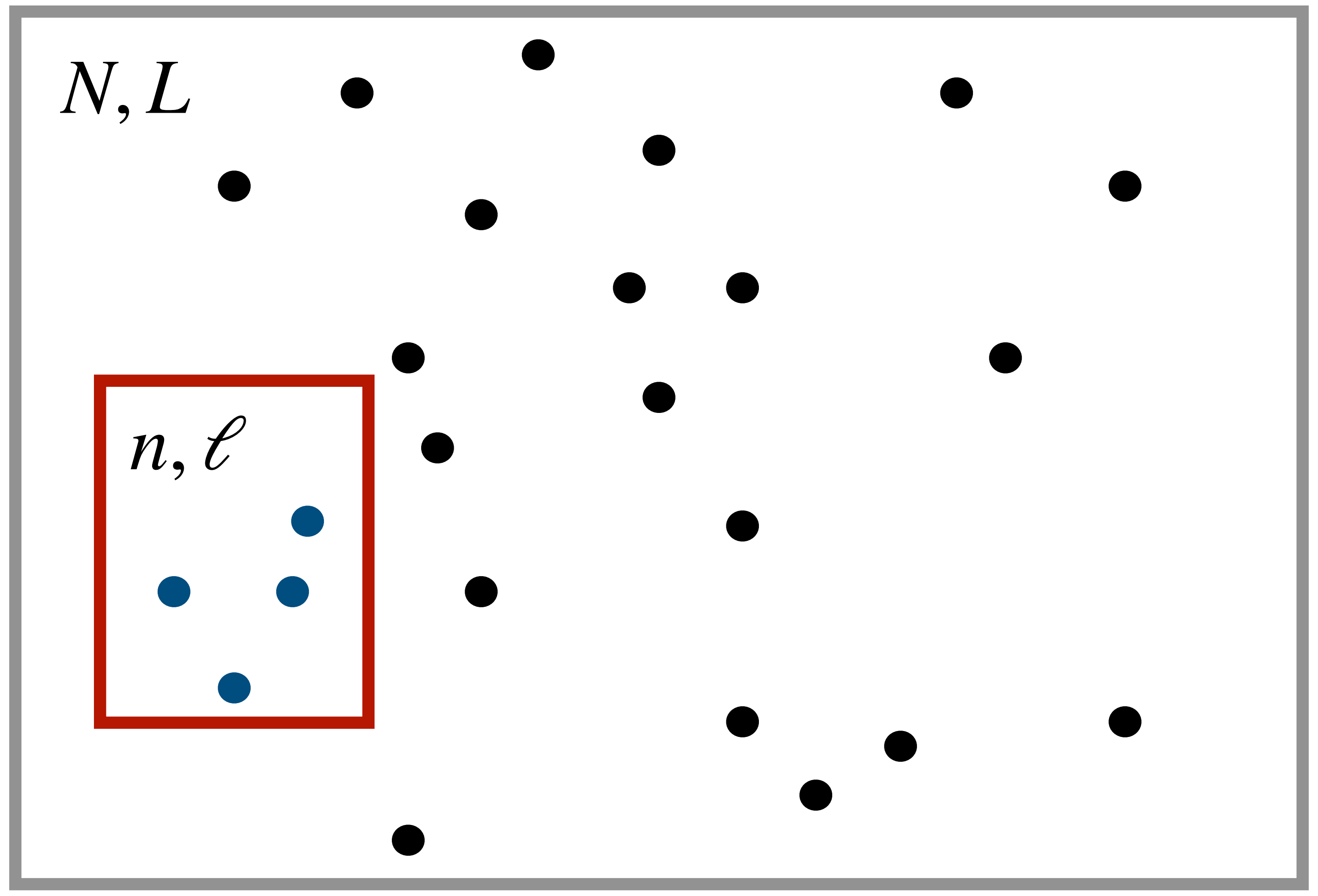
Path integral over space-time trajectories

Prob of finding the initial density profile

Continuity equation

- The probability of finding the initial density profile $\rho(x,0)$, $\mathrm{Prob}[\rho(x,0)] =: e^{-\mathrm{F}[\rho(x,0)]}$, is zero if there is no fluctuation initially. One can statistical mechanically compute it.

- For instance consider a system of size L containing N particles. The probability of finding n particles in a small but still large enough volume ℓ at the position x is given by $P_\ell(n) \sim \exp\left[-\ell I_x(n/\ell)\right]$ where



$$I_x(\rho) = f(\rho) - f(\rho_*) - (\rho - \rho_*) f'(\rho_*), \quad \rho_* := \frac{N}{L}$$

- In particular when $\rho \sim \rho_*$

$$P_\ell(n) \sim \exp\left[-\frac{\chi^{-1}(\rho)}{2}(n - \ell\rho_*)^2\right] = \exp\left[-\frac{\mathfrak{D}(\rho)}{2\sigma(\rho)}(n - \ell\rho_*)^2\right]$$

- One can generalise it to the case where initially the system is in a local equilibrium state with $\rho_{\mathrm{ini}}(x)$. The probability of finding a particular distribution $\rho(x,0)$ is then given neatly by the relative entropy

$$\mathrm{F}[\rho(x,0)] = \underbrace{D(\rho\|\rho_{\mathrm{ini}})}_{\text{Relative entropy}} = \int_{\mathbb{R}} \mathrm{d}x \, \mathrm{Tr}\left(\rho(x,0)\log(\rho(x,0)/\rho_{\mathrm{ini}}(x))\right) = \int_{\mathbb{R}} \mathrm{d}x \, \left(-s(x) + \beta_{\mathrm{ini}}(x)\rho(x) - f_{\mathrm{ini}}(x)\right), \quad \begin{cases} s(x) & \text{Entropy density} \\ f(x) & \text{Free energy density} \\ \beta(x) := \frac{\partial S}{\partial \rho(x)} & \end{cases}$$

- We can reinterpret MFT predictions based on fluctuating hydrodynamics (FHD).

- According to FHD, the mesoscopic dynamics of the system is governed by the Langevin equation

$$\partial_t \rho(x,t) + \partial_x j_{\mathrm{diff}}(x,t) - \partial_x \left(\sqrt{\sigma(\rho)}\eta(x,t)\right) = 0, \quad \langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

- The SCGF then reads

$$\langle e^{\lambda J_T}\rangle \asymp \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathscr{D}\rho(x,t) e^{\lambda J_T} \mathrm{Prob}[\rho(x,0)] \underbrace{\left\langle \delta\left(\partial_t \rho + \partial_x j_{\mathrm{diff}} - \sqrt{\sigma(\rho)}\eta\right)\right\rangle_\eta}_{\text{Averaging over the noise}}$$

- The MFT prediction on $\mathrm{Prob}(\{\rho(x,t), j(x,t)\})$ therefore reproduces the correct fluctuation.

MFT for ballistic transport

- Suppose the system with a single component supports ballistic transport. For instance the hydro equation for the TASEP is given by the Burgers equation

$$\partial_t \rho + \partial_x j_{\rm bal} = 0, \quad j_{\rm bal} := \rho(\rho - 1)$$

- The relevant scale now is the Euler scale, i.e. $x \sim t$, where the effect of fluctuation comes from the initial condition only. The SCGF then is expected to be

$$\langle e^{\lambda J_T} \rangle \asymp \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathscr{D}\rho(x,t) e^{\lambda J_T - \mathrm{F}[\rho(x,0)]} \delta\left(\partial_t \rho + \partial_x j_{\rm bal}\right)$$

- We can derive the same expression by changing the space-time scaling in the first formulation of $\langle e^{\lambda J_T} \rangle$. It is more convenient to rewrite as

$$\langle e^{\lambda J_T} \rangle \asymp \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathscr{D}\rho(x,t)\mathscr{D}H(x,t) e^{-S[\rho(x,t),H(x,t)]}$$

$$S[\rho(x,t),H(x,t)] := -\lambda J_T + \mathrm{F}[\rho(x,0)] + \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathrm{d}t\mathrm{d}x\, H(x,t)(\partial_t \rho(x,t) + \partial_x j_{\rm bal}(x,t))$$

- For large T the path-integral should be dominated by the contribution from the optimal path that minimises the action

$$\langle e^{\lambda J_T} \rangle \asymp e^{-S[\bar{\rho}(x,t),\bar{H}(x,t)]}$$

- The optimal configurations $\bar{\rho}(x,t), \bar{H}(x,t)$ can be obtained by solving $\delta S[\rho(x,t), H(x,t)] = 0$. The resulting set of equations are

$$\begin{aligned}
\lambda\Theta(x) - \beta(x) + \beta_{\text{ini}}(x) - H(x,0) &= 0 \\
-\lambda\Theta(x) + H(x,T) &= 0 \\
\partial_t \rho(x,t) + A[\rho(x,t)]\partial_x \rho(x,t) &= 0 \\
\partial_t H(x,t) + A[\rho(x,t)]\partial_x H(x,t) &= 0
\end{aligned}$$

- The flux Jacobean $A[\rho] := \frac{\partial j}{\partial \rho}$ controls the strength of ballistic transport.
- The time-evolution is the same, but the boundary condition incorporates the effect of biasing the dynamics.
- MFT for ballistic transport generically suffers from singular behaviours, i.e. shocks.
- No Lax condition to be satisfied.
- One way to get around is to start with MFT with diffusive corrections and consider the Euler limit (e.g. WASEP v.s. TASEP).

[Bodineau and Derrida, 2006]

Jensen-Varadhan formalism

- Another approach to obtain the large deviation function $I_{[0,T]}(\rho)$ for the TASEP was formulated by Jensen and Varadhan.

  [Jensen, 2000; Varadhan, 2004]

- JV formalism states that $I_{[0,T]}(\rho)$ is given by

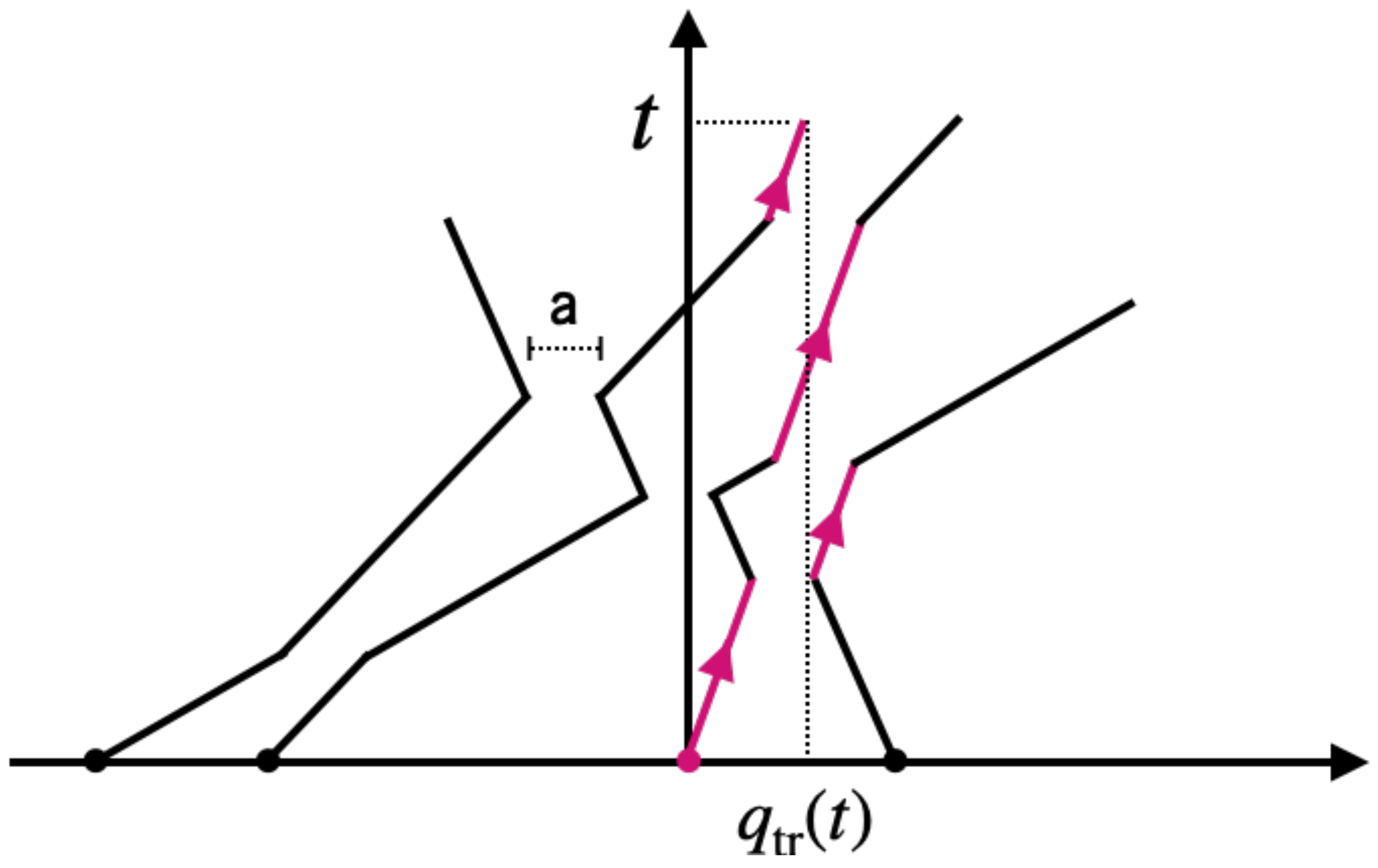
$$I_{[0,T]}(\rho) = \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x (\partial_t s + \partial_x j_s)_-, \quad (a)_- := \min(0,a)$$

- Interestingly, we see that entropic shocks that ensure $\partial_t s + \partial_x j_s \geq 0$ do not contribute. This is natural because $I_{[0,T]}(\rho)$ tells us the cost of not having hydrodynamic configurations, i.e. non-entropic solutions.

- We want to apply the idea of MFT for hyperbolic systems to integrable systems. The absence of shocks in GHD suggests that MFT for GHD could be handled with a better control.

Generalised hydrodynamics

- In integrable systems thermodynamics as well as the dynamics of the system are dictated by the scattering data: particle species, dispersion relation, and the two-body S-matrix.
- We shall consider a diagonally-scattering integrable model with a single species defined on a line. Generalisations to more complicated models are straightforward.
- A kinetic intuition behind GHD is that on a hydrodynamic scale, quasi-particles in integrable systems behave pretty much like tracer particles of hard-rods. [Boldrighini, Dobrushin, and Sukhov, 1983; Spohn, 1991; Doyon and Spohn, 2017; Doyon, TY, and Caux, 2018]



[Cubero, TY, and Spohn, 2021]

- This underlying similarity of kinetics among integrable systems amounts to universal structures of hydrodynamic equations.
- An exact expression of the current average turns out to be instrumental in GHD. [See reviews: Borsi, Pozsgay, and Pristyák, 2021; Cortés Cubero, TY, and Spohn, 2021]

- On the Euler scale, quasi-particles in integrable systems are transported according to the GHD equation

[Castro-Alvaredo, Doyon, and TY, 2016; Bertini, Collura, De Nardis, Fagotti, 2016]

$$\partial_t \rho_\theta(x,t) + A_\theta^{\ \phi}[\rho_\cdot(x,t)]\partial_x \rho_\phi(x,t) = 0, \qquad \begin{cases} \rho_\theta & \text{Density of particle} \\ A_\theta^{\ \phi} := \dfrac{\partial j_\theta}{\partial \rho_\phi} & \text{Flux matrix} \end{cases}$$

- In MFT it is in fact more convenient to work with the Lagrange multipliers β^θ (we are considering a GGE $\varrho \sim e^{-\beta^\theta Q_\theta}$)

$$\partial_t \beta^\theta(x,t) + A_\phi^{\ \theta}[\rho_\cdot(x,t)]\partial_x \beta^\phi(x,t) = 0$$

Note $C_{ij} = \dfrac{\partial \rho_i}{\partial \beta^j}$ and $AC = CA^{\mathrm{T}}$

- To solve initial value problems we shall use the GHD equation in terms of the normal mode

$$\partial_t \epsilon_\theta(x,t) + v_\theta^{\mathrm{eff}}(x,t)\partial_x \epsilon_\theta(x,t) = 0$$

- To go from the equation for β^θ to that for ϵ_θ, we used $(R^{-1})_\phi^{\ \theta}\partial_{t,x}\beta^\phi = \partial_{t,x}\epsilon^\theta$ where R diagonalises the flux matrix: $RAR^{-1} = \mathrm{diag}\, v^{\mathrm{eff}}$.

- One of the crucial properties of the GHD equation is that its solutions always involve neither shocks nor rarefaction waves but only contact discontinuities (CD). CDs can be thought of as shocks without entropy production.

Initial value problems in GHD

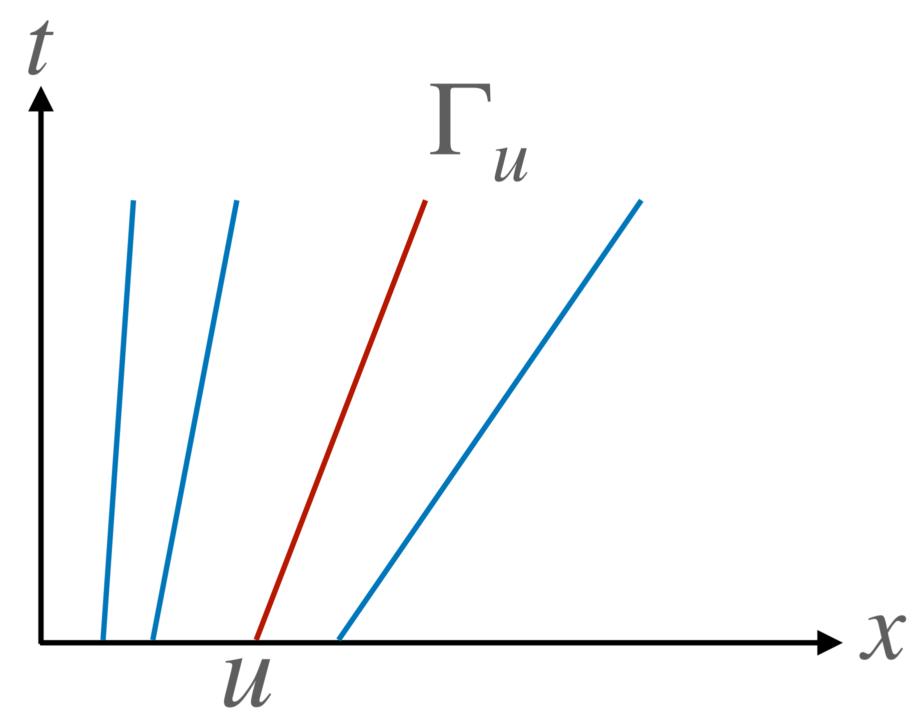
[Doyon, Spohn, and TY, 2017; Doyon 2020]

- Let us start with recalling how the method of characteristics works in a simple case: $\partial_t \rho + v(\rho)\partial_x \rho = 0$ with $\rho(x,0) = \rho_0(x)$.
- For each $x = u$ at $t = 0$, we have the characteristic curve Γ_u along which $\rho(x,t)$ is constant: $\frac{\mathrm{d}x(u,t)}{\mathrm{d}t} = v(\rho(x(u,t),t))$. This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x,t) = \frac{\partial}{\partial t}\rho(x,t) + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}\rho(x,t) = \frac{\partial}{\partial t}\rho(x,t) + v(\rho)\frac{\partial}{\partial x}\rho(x,t) = 0$$

- Furthermore the characteristic curve is straight because clearly $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0$. The equation of characteristics can be solved as $\frac{\mathrm{d}x}{\mathrm{d}t} = v(\rho(x,t)) = v(\rho(u,0)) = v(\rho_0(u))$, i.e.

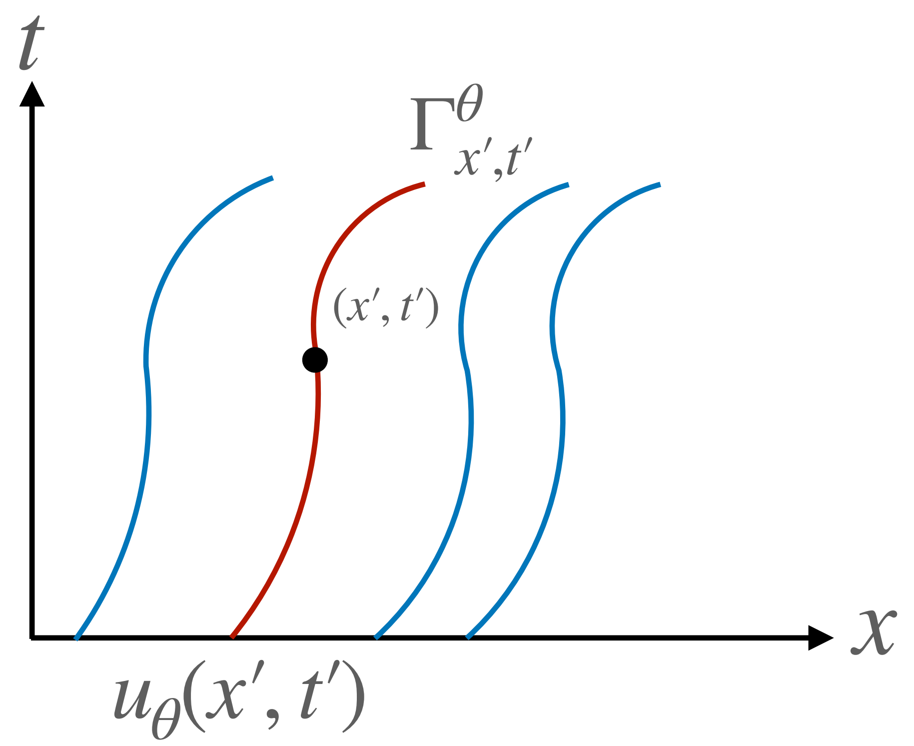
$$x = v(\rho_0(u))t + u$$



- Having $u(x,t)$ by solving the equation, we obtain $\rho(x,t) = \rho(u(x,t),0) = \rho_0(u(x,t))$.
- We want to do the same for GHD.

- The characteristic curve in GHD is defined by $\frac{\mathrm{d}x_\theta(u,t)}{\mathrm{d}t} = v^{\mathrm{eff}}_\theta[\epsilon_\cdot(x_\theta(u,t),t)]$, which immediately implies $\frac{\mathrm{d}\epsilon_\theta(x_\theta(u,t),t)}{\mathrm{d}t} = 0$ with $x_\theta(u,0) = u$.
- The characteristic curve is **not** straight, i.e. $\frac{\mathrm{d}^2x_\theta(u,t)}{\mathrm{d}t^2} \neq 0$. One gets $\epsilon_\theta(x_\theta(u,t),t) = \epsilon_\theta(x_\theta(u,0),0) = \epsilon_\theta(u,0)$.
- In fact it is more convenient to fix the space time (x',t') and then construct a characteristic curve that passes $x = u_\theta(x',t')$ at $t = 0$.
- We thus redefine $u = u_\theta(x,t), x_\theta(u_\theta(x,t),t) = x$ with which we have

$$\epsilon_\theta(x,t) = \epsilon_\theta(u_\theta(x,t),0)$$



- How do we determine $u_\theta(x,t)$? Clearly the characteristic curves being not straight isn't helpful.
- The following observation makes things simpler: by changing the state-dependent coordinate change we get

$$\partial_t\hat{\epsilon}_\theta(q,t) + v^-_\theta\partial_x\hat{\epsilon}_\theta(q,t) = 0, \quad v^-_\theta := v^{\mathrm{eff}}_\theta\mathscr{K}_\theta[n^-]$$

[Doyon, Spohn, and TY, 2017]

- Here $\hat{\epsilon}_\theta(q,t)$ is defined by $\hat{\epsilon}_\theta(q_\theta(x,t),t) = \epsilon_\theta(x,t)$ with

$$q_\theta(x,t) := \int_{x_0}^{x}\mathrm{d}y\,\mathscr{K}_\theta[\epsilon_\cdot(y,t)], \quad \mathscr{K}_\theta[\epsilon_\cdot] := \frac{(p')^{\mathrm{dr}}_\theta[\epsilon_\cdot]}{p'_\theta}, \quad h^{\mathrm{dr}}_\theta := (R^{-\mathrm{T}})_\theta^{\ \phi}h_\phi$$

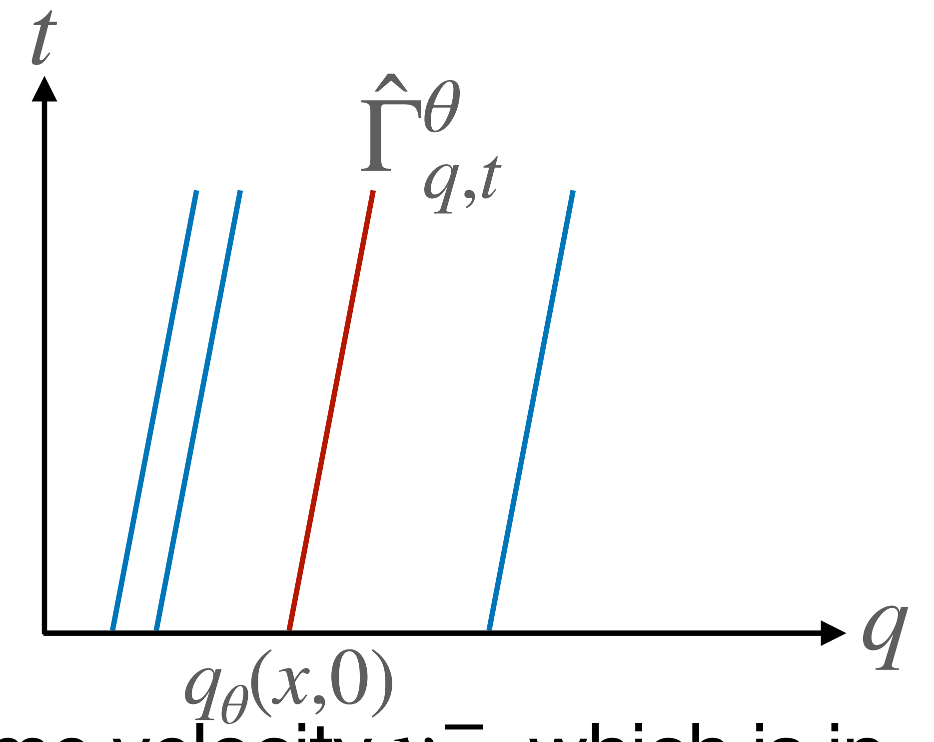
- Quasi-particles are now transported freely according to the above equation but on the state-dependent phase space $\mathrm{d}q\mathrm{d}\theta = \mathscr{K}_\theta[\epsilon(x)]\mathrm{d}x\mathrm{d}\theta$. The asymptotic coordinate x_0 is chosen so that $\rho_\theta(x,t) = \rho_\theta^-$ for all $x_0 \leq x$ at any time $t \in [0,T]$.

- The equation is trivially solved by $\hat{\epsilon}_\theta(q,t) = \hat{\epsilon}(q - v_\theta^- t, 0)$. Using the definition of $u_\theta(x,t)$ i.e. $\epsilon_\theta(x,t) = \epsilon_\theta(u_\theta(x,t),0)$ and $\hat{\epsilon}_\theta(q_\theta(x,t),t) = \epsilon_\theta(x,t)$, it immediately follows that

$$\hat{\epsilon}_\theta(q_\theta(u_\theta(x,t),0),0) = \hat{\epsilon}(q_\theta(x,t) - v_\theta^- t, 0)$$

- We thus get the solution of the characteristics in GHD, which also determines $u_\theta(x,t)$

$$q_\theta(x,t) = v_\theta^- t + q_\theta(u_\theta(x,t),0)$$



- In the q-coordinate space the characteristic lines are all straight. Also importantly they share the same velocity v_θ^-, which is in accordance with the fact that there is no shock in GHD.

- Alternatively the above solution can also be written as

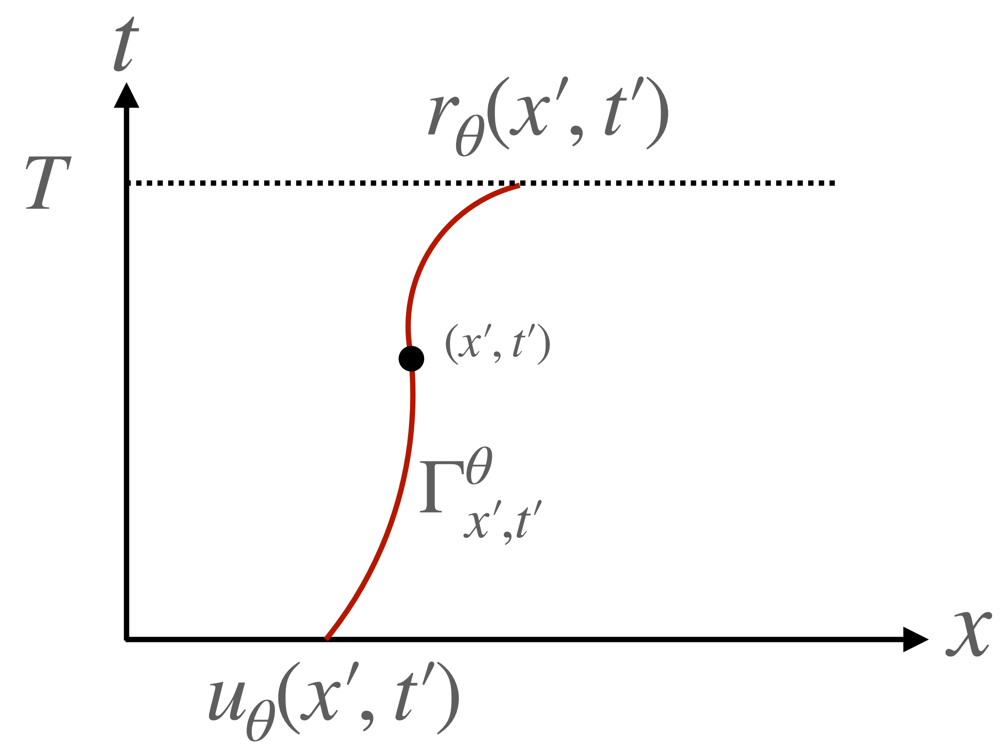
$$\int_{x_0}^{u_\theta(x,t)} \mathrm{d}y\, \mathscr{K}_\theta[\epsilon_\cdot^0(y)] + v_\theta^- t = \int_{x_0}^{x} \mathrm{d}y\, \mathscr{K}_\theta[\epsilon_\cdot(y,t)]$$

- One can also solve the GHD equation in a similar way when the boundary condition at $t = T$ is given rather than the initial condition. In this case the solution is given by $\epsilon_\theta(x,t) = \epsilon_\theta^T(r_\theta(x,t))$ where $r_\theta(x,t)$ satisfies

$$\int_{x_0}^{r_\theta(x,t)} \mathrm{d}y\, \mathscr{K}_\theta[\epsilon_\cdot^T(y)] + v_\theta^-(t-T) = \int_{x_0}^{x} \mathrm{d}y\, \mathscr{K}_\theta[\epsilon_\cdot(y,t)]$$

- These explicit solutions of GHD equations subject to certain boundary conditions turn out to be instrumental in doing MFT for GHD.
- With the aid of exact solutions we can compute quantities of interest such as cumulants.
- It is useful to observe some identities:

$$u_\theta(r_\theta(x,t),T) = u_\theta(x,t), \quad r_\theta(u_\theta(x,t),0) = r_\theta(x,t)$$



MFT for GHD

- We shall generalise the MFT formulated for a single component ballistic transport to GHD. Instead of the current associated to a physical charge Q_{i_*}, we consider that associated to a charge Q_{θ_*} labeled by θ_*, constituting a complete set of charges: $J_T = \int_0^T \mathrm{d}t\, j_{\theta_*}(0,t)$.
- The action to be optimised is then

$$S[\rho(x,t), H(x,t)] = -\lambda J_T + F[\rho(x,0)] + \int_0^T \mathrm{d}t \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}\theta\, H^\theta(x,t)(\partial_t \rho_\theta + \partial_x j_\theta)$$

$$F[\rho(x,0)] := \int_{\mathbb{R}} \mathrm{d}x\, \left(-s(x) + \beta^\theta_{\mathrm{ini}}(x)\rho_\theta(x) - f_{\mathrm{ini}}(x)\right)$$

- The optimisation yields the MFT equations for GHD

$$\begin{aligned} \lambda\delta^\theta_{\theta_*}\Theta(x) - \beta^\theta(x,0) + \beta^\theta_{\mathrm{ini}}(x) - H^\theta(x,0) = 0 \\ -\lambda\delta^\theta_{\theta_*}\Theta(x) + H^\theta(x,T) = 0 \\ \partial_t\beta^\theta(x,t) + A_\phi^{\ \theta}(x,t)\partial_x\beta^\phi(x,t) = 0 \\ \partial_t H^\theta(x,t) + A_\phi^{\ \theta}(x,t)\partial_x H^\phi(x,t) = 0 \end{aligned}$$

- Here $\beta^\theta(x,t) := \beta^\theta[\rho_\cdot(x,t)]$ and $A_\phi^{\ \theta}(x,t) := A_\phi^{\ \theta}[\rho_\cdot(x,t)]$.

- To make use of the exact solutions of GHD, we want to rewrite the MFT equations in terms of normal modes.
- Recall that β^θ and ϵ_θ are related by $(R^{-1})_\phi^{\ \theta}\partial_{t,x}\beta^\phi = \partial_{t,x}\epsilon^\theta$. Motivated by this we define a normal mode associated to H^θ:

$$(R^{-1})_\phi^{\ \theta}\partial_{t,x}H^\phi =: \partial_{t,x}G^\theta$$

- One can show that such G^θ can exist thanks to the compatibility condition $\partial_t\partial_x G^\theta = \partial_x\partial_t G^\theta$.
- In terms of normal modes the MFT equations become

$$\begin{aligned}
\lambda\delta^\theta_{\theta_*}\delta(x) - R_\phi^{\ \theta}(x,0)\partial_x\epsilon^\phi(x,0) + \partial_x\beta^\theta_{\rm ini}(x,0) - R_\phi^{\ \theta}(x,0)\partial_x G^\phi(x,0) = 0 \\
\lambda\delta^\theta_{\theta_*}\delta(x) - R_\phi^{\ \theta}(x,T)\partial_x G^\phi(x,T) = 0 \\
\partial_t\epsilon^\theta(x,t) + v^{\mathrm{eff},\theta}(x,t)\partial_x\epsilon^\theta(x,t) = 0 \\
\partial_t G^\theta(x,t) + v^{\mathrm{eff},\theta}(x,t)\partial_x G^\theta(x,t) = 0
\end{aligned}$$

- We first want to know how $\epsilon^\theta(x,t)$ varies as a function of λ. To this end we need to compute the λ-derivative of $\epsilon_\theta(x,t)$. For simplicity let's consider a homogeneous (i.e. $\partial_x\beta^\theta_{\rm ini}(x,0) = 0$) initial condition. Eventually one obtains

$$\partial_\lambda\epsilon^\theta(x,t) = \partial_\lambda\left(\lambda(R^{-\mathrm{T}})^\theta_{\ \theta_*}(0,0)\Theta(u^\theta) - \lambda(R^{-\mathrm{T}})^\theta_{\ \theta_*}(0,T)\Theta(r^\theta)\right)$$

- This is clearly a total derivative, so one can integrate over λ and obtain

$$\epsilon^{\theta}(x,t) = \lambda (R^{-\mathrm{T}})^{\theta}_{\theta_*}(0,0)\Theta(u^{\theta}(x,t)) - \lambda (R^{-\mathrm{T}})^{\theta}_{\theta_*}(0,T)\Theta(r^{\theta}(x,t))$$

- This is one of the fundamental formulae in MFT for GHD. With this one can compute cumulants. To reiterate the cumulants are defined by $c_n := \lim_{T\to\infty}\frac{\langle J_T^n\rangle^{\mathrm{c}}}{T} = \frac{\mathrm{d}^n F(\lambda)}{\mathrm{d}\lambda^n}\Big|_{\lambda=0}$, e.g.

$$c_1 = \langle j\rangle, c_2 = \int_{\mathbb{R}} \mathrm{d}t\, \langle j(0,t) j(0,0)\rangle^c, c_3 = \int_{\mathbb{R}} \mathrm{d}t\, \langle j(0,t) j(0,0) j(0,0)\rangle^c, \cdots$$

- Note that $e^{TF(\lambda)} \asymp e^{-S[\bar\rho_\cdot(x,t),\bar H_\cdot(x,t)]}$, i.e.

$$F(\lambda) = \lim_{T\to\infty}\frac{1}{T}(\lambda J_T[(\bar\rho_\cdot(x,t)] - \mathrm{F}[\bar\rho_\cdot(x,0)])\,.$$

- When evaluating $\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda}$ with respect to the optimal configuration (i.e. the solution of the MFT eqs),

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = J_T + \lambda\frac{\mathrm{d}J_T}{\mathrm{d}\lambda} - \frac{\mathrm{d}\mathrm{F}[\rho(x,0)]}{\mathrm{d}\lambda} = J_T - \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x \left(\lambda\frac{\delta J_T}{\delta\rho_\theta(x,t)} + \frac{\delta \mathrm{F}[\rho(x,0)]}{\delta\rho_\theta(x,t)}\right)\delta\rho_\theta(x,t) = J_T$$

- Therefore for any λ we have $\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = \lim_{T\to\infty} J_T/T$. Trivially $c_1 = \langle j_{\theta_*}\rangle = \rho_{\theta_*} v^{\mathrm{eff}}_{\theta_*}$.

- Less trivial is c_2. To compute this one needs to use

$$\lim_{\lambda\to 0}\partial_\lambda \rho_\theta(x,t) = (R^{-\mathrm{T}})^{\phi}{}_{\theta_*}\chi_\phi (R^{-\mathrm{T}})^{\phi}{}_{\theta_*}(\Theta(r^\phi(x,t)) - \Theta(u^\phi(x,t)))$$

- This follows from the exact expression of $\partial_\lambda \epsilon^\theta(x,t)$ at $\lambda = 0$ as well as $(RCR^{\mathrm{T}})_{\theta\phi} = \delta_{\theta\phi}\chi_\theta$. The susceptibility χ_θ is defined by $\chi_\theta = \rho_\theta(1-n_\theta)$ with $n_\theta = 1/(1+e^{\epsilon_\theta})$. Since $u^\theta(x,t) = x - v^{\mathrm{eff},\theta}t$, $r^\theta(x,t) = x - v^{\mathrm{eff},\theta}(t-T)$, we can readily compute $c_2 = \frac{\mathrm{d}^2 F}{\mathrm{d}\lambda^2}\Big|_{\lambda\to 0}$

$$c_2 = \frac{\mathrm{d}^2 F}{\mathrm{d}\lambda^2}\Bigg|_{\lambda\to 0} = \frac{\mathrm{d}J_T}{\mathrm{d}\lambda}\Bigg|_{\lambda\to 0} = (R^{-1})_{\theta_*}{}^{\theta}\chi_\theta\,|\,v_\theta^{\mathrm{eff}}\,|\,(R^{-\mathrm{T}})^{\theta}{}_{\theta_*}$$

- The same 2nd cumulant was obtained by different methods. [Doyon and Spohn, 2017; Myers, Bhaseen, Harris, and Doyon, 2018]

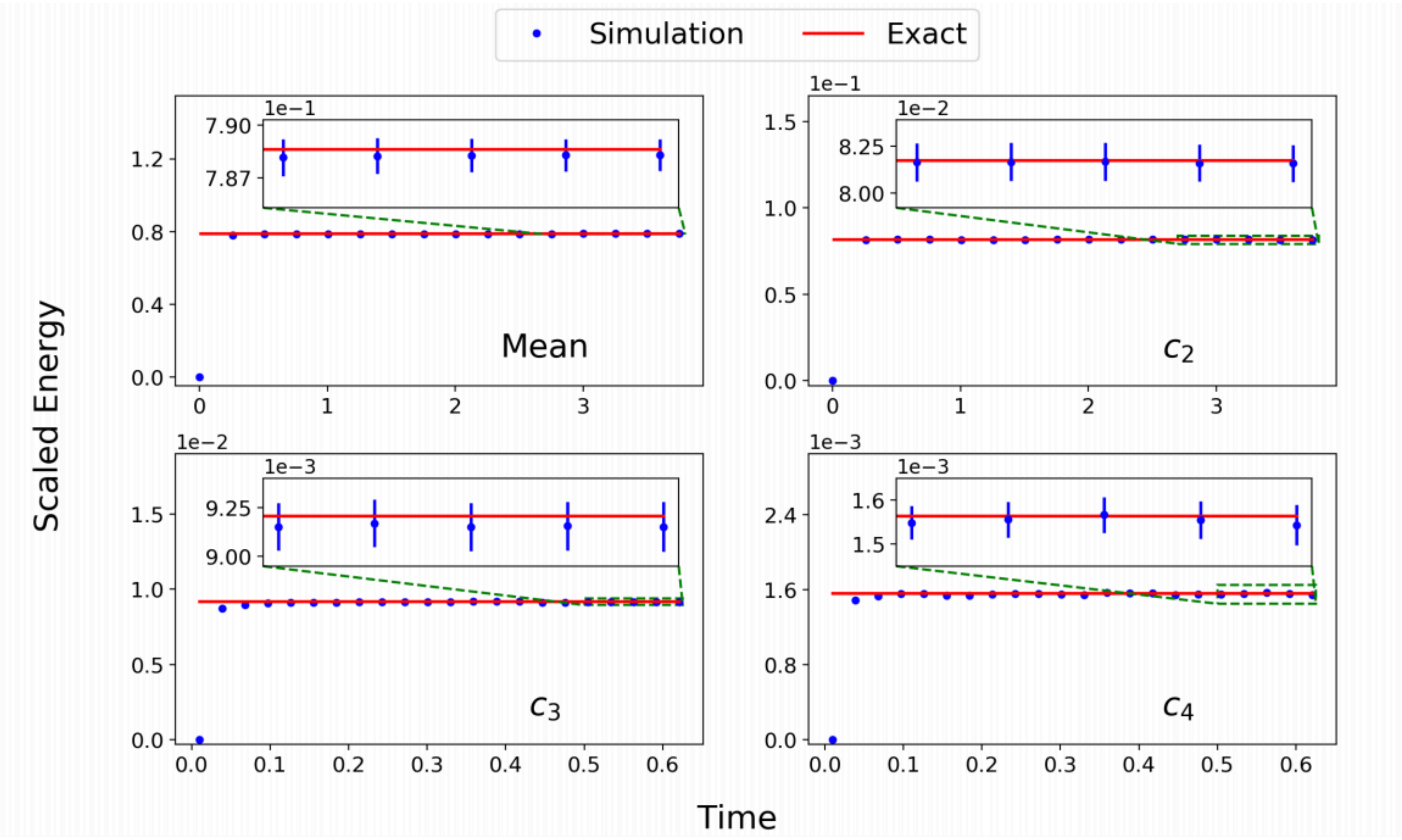
- We can go further and confirm a matching of c_3 with the previously obtained result, which is truly nontrivial. Here we just cite the final result:

$$c_3 = \chi_\phi\,|\,v_\phi\,|\,(R^{-\mathrm{T}})^{\phi}{}_{\theta_*}\Big(s_\phi(n_\phi - 2)((R^{-\mathrm{T}})^{\phi}{}_{\theta_*})^2 + 3(R^{-\mathrm{T}})^{\gamma}{}_{\phi}s_\gamma f_\gamma((R^{-\mathrm{T}})^{\gamma}{}_{\theta_*})^2\Big), \quad s_\theta := \mathrm{sgn}\, v_\theta^{\mathrm{eff}}, \quad f_\theta := 1 - n_\theta$$

- In principle one can compute arbitrary higher cumulants.

- In general cumulants for a physical charge Q_{i_*} can be obtained by replacing $(R^{-\mathrm{T}})^{\theta}_{\theta_*}$ with $(h_{i_*})^{\mathrm{dr}}_{\theta}$.

- Agreements between exact cumulants and numerics were confirmed for hard-rods. [Myers, Bhaseen, Harris, and Doyon, 2018]



- The exact cumulants also agree with the known results in free fermions obtained from the Levitov-Lesovik formula.

[Levitov and Lesovik, 1993]

- What kind of dynamics is described by $\epsilon_\theta(x, t)$?

- Morally speaking, it characterises the dynamics where rare fluctuations in the original dynamics becomes typical.

- There was another approach, the ballistic fluctuation theory (BFT), to study the current large deviation in integrable systems by biasing the dynamics

[Myers, Bhaseen, Harris, and Doyon, 2018; Doyon and Myers, 2019]

$$\langle \mathcal{O} \rangle^{(\lambda)} = \frac{\left\langle e^{\lambda \int_{\mathbb{R}} \mathrm{d}t\, j_{i_*}(0,t)} \mathcal{O} \right\rangle}{\left\langle e^{\lambda \int_{\mathbb{R}} \mathrm{d}t\, j_{i_*}(0,t)} \right\rangle}$$

- With the biased measure the SCGF is given by $\frac{\mathrm{d}}{\mathrm{d}\lambda} F(\lambda) = \langle j_{i_*}(0,0) \rangle^{(\lambda)}$.

- The biased measure turns out to be homogenous, stationary, and clustering, which in turn induces a flow of $\beta_\cdot(\lambda)$: $\langle \mathcal{O} \rangle_{\beta_\cdot(\lambda)} := \langle \mathcal{O} \rangle^{(\lambda)}$.

- $\beta_\cdot(\lambda)$ satisfy a flow equation, which can also be written down for $\epsilon_\cdot(\lambda)$

$$\partial_\lambda \epsilon_\theta(\lambda) = (R^{-\mathrm{T}})_\theta^{\ \theta_*} \mathrm{sgn}\, v_\theta^{\mathrm{eff}}$$

- Could we identify $\epsilon_\theta(\lambda)$ with $\epsilon_\theta(0,t)$?

- To see this we first notice the dynamics described by MFT at $x = 0$ is stationary except at $t = 0, T$, i.e. $\partial_t \epsilon^\theta(0,t) = 0$ for $t \in (0,T)$.

- This can be seen perturbatively in λ. Whenever we take λ-derivatives and set $\lambda = 0$, only terms with t dependence are

$$\delta(-v^{\mathrm{eff},\theta} t), \quad \delta(-v^{\mathrm{eff},\theta}(t-T)), \quad \Theta(-v^{\mathrm{eff},\theta} t), \quad \Theta(-v^{\mathrm{eff},\theta}(t-T))$$

- These are clearly *t*-independent provided $t \in (0,T)$, hence at any order λ^n, $\partial_t \epsilon^\theta(0,t) = 0$ must follow.

- Since the dynamics is stationary at $x = 0$ all the time except when $t = 0, T$, we can define a set of Lagrange multipliers $\beta^\theta(\lambda) := \beta^\theta(0,t)$ that parameterise the stationary state.

- With this we can compute the SCGF $\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \, \langle j_{\theta_*}(0,t) \rangle = \langle j_{\theta_*}(0,0) \rangle$.

- It is highly nontrivial to show $\partial_t \epsilon^\theta(0,t) = 0$ holds non-perturbatively, and in fact there is no reason to expect that it is true away from the vicinity of $\lambda = 0$.
[Doyon and Myers, 2019]

- We expect stationarity of the biased dynamics at $x = 0$ in order to ensure the clustering of correlation functions, which amounts to the existence of cumulants.

Fluctuation theorem

- We can show the fluctuation theorem for the SCGF within the framework of ballistic MFT.
- The theorem can quantify the relation between the probability of positive and negative entropy-producing events:

$$\lim_{T\to\infty}\frac{1}{T}\log\left[\frac{P(S=\sigma T)}{P(S=-\sigma T)}\right]=\sigma$$

- For an initial state such that initially two subsystems are equilibrated with respect to the same set of charges except one charge Q_{i_*} where the associated Lagrange multiplier is biased as $\beta^{i_*}=\beta_L\Theta(-x)+\beta_R\Theta(x)$, the theorem states

$$F(\lambda)=F(\beta_L-\beta_R-\lambda)$$

- To see this, we first rewrite the MFT equations as

$$\lambda\delta^i{}_{i_*}\Theta(x)+\beta^i_{\rm ini}(x)-H^{(\lambda),i}(x,0)=0$$

$$-\lambda\delta^i{}_{i_*}\Theta(x)+H^{(\lambda),i}(x,T)-\beta^{(\lambda),i}(x,T)=0$$

$$\partial_t\beta^{(\lambda),i}(x,t)+A_j{}^i[\rho^{(\lambda)}(x,t)]\partial_x\beta^{(\lambda),j}(x,t)=0$$

$$\partial_tH^{(\lambda),i}(x,t)+A_j{}^i[\rho^{(\lambda)}(x,t)]\partial_xH^{(\lambda),j}(x,t)=0$$

- We assume that there is a time-reversal operator that acts on densities and currents as

$$\mathcal{T}\hat{q}_i(x,t)\mathcal{T}^{-1} := (-1)^i \hat{q}_i(x, T-t), \quad \mathcal{T}\hat{j}_i(x,t)\mathcal{T}^{-1} := (-1)^{i+1}\hat{j}_i(x, T-t)$$

- This implies that the following identities hold

$$\rho_i(x,t) = (-1)^i \tilde{\rho}_i(x, T-t), \quad j_i(x,t) = (-1)^{i+1}\tilde{j}_i(x, T-t)$$

where ρ_i and $\tilde{\rho}_i$ are conjugate to β^i and $\tilde{\beta}^i = (-1)^i\beta^i$, respectively.

- Now let us apply MFT to evaluate the SCGF in a state paramterised by $\tilde{\beta}^i$ with $\beta^{i_*}(x) = \beta_L\Theta(-x) + \beta_R\Theta(x)$. We assume that the state is time-reversal invariant, i.e. $\tilde{\beta}^i = \beta^i$, which implies $\langle e^{\lambda J_T}\rangle_{\beta^i} = \langle e^{\lambda J_T}\rangle_{\tilde{\beta}^i}$.

- To establish the theorem, we replace λ with $\tilde{\lambda} = \beta_L - \beta_R - \lambda$. The resulting MFT equations are

$$\tilde{\lambda}\delta^i{}_{i_*}\Theta(x) + \beta^i_{\text{ini}}(x) - \tilde{H}^{(\tilde{\lambda}),i}(x,0) = 0$$

$$-\tilde{\lambda}\delta^i{}_{i_*}\Theta(x) + \tilde{H}^{(\tilde{\lambda}),i}(x,T) - \tilde{\beta}^{(\lambda),i}(x,T) = 0$$

$$\partial_t\tilde{\beta}^{(\tilde{\lambda}),i}(x,t) + \tilde{A}_j{}^i[\tilde{\rho}^{(\tilde{\lambda})}(x,t)]\partial_x\tilde{\beta}^{(\tilde{\lambda}),j}(x,t) = 0$$

$$\partial_t\tilde{H}^{(\tilde{\lambda}),i}(x,t) + \tilde{A}_j{}^i[\tilde{\rho}^{(\tilde{\lambda})}(x,t)]\partial_x\tilde{H}^{(\tilde{\lambda}),j}(x,t) = 0.$$

- Now, let us define the time-reversed fields

$$\rho_{T,i}(x,t) := (-1)^i \tilde{\rho}_i(x, T-t), \quad j_{T,i}(x,t) := (-1)^{i+1} \tilde{j}_i(x, T-t)$$
$$\beta_T^i(x,t) := (-1)^i \tilde{\beta}^i(x, T-t), \quad H_T^i(x,t) := (-1)^{i+1} \tilde{H}^i(x, T-t)$$

- It is important to note that the time-reversed fields coincide with the original fields: $\rho_{T,i}(x,t) = \rho_i(x,t), j_{T,i}(x,t) = j_i(x,t)$.
- Therefore we have

$$\partial_t \beta_T^{(\tilde{\lambda}),i}(x,t) + A_j{}^i[\rho_T^{(\tilde{\lambda})}(x,t)]\partial_x \beta_T^{(\tilde{\lambda}),j}(x,t) = 0$$
$$\partial_t H_T^{(\tilde{\lambda}),i}(x,t) + A_j{}^i[\rho_T^{(\tilde{\lambda})}(x,t)]\partial_x H_T^{(\tilde{\lambda}),j}(x,t) = 0$$

- Next we turn to the boundary conditions. With a replacement $H_T^{(\tilde{\lambda}),i}(x,t) \mapsto H_T^{(\tilde{\lambda}),i}(x,t) + \beta_L \delta^i{}_{i_*}$, one can rewrite the first boundary condition as $-\lambda \delta^i{}_{i_*}\Theta(x) + H_T^{(\tilde{\lambda}),i}(x,T) = 0$. Likewise another boundary condition also becomes $\lambda \delta^i{}_{i_*}\Theta(x) + \beta^i_{\text{ini}}(x) - H_T^{(\tilde{\lambda}),i}(x,0) - \beta_T^{(\tilde{\lambda}),i}(x,0) = 0$.
- Combining everything the MFT equations parametrised with $\tilde{\beta}$ and the counting parameter λ now read

$$-\lambda \delta^i{}_{i_*}\Theta(x) + H_T^{(\tilde{\lambda}),i}(x,T) = 0$$
$$\lambda \delta^i{}_{i_*}\Theta(x) + \beta^i_{\text{ini}}(x) - H_T^{(\tilde{\lambda}),i}(x,0) - \beta_T^{(\tilde{\lambda}),i}(x,0) = 0$$
$$\partial_t \beta_T^{(\tilde{\lambda}),i}(x,t) + A_j{}^i[\rho_T^{(\tilde{\lambda})}(x,t)]\partial_x \beta_T^{(\tilde{\lambda}),j}(x,t) = 0$$
$$\partial_t H_T^{(\tilde{\lambda}),i}(x,t) + A_j{}^i[\rho_T^{(\tilde{\lambda})}(x,t)]\partial_x H_T^{(\tilde{\lambda}),j}(x,t) = 0.$$

- This is precisely the MFT equations $\rho_i^{(\lambda)}(x,t)$ satisfies, which allows us to identify $\rho_i^{(\lambda)}(x,t)$ and $\rho_{T,i}^{(\tilde{\lambda})}(x,t) = (-1)^i \tilde{\rho}_i^{(\tilde{\lambda})}(x, T-t)$.

- This suggests

$$
\begin{aligned}
\frac{\mathrm{d}F(\tilde{\lambda})}{\mathrm{d}\lambda} = -N^{(\tilde{\lambda})} &= -\int_0^\infty \mathrm{d}x \left(\tilde{\rho}_{i*}^{(\tilde{\lambda})}(x,T) - \tilde{\rho}_{i*}^{(\tilde{\lambda})}(x,0)\right) \\
&= \int_0^\infty \mathrm{d}x \left(\rho_{i*}^{(\lambda)}(x,T) - \rho_{i*}^{(\lambda)}(x,0)\right) \\
&= N^{(\lambda)} \\
&= \frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda}
\end{aligned}
$$

which, after integrating over λ, gives $F(\tilde{\lambda}) = F(\lambda)$.

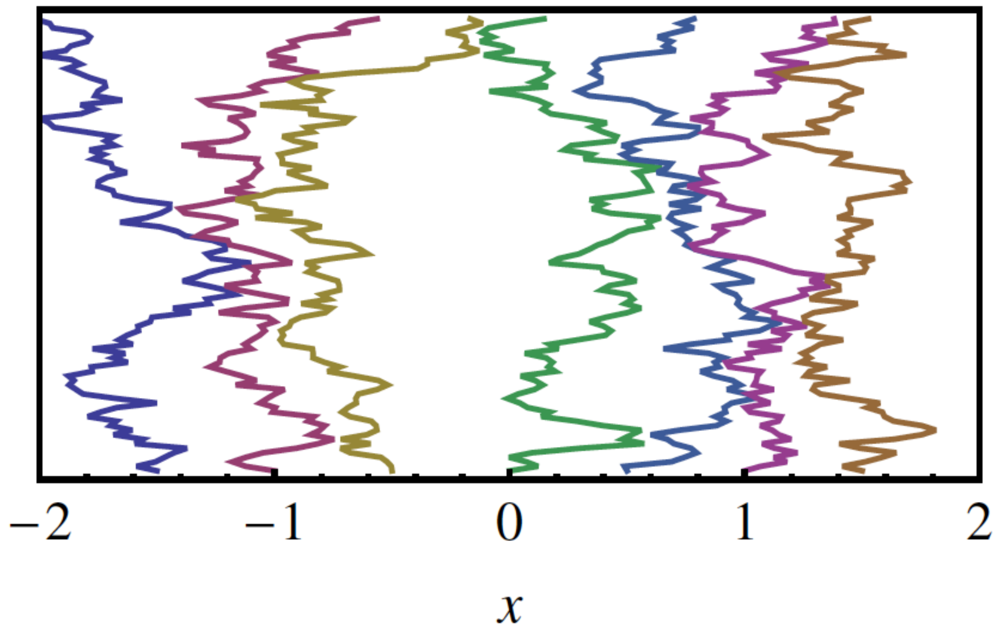
- Loosely speaking, in MFT the time-forward dynamics with the counting parameter λ and the time-backward dynamics with the counting parameter $\tilde{\lambda}$ follow the same dynamics, which amounts to the symmetry of the SCGF.

Conclusion and Outlook

- Large deviation contains far more information about transport than just the average current.
- MFT is a powerful approach that provides a universal framework to study large deviation of both diffusive and hyperbolic systems.
- For generic hyperbolic systems MFT equations could be difficult to handle, but for integrable systems the machinery of GHD allows us to understand a great deal about the structures of MFT equations as well as large deviation.
- The Gallavotti-Cohen type fluctuation theorem can be derive within the framework of MFT for ballistic transport.
- MFT for non-ballistic GHD? Bare transport quantities in GHD?
- Any connection to known SCGFs in integrable impurity systems? Entanglement entropy?
- MFT in its present form does not work when the effect of quantum fluctuations becomes strong. For instance the $\mathrm{Prob}(J_T = Tj)$ decays anomalously in time $\mathrm{Prob}(J_T = Tj) \sim e^{-t^2 I(j)}$ for the spin transport in the XX chain starting from the initial domain-wall condition. Can we extend MFT to incorporate strong quantum fluctuations? [Moriya, Nagao, and Sasamoto, 2019]
- Can we derive MFT for GHD microscopically?

MFT for diffusive systems: an example

- It is in general hard to solve the MFT equations exactly except noninteracting cases.
- An example of solvable cases is brownian particles subject to hard-core repulsion, which corresponds to the case where $\mathfrak{D}(\rho) = 1, \sigma(\rho) = 2\rho$.
- The large deviation of the position of the tagged particle $X_T[\rho]$, which starts at the origin at $t = 0$, was studied in [Krapivski, Mallick, and Sadhu, 2015]. This was made possible by the following identity



$$\int_0^{X_T[\rho]} \mathrm{d}x\, \rho(x,T) = \int_0^{\infty} \mathrm{d}x\, (\rho(x,T) - \rho(x,0)) = J_T$$

- The object of interest now is $\langle e^{\lambda X_T} \rangle$. Upon performing the noise averaging, one arrives at the following

$$\langle e^{\lambda X_T} \rangle \asymp \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathscr{D}\rho(x,t)\mathscr{D}H(x,t)e^{-S[\rho(x,t),H(x,t)]}$$

$$S[\rho(x,t),H(x,t)] := -\lambda X_T + \mathrm{F}[\rho(x,0)] + \int_{(x,t)\in\mathbb{R}\times[0,T]} \mathrm{d}t\mathrm{d}x \left(H\partial_t\rho - \frac{\sigma(\rho)}{2}(\partial_x H)^2 - D(\rho)\partial_x\rho\partial_x H \right)$$

- A priori the scaling of cumulants is nontrivial. A simple observation however suggests that $S_T[\rho, H]$ grows with $\sqrt{T}$. We therefore expect that cumulants also scale with $\sqrt{T}$ rather than T.

[Krapivski, Mallick, Sadhu, 2015]

- In the annealing case where the initial condition fluctuates, one expects $\langle e^{\lambda X_T} \rangle \asymp e^{-S[\rho_{\text{opt}}(x,t),H_{\text{opt}}(x,t)]}$, where the optimised configurations satisfy a set of Hamilton equations with boundary conditions, which we call MFT equations

$$\frac{\lambda}{\rho(Y,T)}\Theta(x) - \beta(x) + \beta_{\text{ini}}(x) - H(x,0) = 0$$

$$-\frac{\lambda}{\rho(Y,T)}\Theta(x) + H(x,T) = 0$$

$$\partial_t H(x,t) + \mathfrak{D}[\rho(x,t)]\partial_x^2 H(x,t) + \frac{\sigma'(\rho(x,t))}{2}(\partial_x H(x,t))^2 = 0$$

$$\partial_t \rho(x,t) - \partial_x \left(\mathfrak{D}[\rho(x,t)]\partial_x \rho(x,t)\right) + \partial_x \left(\mathfrak{D}[\rho(x,t)]\partial_x H(x,t)\right) = 0$$

- Here $Y := X_T[\rho]$ was introduced. In the case of brownian particles with exclusion interaction, i.e. $\mathfrak{D}(\rho) = 1, \sigma(\rho) = 2\rho$, one can solve exploit the structure of MFT equations and compute exact cumulants. [Krapivski, Mallick, Sadhu, 2015]
- In MFT the task of computing the SCGF boils down to the optimisation problem of the action.
- In the presence of interaction, e.g. SSEP ($\mathfrak{D}(\rho) = 1, \sigma(\rho) = 2\rho(1-\rho)$), brute force computations seem not possible.
- Recently inverse scattering method was employed to solve MFT-like equations, which emerge when studying the short-time large deviation behaviour of the KPZ equation. [Krajenbrink and Le Doussal, 2021]