KAWS '22

Tutorial (S-matrix)

Instructor: Prof. Shiraz Minwalla

– Suman

Take away message:

- First Shiraz motivated few "conjectures" and "CRG". ($|T| \leq s^2$)
- Next he gave a proof of CRG using bound on chaos.
- After that he started checking which interaction satisfies this CRG? That needs a full classification of S-matrix. It was done with gradation in momenta.
- One can do another classification at the Lagrangian level with derivative gradation, which has one to one correspondance with S-matrix list.
- These two classifications are indeed isomorphic.
- Result: If any S-matrix violates 'CRG' then the corresponding Lagrangian is disallowed. For $D \leqslant 6$, only Einstein term survives!

Plan:

1. Classification of Lagrangians (Plythestic Counting)
2. Enumeration of Bare Module
3. Time Ordering in Chaos Bound

Classification of Lagrangians (Plythestic Counting)

A detour to SUSY QM :

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + i\bar{\psi}\dot{\psi} - \bar{\psi}\psi .$$

$$H = a_B^\dagger a_B + a_F^\dagger a_F$$

Vac : $H|0\rangle = Q|0\rangle = \bar{Q}|0\rangle = 0$

States : $|x_B, n\rangle \equiv (a_B^\dagger)^n |0\rangle$ $\qquad |x_F, m\rangle \equiv (a_B^\dagger)^m a_F^\dagger |0\rangle$

Partition fn : $Z(x) = \mathrm{Tr}\, x^H = \underbrace{1}_{\text{vac}} + \underbrace{(x + x^2 + x^3 + \cdots)}_{\text{bosons}} + \underbrace{(x + x^2 + x^3 + \cdots)}_{\text{fermions}}$

$$= \frac{1+x}{1-x} \qquad x = e^{-\beta}$$

A very well known result.

A trick to calculate the same :

Note : This trick is exclusive to free theory.

* The spectrum is generated by action of all the creation operators on the vac, $|0\rangle$.

* These operators commute and act independently.

* It is convenient to compute the partition fn over "single letters" i.e. on the state with a single particle.

Named as "single letter partition fn" $z(x)$

* Useful to compute it for bosons and fermions seperately. (denoted by $z_B(x)$ and $z_F(x)$)

* In our example only one bosonic and one fermionic creation operator.

$$z(x) = z_B(x) + z_F(x)$$

$$z_B(x) = \mathrm{Tr}_{\text{bosonic letters}}\, x^H = x$$

$$z_F(x) = \mathrm{Tr}_{\text{fermionic letters}}\, x^H = x$$

* The partition fn. over bosonic multiparticle states is obtained by,

$$z_B(x) = x \rightarrow \underline{\frac{1}{1-x}} \equiv Z_B(x) \; ; \qquad z_F(x) = x \rightarrow \underline{(1+x)} \equiv Z_F(x)$$

Full partition fn. $Z = Z_B Z_F = \dfrac{1+x}{1-x}$ (matches with previous calculation)

$(a_B^\dagger)^n (a_F^\dagger)^m$ $\qquad \underline{m = 0, 1} \qquad \underline{n = 0, 1, 2, \ldots}$

Z_B correspond to sum over all the states obtained by acting with any no. of bosonic creation operators.

Z_F correspond to state with no fermion and a single fermion.

* If theory has multiple types of creation operators, single letter partition fn. would have more terms.

* To compute the full partition fn. replace the single letter to multiparticle and then take the product.

Called as "Plethystic exponentiation" i.e. PE.

Definition:

$$PE[f(x_i)] = \exp\Big[\sum_{n=1}^{\infty} \frac{1}{n} \underbrace{f(x_i^n)}_{\hookrightarrow\, x^n}\Big] \; ; \; \widetilde{PE}[f(x_i)] = \exp\Big[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(x_i^n)\Big]$$

Then,

$$\boxed{Z_B(x) = PE[z_B(x)] \,, \quad Z_F(x,f) = \widetilde{PE}[z_F(x,f)] \,, \quad Z = Z_B Z_F}$$

Check: $Z_B(x) = PE[x] = \exp\Big[\sum_{n=1}^{\infty} \frac{x^n}{n}\Big] = \exp[-\log(1-x)] = \frac{1}{1-x}$

$Z_F(x) = \widetilde{PE}[x] = \exp\Big[-\sum_{n=1}^{\infty} \frac{(-x)^n}{n}\Big] = \exp[\log(1+x)] = 1+x.$

$$\Rightarrow \qquad Z = Z_B Z_F = \frac{1+x}{1-x}$$

$z_B(x) = x$

$z_F(x) = x$

Advantages:

* One can calculate it individually for each fields, then multiply it later to get the full partition fn.

Use this trick for Global Symmetry:

Consider, $L = \frac{1}{2}|\dot{x}_i|^2 - \frac{1}{2}|x_i|^2 + i\bar{\psi}_i\dot{\psi}_i - \bar{\psi}_i\psi_i$ $\quad i = 1 \ldots N$

Symmetry: $SO(2N)$.

Corresponding interacting theory (preserving SUSY)

$$L = \frac{1}{2}|\dot{x}_i|^2 - \frac{1}{2}\left|\frac{\partial W}{\partial x_i}\right|^2 + i\bar{\psi}_i\dot{\psi}_i - \frac{\partial^2 W}{\partial x_i \partial x_j}\bar{\psi}_i\psi_j$$

* free theory : $W = |x_i|^2$.

Using our new trick, the free theory has,

2N bosonic creation operators $a^\dagger_{B,i}$

and 2N fermionic creation operators $a^\dagger_{F,i}$

This means,

$$z_B(x) = 2N x \qquad\qquad z_F(x) = 2N x$$

Note,
$z_B(x^n) = 2N\, x^n$

$$\Rightarrow \quad PE[2Nx] = \exp\left[\sum_{n=1}^{\infty} \frac{2N x^n}{n}\right] = \frac{1}{(1-x)^{2N}}$$

$$\widetilde{PE}[2Nx] = \exp\left[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, 2N x^n\right] = (1+x)^{2N}$$

$$\Rightarrow \quad Z(x) = \left(\frac{1+x}{1-x}\right)^{2N}$$

This matches with the expected answer.

Sometimes it is convenient to keep track of chemical potentials other than the fugacity.

Now we will calculate,

$$Z(x, a_i) = \mathrm{Tr}\; x^H a_i^{J_i} \qquad i = 1, \ldots, N$$

J_i : $SO(2N)$ Cartans.

* Boson transform in fundamental repⁿ of SO(2N).

$$\Rightarrow \quad z_B(x, a_i) = x\left(a_1 + \frac{1}{a_1} + \cdots + a_N + \frac{1}{a_N}\right) = x\,\chi_{\text{fund}}(a_i)$$

$\chi_{\text{fund}}(a_i)$: character of the fundamental repⁿ.

$$Z_B(x) = \text{PE}\left[z_B(x, a_i)\right] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} z_B(x^n, a_i^n)\right]$$

$$= \exp\left[\sum_{n=1}^{\infty} \frac{x^n}{n}\left(a_1^n + \frac{1}{a_1^n} + \cdots + a_N^n + \frac{1}{a_N^n}\right)\right]$$

$$= \exp\left[-\log(1 - x a_1) - \log\left(1 - \frac{x}{a_1}\right) - \cdots - \log(1 - x a_N) - \log(1 - x/a_N)\right]$$

$$\boxed{Z_B(x) = \prod_{i=1}^{N} \frac{1}{(1 - x a_i)\left(1 - \frac{x}{a_i}\right)}}$$

Similarly,

and $z_F(x,a_i) = x\left(a_1 + \frac{1}{a_1} + \cdots + a_N + \frac{1}{a_N}\right)$

$$Z_F(x, a_i) = \widetilde{PE}[z_F(x,a_i)] = \prod_{i=1}^{N} (1 + x a_i)\left(1 + \frac{x}{a_i}\right)$$

$$\Rightarrow Z(x,a_i) = \prod_{i=1}^{N} \frac{(1+x a_i)(1+x/a_i)}{(1-x a_i)(1-x/a_i)}$$

Check : if $a_i = 1 \ \forall i$ then, $Z(x, a_i = 1) = \left(\frac{1+x}{1-x}\right)^{2N}$

Advantage of keeping track of Cartan charges: We can compute partition fn over states with a given repn of global symmetry. This uses orthogonality

$$\frac{1}{|W|} \oint \prod_{i=1}^{N} \frac{da_i}{2\pi i\, a_i} \Delta(a_i)\ \chi_R(a_i)\ \chi_{R'}(a_i) = \delta_{RR'}$$

N : rank of the group.

$|W|$: cardinality of the associated Weyl group.

$\Delta(a_i)$: Van-der-Monde determinant.

Projection to a particular representation:

So, projection onto states with given repⁿ,

$$Z(x,a_i)\Big|_R = \frac{1}{|W|}\oint \prod_{i=1}^{N} \frac{da_i}{2\pi i\, a_i}\, \Delta(a_i)\, Z(x,a_i)\, \chi_R(a_i)$$

Particularly to project to the singlet, we have to use the following reduced formula, with $\chi_{\text{singlet}} = 1$

$$Z(x,a_i)\Big|_R = \frac{1}{|W|}\oint \prod_{i=1}^{N} \frac{da_i}{2\pi i\, a_i}\, \Delta(a_i)\, Z(x,a_i)$$

Reference: A very good review for this plythestic counting used to calculate SUSY Index is,

2006.13630 , A. Gadde

Implementation of this trick:

Our goal: We will count no. of independent local lagrangians
- quartic in ϕ
- graded by no. of derivatives.
- upto equation of motion. $\partial^2\phi = 0$.

The space of such operators for scalars is spanned by,

$$\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\phi \quad , \quad \text{for } l = 0, 1, \ldots \quad \text{subjected to } \partial_\mu \partial^\mu \phi = 0.$$

This has SO(D) symmetry with only bosons. So, we will directly write the answer from previous study that,

$$i_s(x, a_i) = \mathrm{Tr}\, x^\Delta a_i^{J_i} \overset{?}{=} \frac{1}{\prod_{i=1}^{D/2} (1 - x a_i)(1 - x/a_i)} = \mathbb{D}(x, a_i)$$

This answer is not exactly correct. Two reasons,

- Previous calculation was done for SO(2N) symmetry i.e. when D is even. For odd D formula changes a bit but not significantly.

* Such operators are generated by acting with an arbitrary no. of derivatives on $\partial^2\phi$, so their partition fn. is,

$$x^2 D(x, a_i) \qquad \underline{\partial_{\mu_l} \cdots \partial_{\mu_3}\partial_{\mu_2}\partial_{\mu_1}\partial^2\phi}$$

We need to subtract this part and we get,

$$i_s(x, a_i) = (1-x^2)\, D(x, a_i)$$

Till now we kept track of Cartans of SO(D) bcz. we will eventually need to project polynomials built out of scalar letters to the space of SO(D) singlets below.

Now we will construct all possible lagrangians and then we will subtract out the total derivatives.

Multiletter partition function :

$$\sum_{k=1}^{\infty} t^k\, i_s^{(k)}(x, a) = \exp\left[\sum_{n=1}^{\infty} \frac{t^n}{n}\, i_s(x^n, a_i^n)\right]$$

$i_s^{(k)}(x, a_i)$: k-letter partition fn. Note, $i_s^{(1)} = i_s$

4-letter partition fn :

$$i_s^{(4)}(x,a) = \frac{1}{24}\left[i_s^4(x,a) + 6 i_s^2(x,a)\, i_s(x^2,a^2) + 3 i_s^2(x^2,a^2) + 8\, i_s(x,a)\, i_s(x^3,a^3) + 6 i_s(x^4,a^4) \right]$$

This includes operators which are total derivative and they should be excluded.

Total derivative exclusion:

$$\boxed{\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}}\left(\quad\right)$$

In CFT language, this means we only want to calculate primary scalar quartic operators.

Assuming $\nexists$ null states, if the character of conformal primary is $P(x,a)$ then the character over its entire multiplet is given by $P(x,a)\,\mathbb{D}(x,a) = i_s^{(4)}$

Since $\mathbb{D}(x,a)$ encodes the contribution coming from the tower of derivatives.

So, the final answer i.e. polynomials of $\partial^l \varphi$, modulo total derivatives is given by

$$i_s^{(4)}(x,a) / \mathbb{D}(x,a)$$

This gives counts for all indexed structures. To project onto $SO(D)$ scalars / singlets we need to integrate over the Cartan chemical potentials.

$$I_s^D(x) := \oint \prod_{i=1}^{D/2} da_i \, \Delta(a_i) \, \frac{i_s^{(4)}(x,a_i)}{\mathcal{D}(x,a_i)}$$

This integral is hard to do analytically but numerical integration easily generalizes to

dimension	Scalar Partition fn
$D \geqslant 4$	$\mathcal{Z}$
$D = 3$	$(1+x^9)\,\mathcal{Z}$
$D = 2$	$(1-x^6)\,\mathcal{Z}$

$$\mathcal{Z} \equiv \frac{1}{(1-x^4)(1-x^6)}$$

This counts both parity even and odd.

The single letter partition fn for scalar can be thought of as,

$$i_s(x,a_i) = 1 + x\,\chi_{\square} + x^2\,\chi_{\square\square} + x^3\,\chi_{\square\square\square} + \cdots$$

$\chi_{\square}$: represent spin 1 of $SO(D)$; $\chi_{\square\square}$: spin 2; $\chi_{\square\square\square}$: Spin 3

$\chi_R(a_i)$ is the character of repn. R of $SO(D)$.

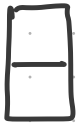
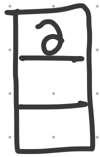
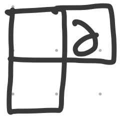
Plythestic for Photon:

Basic Structure : act derivatives on $F_{\mu\nu}$

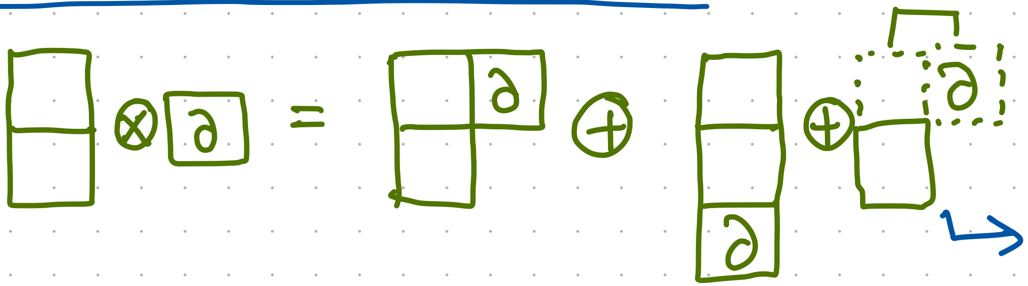
subject to $\partial^{\mu} F_{\mu\nu} = 0$. Equation of motion.

$\partial_{[\rho} F_{\mu\nu]} = 0$ Bianchi Identity.

and $(\partial^{\mu}\partial_{\mu}) F_{\rho\lambda} = 0$ photon has null momenta

$F_{\mu\nu}$:  , $\partial_{[\rho} F_{\mu\nu]}$:  $\partial_{\rho} F_{\mu\nu}$:  , ...

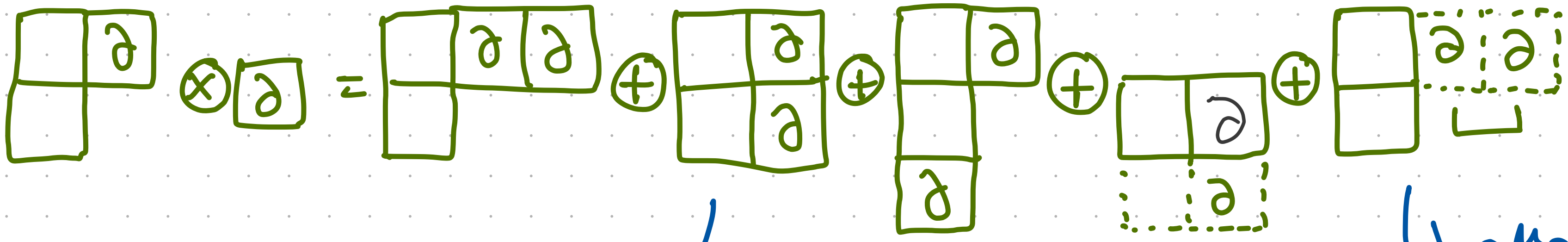
Take one derivative on $F_{\mu\nu}$:



↳ Equation of motion $= 0$

↳ This is the Bianchi identity $= 0$

Action of another derivative :

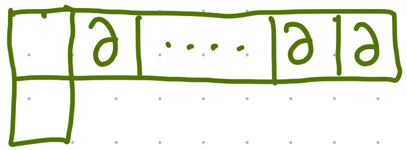


$\hookrightarrow \partial^\mu \partial_\mu (\ ) = 0$

Bianchi $\quad \partial^\mu F_{\mu\nu} = 0$

$= 0 \because [\partial, \partial] = 0$

It can be shown that the only photon structure that survives are,



Another reason: there can be only one derivative antisymmetric with polarization, others have to be sym. since $[\partial, \partial] = 0$ and we get,

$$i_V(x, a) = x\, \chi_{(1,1)} + x^2 \chi_{(2,1)} + x^3 \chi_{(3,1)} + \cdots$$

First term is $F_{\mu\nu}$ has one derivative $\Rightarrow x$

2nd term is $(\partial_\mu F_{\rho\lambda} + \partial_\rho F_{\mu\lambda} + \partial_\lambda F_{\rho\mu})$ has 2 derivatives $\Rightarrow x^2$

$\vdots$

all of them has coefficient 1 since there are only one structure at each derivative order.

✱ These are precisely the rep$^{\underline{n}}$s of vector spherical harmonics.

Now we will do this $i_V(x,a)$ sum, $\boxed{\chi_{R_1\otimes R_2} = \chi_{R_1}\cdot\chi_{R_2}}$

Till now we know,

$$i_S(x,a_i) = 1 + x\,\chi_{\ydiagram{1}} + x^2\,\chi_{\ydiagram{2}} + x^3\,\chi_{\ydiagram{3}} + x^4\,\chi_{\ydiagram{4}} + \cdots$$

$$\underline{i_V(x,a_i) = \quad x\,\chi_{\ydiagram{1,1}}} + x^2\,\chi_{\ydiagram{2,1}} + x^3\,\chi_{\ydiagram{3,1}} + x^4\,\chi_{\ydiagram{4,1}} + \cdots$$

and $i_S(x,a_i) = (1-x^2)\, D(x,a_i)$ $\qquad \underline{\ydiagram{1}\otimes\ydiagram{1}} = \underline{\ydiagram{2}\oplus\ydiagram{1,1}\oplus\cdot}$

Note, $i_S(x,a_i)(x\chi_{\ydiagram{1}}) = \underline{(x\,\chi_{\ydiagram{1}})(1} + x\,\chi_{\ydiagram{1}} + x^2\,\chi_{\ydiagram{2}} + x^3\,\chi_{\ydiagram{3}} + x^4\,\chi_{\ydiagram{4}} + \cdots)$

$$= x\,\chi_{\ydiagram{1}} + x^2\,\chi_{\ydiagram{2}} + x^3\,\chi_{\ydiagram{3}} + x^4\,\chi_{\ydiagram{4}} + \cdots$$
$$+ x^2 + x^3\,\chi_{\ydiagram{1}} + x^4\,\chi_{\ydiagram{2}} + x^5\,\chi_{\ydiagram{3}} + \cdots$$
$$+ x^2\,\chi_{\ydiagram{1,1}} + x^3\,\chi_{\ydiagram{2,1}} + x^4\,\chi_{\ydiagram{3,1}} + \cdots$$

first line : traceless symmetrised product.
2nd line : taking the trace.
3rd line : antisymmetric product.

$$\Rightarrow \quad i_s(x,a_i)\,(x\,\chi_\square) = \left(i_s(x,a_i)-1\right) + x^2\, i_s(x,a_i) + x\, i_v(x,a_i)$$

$$\Rightarrow \quad \underline{\underline{i_v(x,a_i)}} = \frac{1}{x}\left[x(1-x^2)\chi_\square - (1-x^4)\right]\underline{\underline{\mathbb{D}(x,a_i)}} + \frac{1}{x}.$$

Note, $\chi_\square$ can be easily calculated from Taylor expansion of $i_s(x,a_i)$.

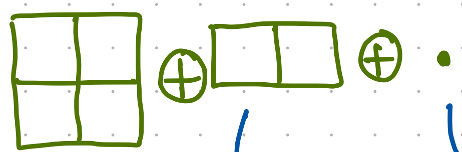
Gravitons:

$$\underline{R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 0} \qquad \underline{R,\ R_{\mu\nu} = 0}$$

For graviton, constraints are,

$$\partial^\mu R_{\mu\nu\alpha\beta} = 0 \qquad \text{EOM}$$

$$\partial_{[\rho} R_{\mu\nu]\alpha\beta} = 0 \qquad \text{Bianchi}$$

$R_{\mu\nu\alpha\beta}$: 

Ricci scalar $= 0$

Ricci tensor $= 0$

Weyl tensor

On-shell

a similar procedure gives,

$$x^2\, i_t(x,a_i) = \left[x^2(1-x^2)(1+\underline{\chi_{\square\square}}) - x(1-x^4)\underline{\chi_\square}\right]\mathbb{D}(x,a_i) + x^2\,\underline{\underline{\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}} + x\,\underline{\chi_\square}$$

* Now we have the "single letter partition fn". for scalar, photon and graviton.

* Next we have to calculate the 4-letter partition fn.

* Then devide by a factor to subtract the total derivative terms.

* After that integrate over the Harr measure to project to singlets i.e. to construct scalar lagrangians.

All these are explained above for scalars and the results are shown in 3rd lecture.

<u>But there is one subtlety!</u>

Partition fn for photon is,

$$\mathcal{D} = \frac{1}{(1-x^4)(1-x^6)}$$

dimension	photon partition fn
$d \geq 10$	$x^4(2+3x^2+2x^4)\mathcal{D}$
$d=9$	$x^4(2+3x^2+2x^4)\mathcal{D} - x^5$ (a miscount?!)
$\vdots$	$\vdots$
$d=3$	$x^4(1+x^2+x^5-x^6)\mathcal{D}$

Plethystic counting proceded in three steps;

Step 1: Computed the partition fn built out of 4-letters [gauge inv. + On-shell]

Step 2: Subtracted the total derivatives by division of $\mathcal{D}$ of Step 1.

Step 3: Integrate step 2 over SO(D) to isolate SO(D) singlets.

In D=9 step 2 has a mistake in assuming that there is no null state. But this is wrong and

$J = * \tilde{F}\wedge\tilde{F}\wedge\tilde{F}\wedge\tilde{F}$ is an identically conserved current.

$\Rightarrow \boxed{\partial_\mu J^\mu} = 0$ (has total 5 ∂_μ's)

$\Rightarrow$ we need to subtract x^5 from the answer.

This gives a way to count and list all possible 4-pt interactions graded with derivative.

Enumeration of Bare Module

Similarly one can count no. of S-matrix structure constructed from Maldemstam variables. The authors have shown a very explicit one-to-one connection b/w these two lists.

Here we will roughly state how the s-matrix are constructed. You have seen this in previous lecture.

Scalars:

* fn of only momenta. i.e. (s,t) $\qquad u+t+s=0$

Parity odd S-matrix are given by $\epsilon_{\mu\nu\rho}\, p_1^{\mu} p_2^{\nu} p_3^{\rho}$ times $f(s,t)$.

Note, it is only valid in $D=3$.

You have seen, $\quad S^4/_{\mathbb{Z}_2 \otimes \mathbb{Z}_2} = S^3$

$\mathbb{Z}_2 \otimes \mathbb{Z}_2$ keeps (s,t,u) invariant.

So, S^3 is the only non-trivial part.

Action of S^3 on polynomial of (s,t) is best realised as permutation of (s,t,u) subjected to $\underline{s+t+u=0}$.

S^3 invariant polynomials

degree	sym	mixed sym	anti sym
0	$\underline{1}$		
1	$s+t+u$ ~ 0 ,	$2s-t-u$ $\sim s$, $2t-s-u$ $\sim t$	
2	$s^2+t^2+u^2,\ st+tu+us$	$2s^2-t^2-u^2,\ 2t^2-s^2-u^2$ $2st-tu-us,\ 2tu-st-us$	
3	3 sym	3 mixed	$s^2t+t^2u+u^2s-st^2-tu^2-us^2.$

It is useful to count all polynomials by their rep$\underset{\cdot\cdot}{\text{n}}$s into a partition f$\underset{\cdot\cdot}{\text{n}}$.

$$z(x) := \operatorname{Tr} x^{2\Delta} = \frac{1}{1-x^2}.$$

(Ignoring the constraint $s+t+u=0$)

Δ: momentum homogeneity.

$$\tilde{Z}(x) := Z_{\text{no-sym}} = z(x)^3 \qquad \tilde{Z}_{1_S, 1_A}(x) = \frac{1}{6} z(x)^3 \pm \frac{1}{2} z(x) z(x^2) + \frac{1}{3} z(x^3)$$

(using multi-letter partition f$\underset{\cdot\cdot}{\text{n}}$)

$$\tilde{Z}_{2_M}(x) = \frac{\tilde{Z}(x) - \tilde{Z}_{1_S}(x) - \tilde{Z}_{1_A}(x)}{2}$$

$\forall R$ $$\tilde{Z}_R(x) = \sum_m n_R(m)\, x^{2m}$$

$n_R(m)$: no of S_3 repn of type R at degree m.

Till now we have not removed the condition $s+t+u=0$.

That can be done very easily. We also did it before.

polynomials we don't want have the form

$$f(s,t) \times (s+t+u)$$

which has partition fn $\underline{\underline{x^2}}\,\tilde{Z}_R(x)$ which should be subtracted

$$\Rightarrow \quad Z_R(x) = (1-x^2)\,\tilde{Z}_R(x)$$

Here $Z_R(x)$ denotes the partition fn over polynomials in the repn R with the constraint $s+t+u=0$.

Using these, similar way one can calculate the following partition fn's,

$$Z_{1_S}(x) = \mathcal{D} \quad , \quad Z_{1_A}(x) = x^6 \mathcal{D} \ , \quad Z_{2_M}(x) = (x^2+x^4)\mathcal{D}$$

$$\mathcal{D} = \frac{1}{(1-x^4)(1-x^6)}$$

Check: $\tilde{Z}_{1_S}(x) = \frac{1}{6} z(x)^3 + \frac{1}{2} z(x) z(x^2) + \frac{1}{3} z(x^3)$

where $z(x) = \frac{1}{1-x^2}$

$$\Rightarrow Z_{1_S}(x) = (1-x^2)\left[\frac{1}{6}\frac{1}{(1-x^2)^3} + \frac{1}{2}\frac{1}{(1-x^2)(1-x^4)} + \frac{1}{3}\frac{1}{1-x^6}\right]$$

$$= \frac{1}{6}\frac{1}{(1-x^2)^2} + \frac{1}{2(1-x^4)} + \frac{1}{3(1+x^2+x^4)} \qquad \text{(H.W.)}$$

$$= \frac{1}{(1-x^4)(1-x^6)} \Big) = \mathcal{D}$$

Another way :

1_S repⁿ. polynomials are, $(s^2+t^2+u^2)^m \, (stu)^n$
(Completely symmetric)

$(s^2+t^2+u^2)$ ↓ single letter $\qquad$ (stu) ↓ another single letter

$$z_1(x) = x^4 \qquad\qquad z_2(x) = x^6$$

$$Z_1 = PE[z(x)] = \frac{1}{1-x^4} \qquad\qquad Z_2 = PE[z(x)] = \frac{1}{1-x^6}$$

$\Rightarrow$ Total partition fn, $Z_{1_S} = Z_1 Z_2 = \dfrac{1}{(1-x^4)(1-x^6)}$

1_A repⁿ. polynomials are of the form,

$$\underbrace{(s^2t - t^2s - s^2u + su^2 - u^2t + t^2u)}_{\sim x^6} \underbrace{\underbrace{(s^2+t^2+u^2)^m}_{\sim \frac{1}{1-x^4}} \underbrace{(stu)^n}_{\sim \frac{1}{1-x^6}}}_{\mathcal{D}}$$

(Since only one structure)

$\Rightarrow Z_{1_A}(x) = x^6 \mathcal{D}$ $\qquad$ etc.

Enumeration of photon bare module:

You have seen decomposition of photon polarizations into parallel and perpendicular to the scattering plane.

The stabilizer group of the scattering process is $SO(D-3)$.

Under $SO(D-3)$, the polarization of photon takes value in the space $\rho = (\underset{\text{Scalar}}{s} \oplus \underset{\text{vector}}{v})$

Scalar & vector under $SO(D-3)$.

So, 4-photon scattering space is, $(s\oplus v)^{\otimes 4}\Big|_{\mathbb{Z}_2\otimes \mathbb{Z}_2}$

i.e. projected to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$.

A very well known formula for this projection is,

$$\rho^{\otimes 4}\Big|_{\mathbb{Z}_2\times\mathbb{Z}_2} = \rho^4 \ominus 3(S^2\rho \otimes \Lambda^2 \rho)$$

(Previously proved in a very sophisticated way.)

A 'physics' proof of $\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} = \rho^4 \oplus 3(S^2\rho \otimes \Lambda^2\rho)$?⇒

Consider a 'single particle partition fn'.

$$\mathrm{Tr}_\rho\left(\prod_i a_i^{J_i}\right) = \sum_i \langle i| \prod_k a_k^{J_k} |i\rangle = z(a_i)$$

Now consider a 'two identical particle partition fn',

Decompose ρ^2 into two parts $S^2\rho$ and $\Lambda^2\rho$.

$$\mathrm{Tr}_{S^2\rho}\left(\prod_i a_i^{J_i}\right) = \sum_{i_1 i_2} \langle i_1 i_2| \left(\prod_m a_m^{J_m}\right)\left(\frac{1+P_{(12)}}{2}\right)|i_1 i_2\rangle = \frac{z^2(a_i) + z(a_i^2)}{2}$$

$$\mathrm{Tr}_{\Lambda^2\rho}\left(\prod_i a_i^{J_i}\right) = \sum_{i_1 i_2} \langle i_1 i_2| \left(\prod_m a_m^{J_m}\right)\left(\frac{1-P_{(12)}}{2}\right)|i_1 i_2\rangle = \frac{z^2(a_i) - z(a_i^2)}{2}$$

where we have used the fact,

$$\langle i_1 i_2| \left(\prod_m a_m^{J_m}\right) P_{(12)} |i_1 i_2\rangle = \langle i_1 i_2| \prod_m a_m^{J_m} |i_2 i_1\rangle = \delta_{i_1,i_2} \langle i_1| \prod_m (a_m^2)^{J_m} |i_1\rangle$$

Next consider, the Hilbert space $\mathcal{S}$ of four distinguishable particles,

$$\mathrm{Tr}_{\mathcal{S}^{\otimes 4}}\left(\prod_m a_m^{J_m}\right) = \sum_{i_1 i_2 i_3 i_4} \langle i_1 i_2 i_3 i_4 | \prod_m a_m^{J_m} | i_1 i_2 i_3 i_4 \rangle = z^4(a_m)$$

Now we want to project to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$,

$$\Rightarrow \mathrm{Tr}_{\mathcal{S}^{\otimes 4}}\Big|_{\mathbb{Z}_2 \otimes \mathbb{Z}_2}\left(\prod_m a_m^{J_m}\right) = \sum_{\substack{i_1 i_2 \\ i_3 i_4}} \langle i_1 i_2 i_3 i_4 | \prod_m a^{J_m} \left(\frac{1 + P_{(2143)} + P_{(3412)} + P_{(4321)}}{4}\right) | i_1 i_2 i_3 i_4 \rangle$$

$$= \frac{1}{4} \sum_{\substack{i_1 i_2 \\ i_3 i_4}} \Big(\langle i_1 i_2 i_3 i_4 | a^J | i_1 i_2 i_3 i_4 \rangle + \langle i_1 i_2 i_3 i_4 | a^J | i_2 i_1 i_4 i_3 \rangle + \langle i_1 i_2 i_3 i_4 | a^J | i_3 i_4 i_1 i_2 \rangle + \langle i_1 i_2 i_3 i_4 | a^J | i_4 i_3 i_2 i_1 \rangle \Big)$$

1^{st} term : $z^4(a)$ we checked.

2^{nd} term : $\sum_{i_1 i_2 i_3 i_4} \delta_{i_1 i_2} \delta_{i_3 i_4} \langle i_1 i_3 | (a^2)^J | i_1 i_3 \rangle = z^2(a^2)$

3^{rd} term : $\sum_{i_1 i_2 i_3 i_4} \delta_{i_1 i_3} \delta_{i_2 i_4} \;\ldots\ldots = z^2(a^2)$

$$\Rightarrow \mathrm{Tr}_{\rho^{\otimes 4}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}}\left(a_i^{J_i}\right) = \frac{z^4(a_i) + 3\, z^2(a_i^2)}{4}$$

$$= \underbrace{z^4(a_i)}_{\mathrm{Tr}_{\rho^{\otimes 4}}(a^J)} - 3\underbrace{\left(\frac{z^2(a_i)+z\,(a_i^2)}{2}\right)}_{\mathrm{Tr}_{S^2\rho}(a^J)} \times \underbrace{\left(\frac{z^2(a_i)-z(a_i^2)}{2}\right)}_{\mathrm{Tr}_{\Lambda^2\rho}(a^J)}$$

$$\Rightarrow \mathrm{Tr}_{\rho^{\otimes 4}|_{\mathbb{Z}_2\otimes\mathbb{Z}_2}}\left(\prod_i a_i^{J_i}\right) = \mathrm{Tr}_{\rho^{\otimes 4}}\left(\prod_i a_i^{J_i}\right) - 3\, \mathrm{Tr}_{S^2\rho}\left(\prod_i a_i^{J_i}\right) \mathrm{Tr}_{\Lambda^2\rho}\left(\prod_i a_i^{J_i}\right)$$

So, the partition $\underline{fn}$ matches exactly in both sides of

$$\rho^{\otimes 4}\Big|_{\mathbb{Z}_2\otimes\mathbb{Z}_2} = \rho^{\otimes 4} - 3\, S^2\rho \otimes \Lambda^2\rho$$

Time Ordering in Chaos Bound

Bound on Chaos: Review

• **Statement:** Out of time order thermal four-point function in a large N theory cannot grow faster with time than $e^{\frac{2\pi t}{\beta}}$ where β is the inverse temperature of the ensemble.

• In Appendix A of the same paper, the authors explained that, in the special case of conformal large N field theories, their bound also constrains the growth of ordinary time-ordered correlators in the Regge limit on the Causally Regge sheet.

• Large N CFT in Euclidean $\Rightarrow$ 'angular quantization' (i.e., angular coordinate θ is Euclidean time and radial coordinate r is space) $\Rightarrow$ θ is periodic with periodicity 2π. $\Rightarrow$ The theory in this quantization is effectively thermal with $\beta = 2\pi$.

Bound on Chaos: Review

- In the angular quantization the path integral computes,

$$G_{\text{norm}} = \frac{\langle O_1 O_4 O_2 O_3 \rangle}{\langle O_2 O_1 \rangle \langle O_4 O_3 \rangle}$$

(ordered in θ)

Operators	r	θ
O_3	1	0
O_2	x (< 1)	$i(\tau - i\epsilon)$
O_4	1	π
O_1	x	$\pi + i(\tau - i\epsilon)$

- G_{norm} have a simple representation in the quantization of the same theory in usual Minkowski time (in the plane $R^{1,1}$ obtained by starting with the plane R^2 and performing the usual analytic continuation to go to $R^{1,1}$)
- Operators O_2 and O_1 which are both inserted at Rindler time $\tau - i\epsilon$ are respectively inserted at Minkowski time

$$t_M = \pm x \sinh(\tau - i\epsilon) \simeq \pm x \sinh \tau \mp i\epsilon \qquad \Longrightarrow \; O_2 > O_1$$

$$\text{for } O_3 \& O_4, \quad t_M = 0 \qquad \Longrightarrow \; O_2 > (O_4 \& O_3) > O_1$$

- **Reason:** Euclidean $(x_E, t_E) \equiv (r cos\theta, r\sin\theta)$ & $t_M = -i \; t_E = -i \; r\sin\theta$

Bound on Chaos

- In Minkowski space, the normalized correlator takes the form,

$$G_{\rm norm} = \frac{\langle B(\tau)O_2(x,x)B^{-1}(\tau)O_4(-1,-1)O_3(1,1)B(\tau)O_1(-x,-x)B^{-1}(\tau)\rangle}{\langle B(\tau)O_2(x,x)B^{-1}(\tau)B(\tau)O_1(-x,-x)B^{-1}(\tau)\rangle\langle O_4(-1,-1)O_3(1,1)\rangle}$$

$B(\tau)$: boost operator by rapidity τ.

- Operator O_m with weight under boost λ_m follows

$$B(\tau)O_2(x,x)B^{-1}(\tau) = e^{\lambda_2\tau}O_2(xe^{-\tau}, xe^{\tau})$$
$$B(\tau)O_1(-x,-x)B^{-1}(\tau) = e^{\lambda_1\tau}O_1(-xe^{-\tau}, -xe^{\tau})$$

The normalized four point function simplifies to,

$$G_{\rm norm} = \frac{\langle O_2(xe^{-\tau}, xe^{\tau})O_4(-1,-1)O_3(1,1)O_1(-xe^{-\tau}, -xe^{\tau})\rangle}{\langle O_2(xe^{-\tau}, xe^{\tau})O_1(-xe^{-\tau}, -xe^{\tau})\rangle\langle O_4(-1,-1)O_3(1,1)\rangle}$$

Note: For large enough τ it is in Causally Regge sheet and

$$\boxed{\sigma = 4e^{-\tau}}$$

'Physics' motivation for Plythestic Exponentiation:

Bosonic partition fn:

n_α : occupation no.

E_α : energy of level α

$$\sum_{\{n_\alpha\}} \prod_\alpha e^{-\beta n_\alpha E_\alpha}$$

[unconstrained sum]

$$= \prod_\alpha \sum_{\{n_\alpha\}} e^{-\beta n_\alpha E_\alpha} = \prod_\alpha \frac{1}{(1-e^{-\beta E_\alpha})}$$

$E_\alpha = \hbar\omega(\alpha + \frac{1}{2}) \propto \alpha$ Then,

$$= \prod_{n=1}^{\infty} \frac{1}{(1-e^{-\beta n E})} = e^{-\sum_n \log(1-e^{-\beta E n})}$$

$$= e^{\sum_{n=1}^{\infty} \frac{1}{n}\left(e^{-\beta E}\right)^n}$$