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# Supplementary Lecture Note
# Optical Coherence and Higher Order Correlations

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## Higher order coherence, Hanbury Brown-Twiss effect and photon statistics

First order coherence is described in terms of two-time amplitude-amplitude correlation function which reduces to intensity when the two times are equal. In this note, we will discuss second order (in terms of intensity) coherence and related concepts. We are dealing with complex random variables. Product of two random variables is also a random variable. Suppose, the random variable $\hat{V}(t,\tau)$ is a quadratic product of random variables $\hat{A}(t)$ at two times, $\hat{V}(t,\tau)=\hat{A}(t)\hat{A}(t+\tau)$, which is complex for any values of the two times. Then in order to relate to an observable quantity, we need to seek ensemble average of the quantity $\hat{V}(t,\tau)^*\hat{V}(t,\tau)$ or $\hat{V}(t,\tau)\hat{V}(t,\tau)^*$. Note that in quantum mechanics, all these variables are operators and hence ordering of them is important. According to stationarity, this average is independent of absolute time *t*, but it is a function of time difference or time delay $\tau$ only.

Second order coherence was experimentally discovered by Hanbury Brown and R. Twiss in early 50s. Hanbury Brown was an astrophysicist and R. Twiss was a mathematician. They built up a new kind of interferometer based on second order coherence and measured more accurately the diameter of binary stars. Here we describe their experiment.

In the so-called intensity-intensity (or sometimes also called photon-photon) correlation or second order coherence, the two light beams emerging out of beam splitter or reflected from two mirrors (in case of stellar interferometer ) are first detected at two photo-detectors or photo-tubes and the current output from the detectors are merged together in a correlator or integrator circuit. The dc parts of the two currents are cut-off and only the ac parts are integrated. The ac current output is proportional to the intensity fluctuation in the light beam. So, for $\tau = 0$, the correlator measures the correlation between two intensity fluctuations.

Classically, this correlation can be expressed as

$$C^{(2)}(\tau=0)=\langle (I-\langle I\rangle)^2\rangle=\langle I^2\rangle-\langle I\rangle^2 \tag{2.1}$$

$$C^{(2)}(\tau)=\langle \hat{A}^*(t)\hat{A}^*(t+\tau)\hat{A}(t+\tau)\hat{A}(t)\rangle-\langle \hat{A}^*(t)\hat{A}(t)\rangle\langle \hat{A}^*(t+\tau)\hat{A}(t+\tau)\rangle$$

Or, using the staionarity condition, we have

$$C^{(2)}=\langle \hat{A}^*(0)\hat{A}^*(\tau)\hat{A}(\tau)\hat{A}(0)\rangle-\langle \hat{A}^*(0)\hat{A}(0)\rangle^2$$

In quantum theory of coherence, $\hat{A}^*$ and $\hat{A}$ are quantum mechanical operators, and $\hat{A}^*$ has to be replaced by $A^\dagger$ and the order (normal order) of the operators should remain same as shown in the above equations. However, in classical theory of coherence, $\hat{A}^*$ and $\hat{A}$ are classical random variables, so the order of these variables does not matter. Note that in quantum theory, $A^\dagger$ and $\hat{A}$ are field operators, for a single-mode field $A^\dagger$ and $\hat{A}$ are proportional to the photon creation and annihilation operators, respectively. One defines a normalized second order coherence function by

$$g^{(2)}(\tau)=\frac{\langle \hat{A}^*(0)\hat{A}^*(\tau)\hat{A}(\tau)\hat{A}(0)\rangle}{\langle \hat{A}^*(0)\hat{A}(0)\rangle\langle \hat{A}^*(\tau)\hat{A}(\tau)\rangle}=\frac{\langle \hat{A}^*(0)\hat{A}^*(\tau)\hat{A}(\tau)\hat{A}(0)\rangle}{\langle \hat{A}^*(0)\hat{A}(0)\rangle^2} \tag{2.2}$$

Classically, $g^{(2)}(\tau)$ can not become less than 1. It is easy to understand if we look at Eq.(2.1). $g^{(2)}(\tau=0)<1$ means $C^{(2)}(0)<0$ , but since $C^{(2)}(0)$ is average value of the square of the quantity $I-\langle I\rangle$ , it can not be negative. But, $g^{(2)}(\tau)$ of resonance fluorescence light (light emitted by laser driven two-level atoms) was experimentally found to be less than unity, violating classical concept of coherence. Light for which $g^{(2)}(\tau)<1$ is called anti-bunched light, because two photons arriving at the same time is minimum. Anti-bunching of photons is is purely non-classical property of light having no classical analogue.

## Quantum theory of coherence, coherent state, and photon statistics

As we have already mentioned, quantum theory of coherence is based on the correlation functions of quantized field operators at different space-time points. A field is called coherent if all the normalized coherence functions of all orders are unity. Usually, first and second order coherence functions are of most importance. It follows from Eq.(2.2), that when $g^{(2)}(\tau)=1$ the un-normalized function $G^{(2)}(\tau)=\langle \ddot{A}^{\dagger}(0)\ddot{A}^{\dagger}(\tau)\hat{A}(\tau)\hat{A}(0)\rangle$ is separable in the form

$$G^{(2)}(\tau)=\langle \ddot{A}^{\dagger}(0)\ddot{A}^{\dagger}(\tau)\hat{A}(\tau)\hat{A}(0)\rangle=\langle \ddot{A}^{\dagger}(0)\hat{A}(0)\rangle\langle \ddot{A}^{\dagger}(\tau)\hat{A}(\tau)\rangle \qquad (2.3)$$

So, a field state is called coherent if the un-normalized higher order coherence functions (in particular, second order function) or correlation functions are separable in lower order (first order) functions (Glauber, 1963). Apart from this separability, a coherent state has the properties:

1. It is a minimum uncertainty state meaning that the product of the uncertainty (or fluctuation) in two canonical conjugate variables (such as, generalized position and momentum variables) is minimum (for instance, in case of position and momentum variables, it is equal to $\hbar/2$ ).
2. It is an eigenstate of photon annihilation operator. Since annihilation operator is a non-hermitian operator, the eigenvalue is complex.
3. The photon distribution for a coherent state is Poissonian.

## Photon distribution

$G^{(2)}(0)$ gives information about the photon distribution of the field. Let $\hat{a}\equiv\hat{A}$ and $a^{\dagger}\equiv A^{\dagger}$ denote photon annihilation and creation operator, respectively, of a single-mode field.

Any field state can be expanded in the basis of number states. A number $|n>$ is a state of photon number *n*. It is an eigenstate of the number operator $\hat{n}=a^{\dagger}\hat{a}$ with eigenvalue n. We have the relations

$$\hat{n}|n>=n|n>, \quad \hat{a}|n>=\sqrt{n}|n-1> \quad a^{\dagger}|n>=\sqrt{n+1}|n+1>$$

$$[\hat{a},\hat{n}]=\hat{a}, \qquad [\hat{a}^{\dagger},\hat{n}]=-\hat{a}^{\dagger}$$

A coherent state $|\alpha>$ can be expressed in number state basis in the following form

$$|\alpha>=\exp[-|\alpha|^2/2]\Sigma_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n> \qquad (2.5)$$

The photon distribution p(n) for a coherent state is given by

$$p(n)=|<n|\alpha>|^2=\exp[-|\alpha|^2]|\frac{\alpha|^{2n}}{n!} \qquad (2.6)$$

The average photon number $\langle n\rangle$ in the coherent state is given by

$$\langle n\rangle=\langle\alpha|\hat{n}|\alpha\rangle=|\alpha|^2 \qquad (2.7)$$

Substituting Eq.(2.7) in Eq.(2.6) we have

$$p(n)=\exp[-\langle n\rangle]\frac{\langle n\rangle^n}{n!} \qquad (2.8)$$

which is in the form of Poisson distribution. It is easy to prove that for this distribution $g^2(0)=1$ . In fact, for a coherent state, the normalized second order coherence function is unity for any time delay. Laser is a close realization of a coherent state. A coherent state is said to be a quantum state with minimum classical behavior or a classical state with maximum quantum behavior that is possible for a classical state.

The photon number variance $\Delta n$ is defined by

$$(\Delta n)^2=\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2 \qquad (2.9)$$

It is easy to prove that (do it yourself) that from $g^2(0)=1$ and using operator algebra, for a coherent state one obtains

$$(\Delta n)^2 = \langle n \rangle \tag{2.10}$$

For a thermal state, or bunched or chaotic light ( $g^2(\tau) \geq 1$ ) .