A non-reciprocal classical Heisenberg chain: Conservation laws, emergent hydrodynamics and chaos

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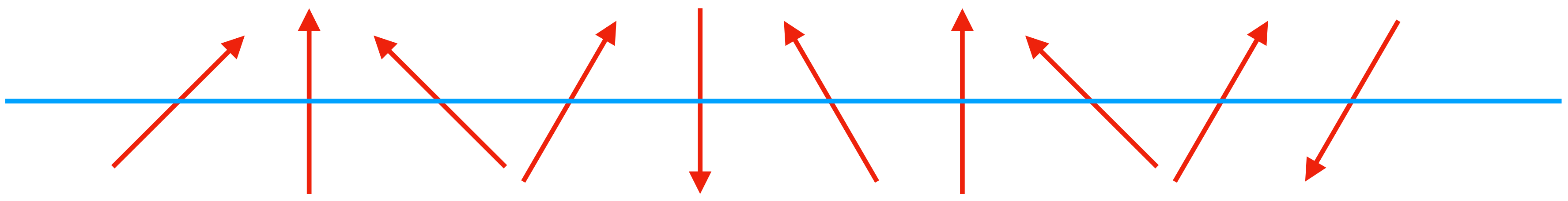
Classical Heisenberg chain

3D rotors of fixed length on a 1D lattice



Hamiltonian

$$H = -\sum_i \mathbf{S}_i . \mathbf{S}_{i+1}$$

Dynamics

$$\frac{d\mathbf{S}_i}{dt} = \{\mathbf{S}_i, H\}$$

$$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times (\mathbf{S}_{i+1} + \mathbf{S}_{i-1})$$

Classical Heisenberg Chain (CHC)

$$H = -\sum_i \mathbf{S}_i.\mathbf{S}_{i+1}$$

Quantum Heisenberg Chain (QHC)

$$\hat{H} = -\sum_i \hat{\mathbf{S}}_i.\hat{\mathbf{S}}_{i+1}$$

CHC is not an integrable system

QHC is integrable only for $S = 1/2$

CHC is thus expected to thermalise at arbitrary energies

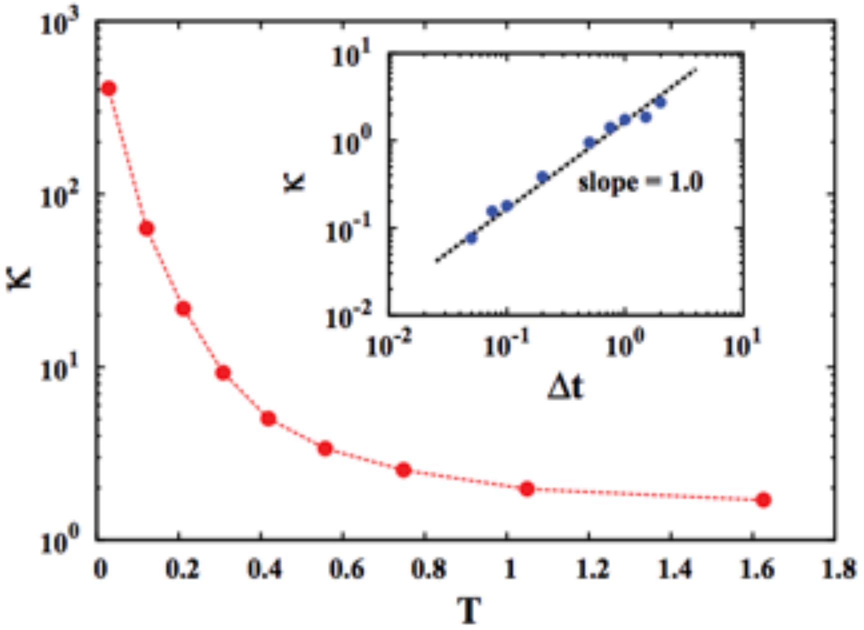
Classical Heisenberg Chain (CHC)

Conserved quantities

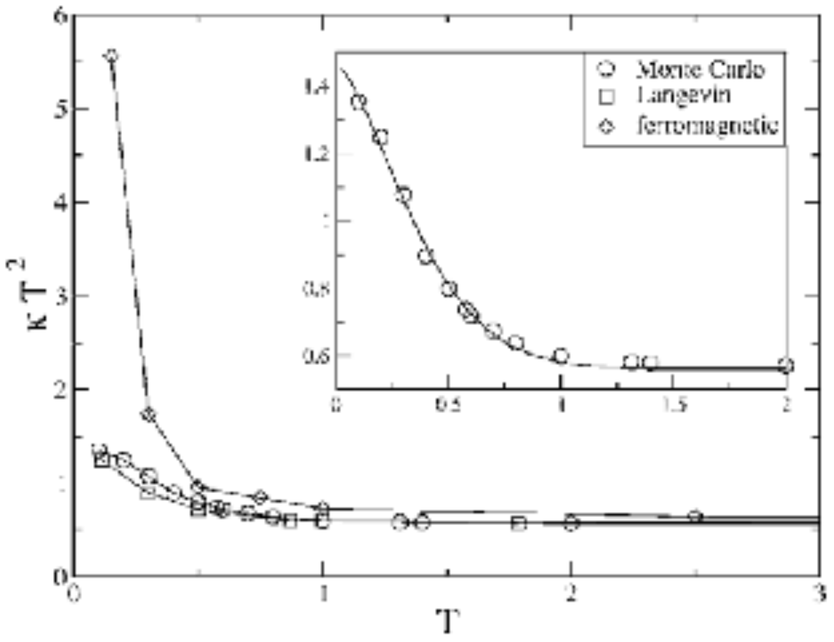
Energy $H = -\sum_i \mathbf{S}_i . \mathbf{S}_{i+1}$ **Magnetisation** $\mathbf{M} = \sum_i \mathbf{S}_i$

Evidence of thermalisation

Diffusion at all energies observed numerically



Bagchi and Mohanty,
Phys. Rev. B 86, 214302 (2012)



Savin, Tsironis and Zotos,
Phys. Rev. B 72, 140402R (2005)

Chaos spreading in the CHC

Spatial spreading of chaos can be characterised by the decorrelator

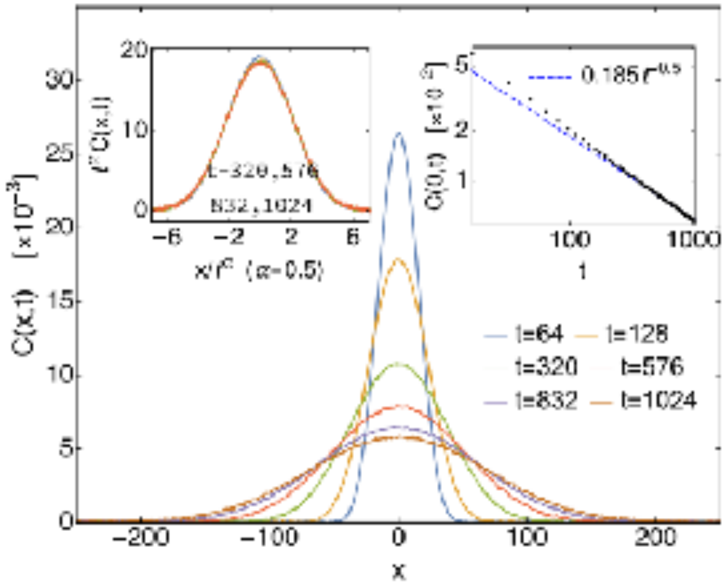
[analogue of the Out of Time Ordered Correlator (OTOC) for quantum systems]

$$D(x,t) = 1 - \langle \mathbf{S}_a(x,t).\mathbf{S}_b(x,t)\rangle$$

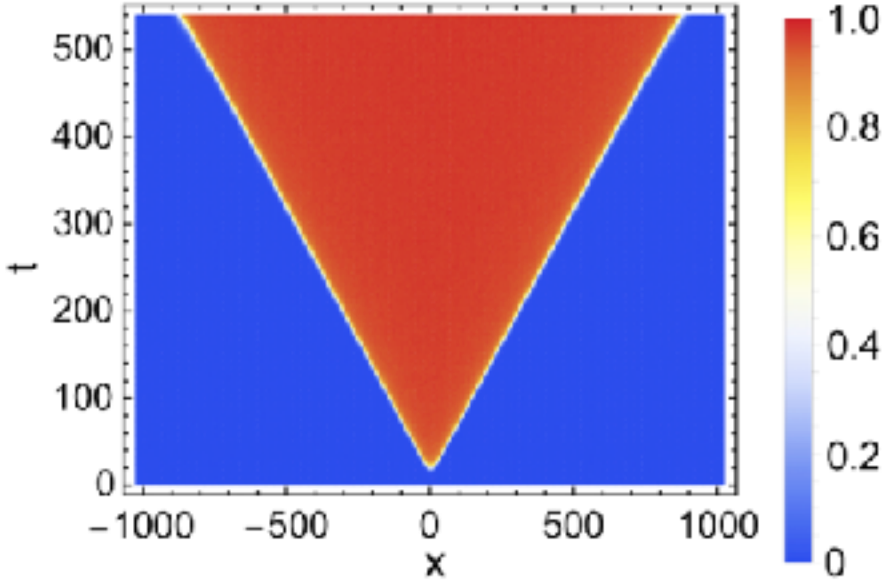
$a\&b$ **correspond to a pair of "close" initial states**

Correlator $C(x,t) = \langle \mathbf{S}(x,t).\mathbf{S}(0,0)\rangle$

$\langle \ldots \rangle$ **average over initial states drawn from infinite temperature ensemble**



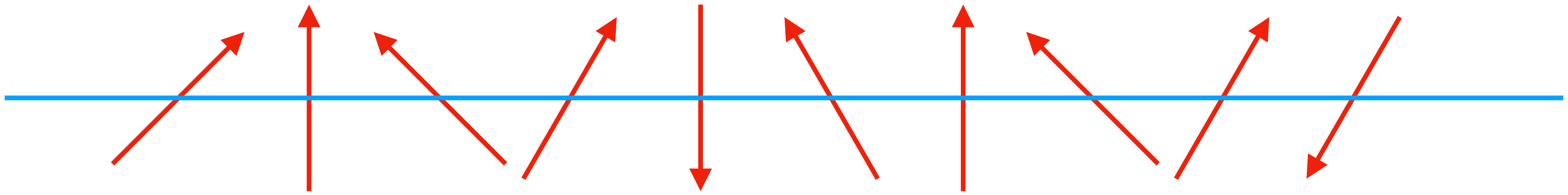
Numerically obtained heat map of $D(x,t)$



Ballistic spread of chaos with a "butterfly" velocity. Spin transport diffusive

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, Ray and Bhattacharjee
Phys. Rev. Lett., 121 024101 (2018)

Non-reciprocal model



$$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times \left(\mathbf{S}_{i+1} - \mathbf{S}_{i-1}\right)$$

Previously studied in the context of a coarse grained model with a drive that induces a current

Das, Rao and Ramaswamy, Europhys. Lett. 60(3) 418 (2002)

Non-reciprocal dynamics: Torque on i due to i+1 is equal (and not equal and opposite) to torque on i+1 due to i

Cannot be derived from a Hamiltonian

Is there any notion of thermalisation?

What is the nature of chaos spreading in this model?

Non-reciprocal model

$$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times (\mathbf{S}_{i+1} - \mathbf{S}_{i-1})$$

The model has conserved quantities

Staggered magnetisation $\mathbf{N} = \sum_i (-1)^i \, \mathbf{S}_i$

Pseudo-energy $\tilde{H} = -\sum_i (-1)^i \, \mathbf{S}_i . \mathbf{S}_{i+1}$

Note however that

$$\frac{d\mathbf{S}_i}{dt} \neq \{\mathbf{S}_i, \tilde{H}\}$$

The model is manifestly non-Hamiltonian

But there is a Liouville theorem for the dynamics

Hanai, arXiv:2208.08577 (2022)

Recent interest in classical non-reciprocal systems

Bachelard, Piovela and Gupta, Phys. Rev. E 99, 010104(R) (2019)

Subdiffusive behaviour with long range coupling

Fruchart, Hanai, Littlewood and Vitelli, Nature 592 363 (2021)

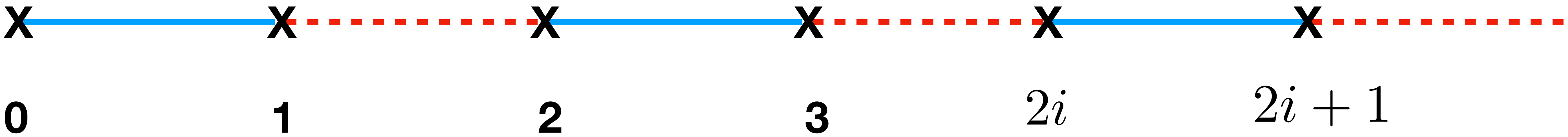
Time dependent phases in which spontaneously broken symmetries are dynamically restored

McRoberts, Zhao, Moessner and Bukov, arXiv:2208.09005 (2022)

Analogue of prethermalisation in driven systems

Emergent hydrodynamics

$$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times (\mathbf{S}_{i+1} - \mathbf{S}_{i-1})$$



Staggered magnetisation

$$\mathbf{N}_i = \frac{\mathbf{S}_{2i} - \mathbf{S}_{2i+1}}{2}$$

$\sum_i \mathbf{N}_i$ **conserved**

Magnetisation

$$\mathbf{M}_i = \frac{\mathbf{S}_{2i} + \mathbf{S}_{2i+1}}{2}$$

$\sum_i \mathbf{M}_i$ **not conserved**

Pseudo-energy

$$E_i = -\frac{1}{2}\left[\mathbf{S}_{2i}.\left(\mathbf{S}_{2i-1} - \mathbf{S}_{2i+1}\right)\right]$$

$\sum_i E_i$ **conserved**

Emergent hydrodynamics

Coarse grain and retain gradient terms only to first order

Staggered magnetisation

$$\partial_t \mathbf{N} = -\partial_x \left(\mathbf{M}\times\mathbf{N}\right)$$

Magnetisation

$$\partial_t \mathbf{M} = -\mathbf{N}\times\partial_x\mathbf{N} + \mathbf{M}\times\partial_x\mathbf{M}$$

Staggered energy

$$E = -\tfrac{1}{2}\left(\mathbf{N}+\mathbf{M}\right).\partial_x\left(\mathbf{N}-\mathbf{M}\right)$$

$$\partial_t E = -\partial_x\left[\tfrac{1}{2}\left(\mathbf{M}-\mathbf{N}\right).\left(\mathbf{M}+\mathbf{N}\right)\times\partial_x\left(\mathbf{N}+\mathbf{M}\right)\right]$$

Self-consistent calculation

Assume that higher wavenumber modes (higher gradient terms) provide a bath for the low wavenumber hydrodynamic modes and "renormalised" couplings

$$\partial_t \mathbf{N} = -\lambda_{MN}\partial_x\left(\mathbf{M}\times\mathbf{N}\right) + \vec{\eta}_N$$

Effective coupling **Conserving noise**

$$\partial_t \mathbf{M} = -\lambda_N \mathbf{N}\times\partial_x\mathbf{N} + \lambda_M \mathbf{M}\times\partial_x\mathbf{M} + \vec{\eta}_M$$

Effective coupling **Non-conserving noise**

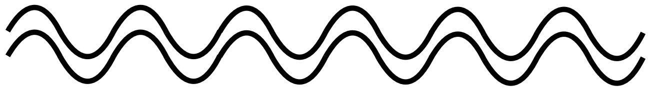
which gives rise to

$$\partial_t \mathbf{N} = D\partial_x^2\mathbf{N} + \vec{\eta}_N \qquad \partial_t \mathbf{M} = -\frac{\mathbf{M}}{\tau} + \vec{\eta}_M$$

Diffusion **Relaxation**

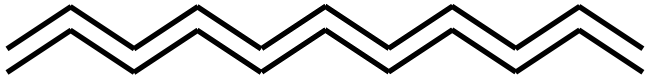
Self-consistent relations can be obtained for the diffusion constant, relaxation time, coupling constants and the strengths of the noise

Propagator for $\mathbf{M}$**,** $G_M(q,\omega) = \frac{1}{-i\omega - \Sigma_M(q,\omega)}$



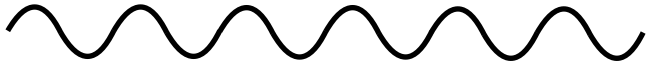
$$\Sigma_M(q=0,\omega=0) = -\frac{1}{\tau}$$

Propagator for $\mathbf{N}$**,** $G_N(q,\omega) = \frac{1}{-i\omega - \Sigma_N(q,\omega)}$



$$\Sigma_N(q \to 0,\omega=0) = -Dq^2$$

Free propagator $G_0(q,\omega) = \frac{1}{-i\omega + \delta}$



Non-conserving noise

$$\langle \eta_M^\alpha(q,\omega)\eta_M^\beta(q',\omega')\rangle = A_M\delta(q+q')\delta(\omega+\omega')\delta_{\alpha\beta}$$ **X**

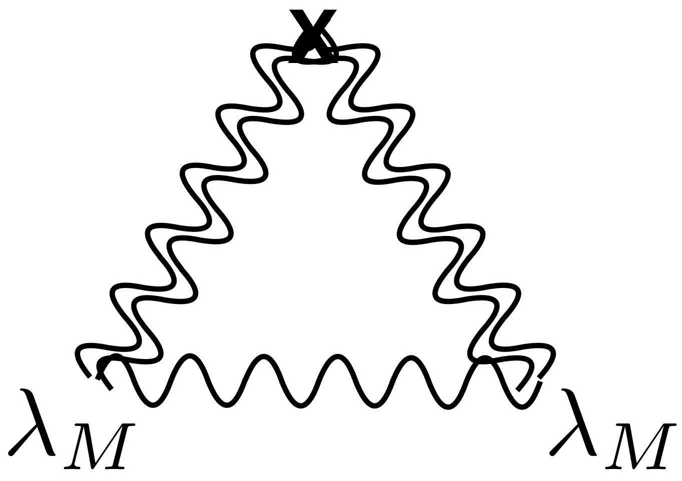
Conserving noise

$$\langle \eta_N^\alpha(q,\omega)\eta_N^\beta(q',\omega')\rangle = A_N q^2\delta(q+q')\delta(\omega+\omega')\delta_{\alpha\beta}$$ **+**

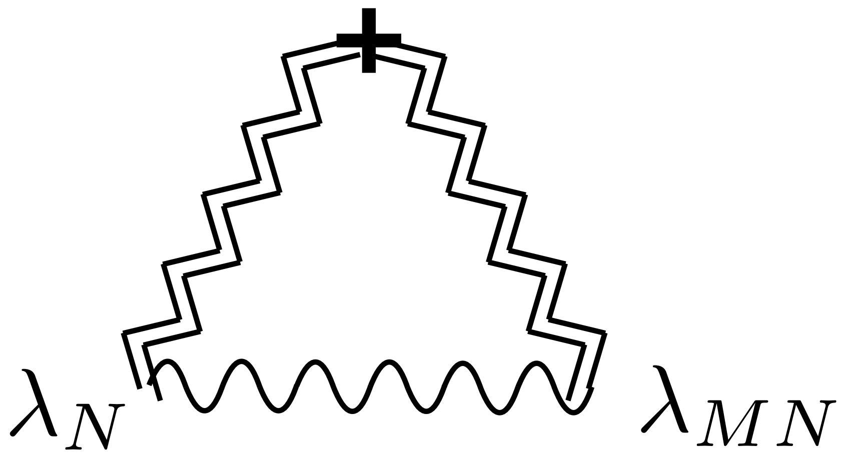
Self-consistent diagrammatic calculation

a la **Ma and Mazenko, Phys. Rev. B 11 4077 (1975)**

Diagrams contributing to $\Sigma_M(q=0,\omega=0)$

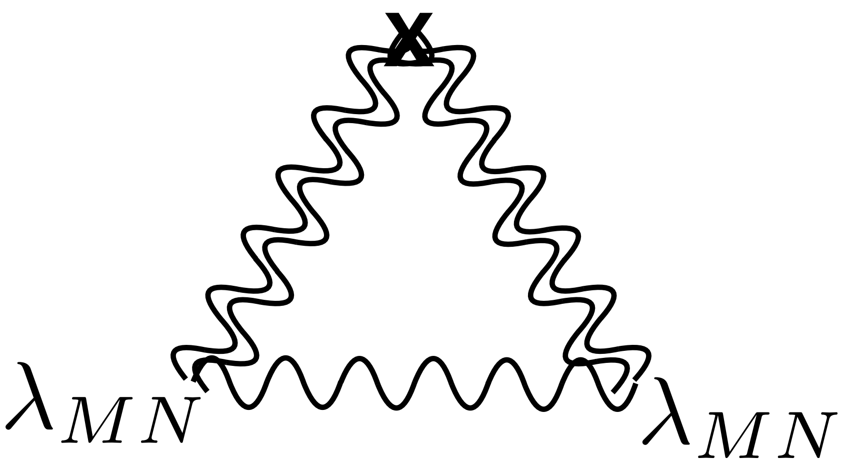


$\sim -\lambda_M^2 A_M \tau^2 \Lambda^3$

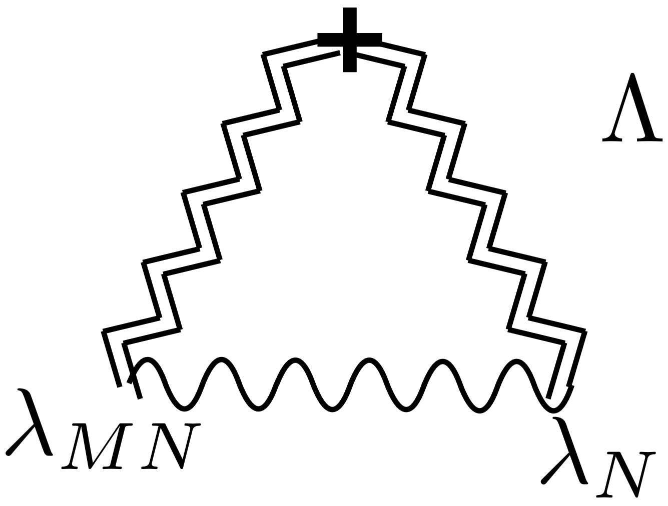


$\sim -\lambda_M \lambda_{MN} A_N \Lambda / D^2$

Diagrams contributing to $\Sigma_N(q\to 0,\omega=0)$



$\sim -q^2 \lambda_{MN}^2 \sqrt{\frac{\tau^3}{D}}$



$\sim -q^2 \lambda_M \lambda_{MN} A_N \sqrt{\frac{\tau}{D^3}}$

Λ **-ultraviolet cutoff**

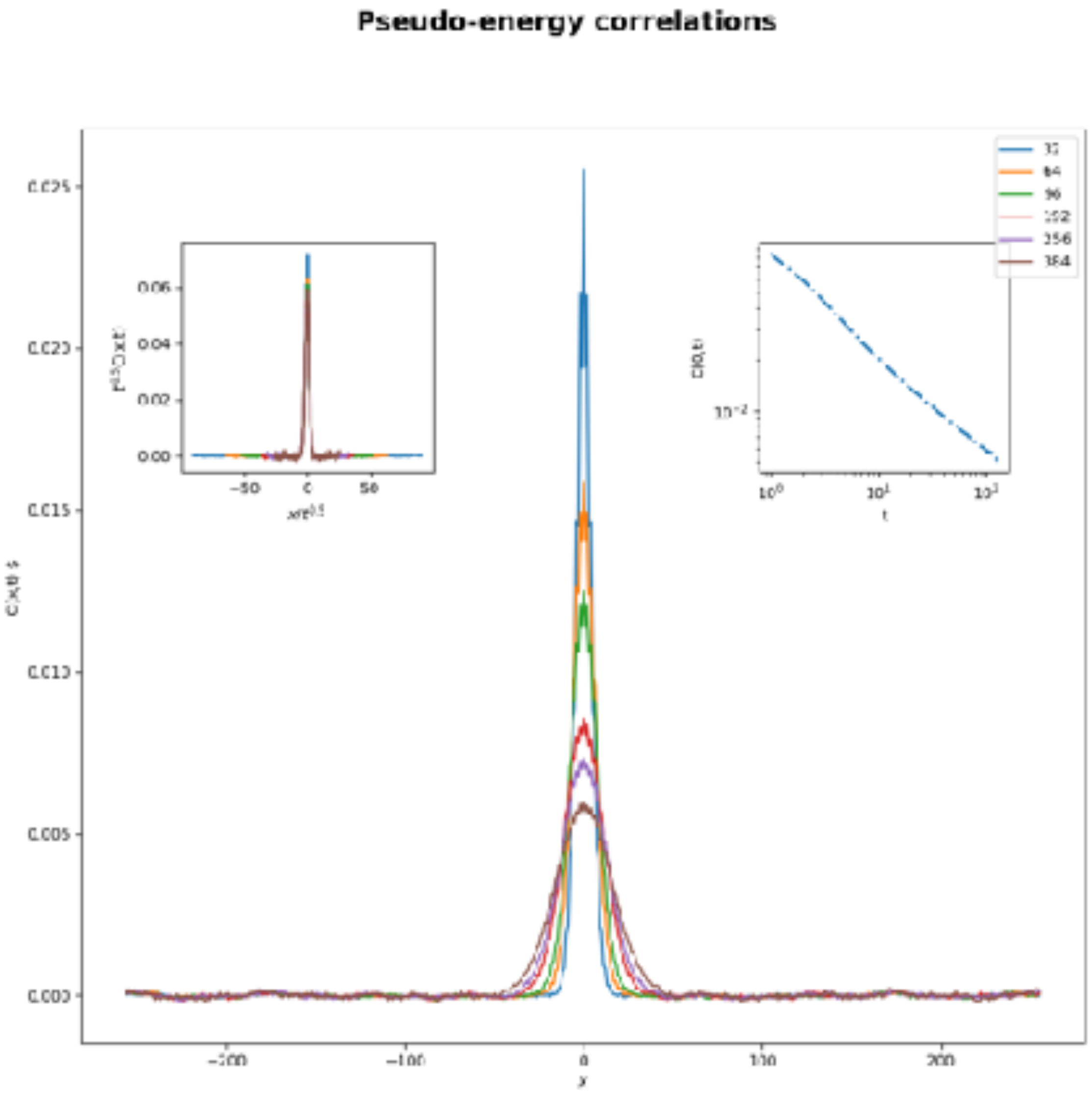
A slight correction to the version presented during the talk

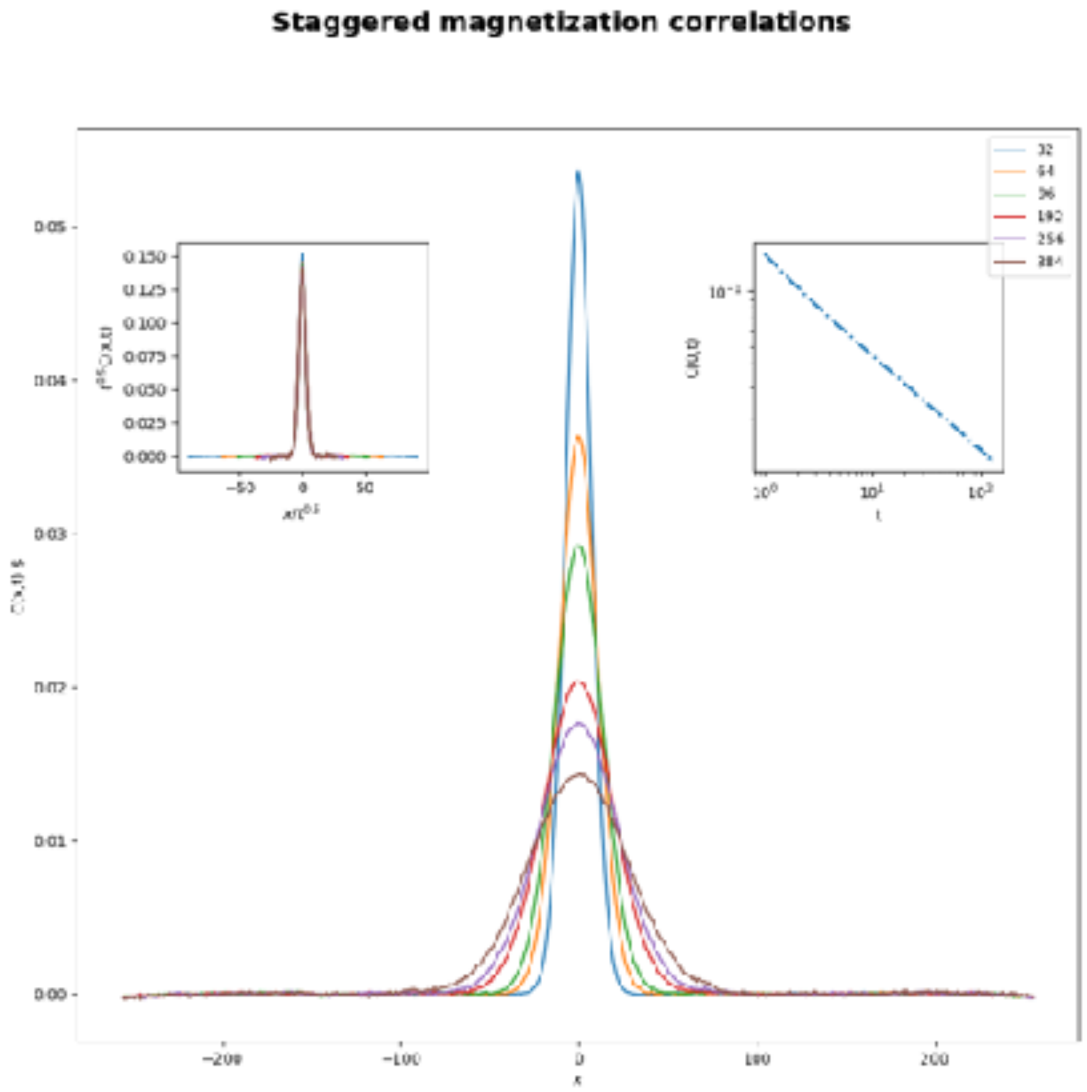
Non-reciprocal model

Numerical data for system size - 2048

Averaged over 5000 initial conditions

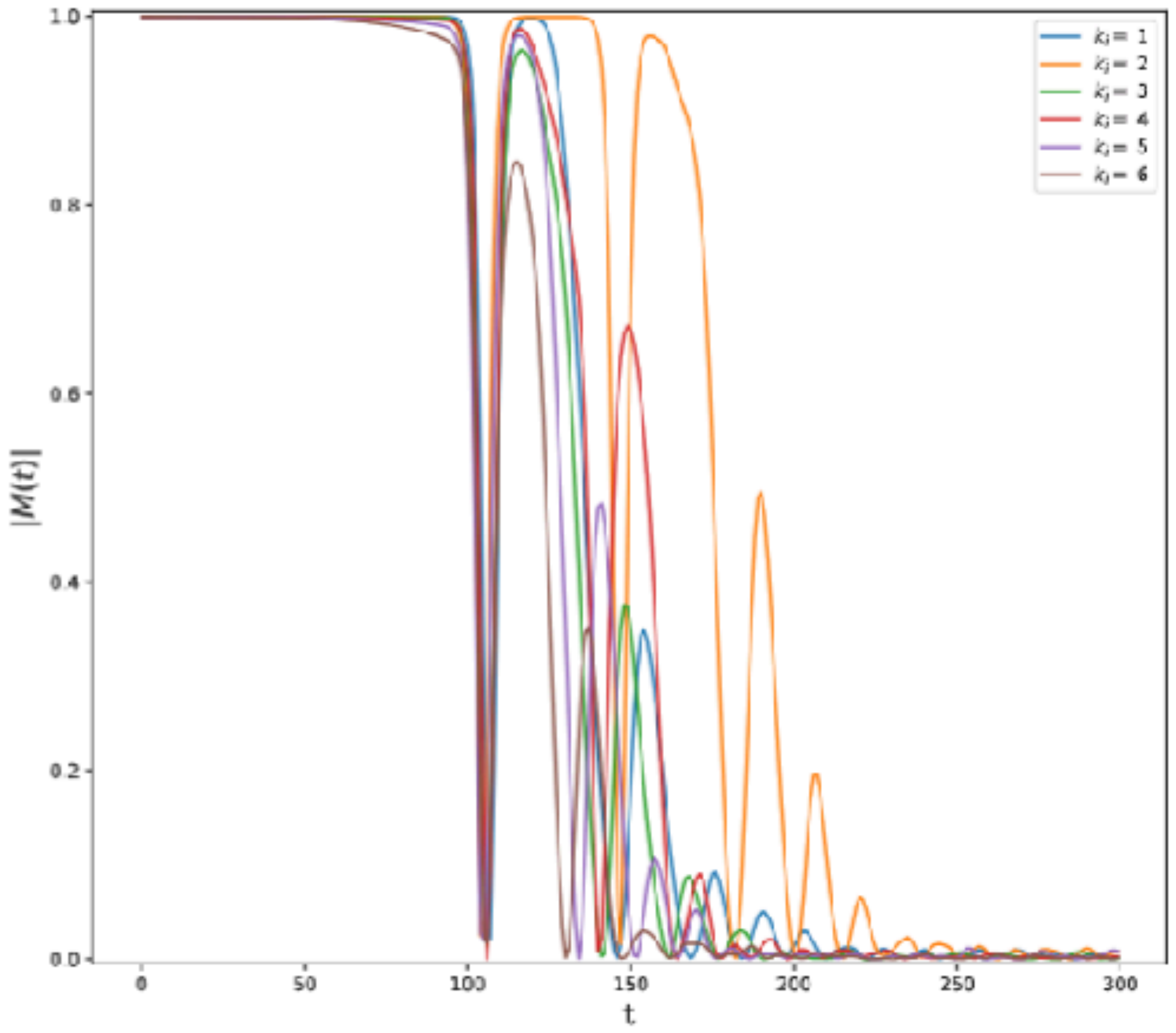
All possible initial states weighted equally


Pseudo-energy correlations


Staggered magnetization correlations

Staggered magnetisation and staggered energy display diffusion

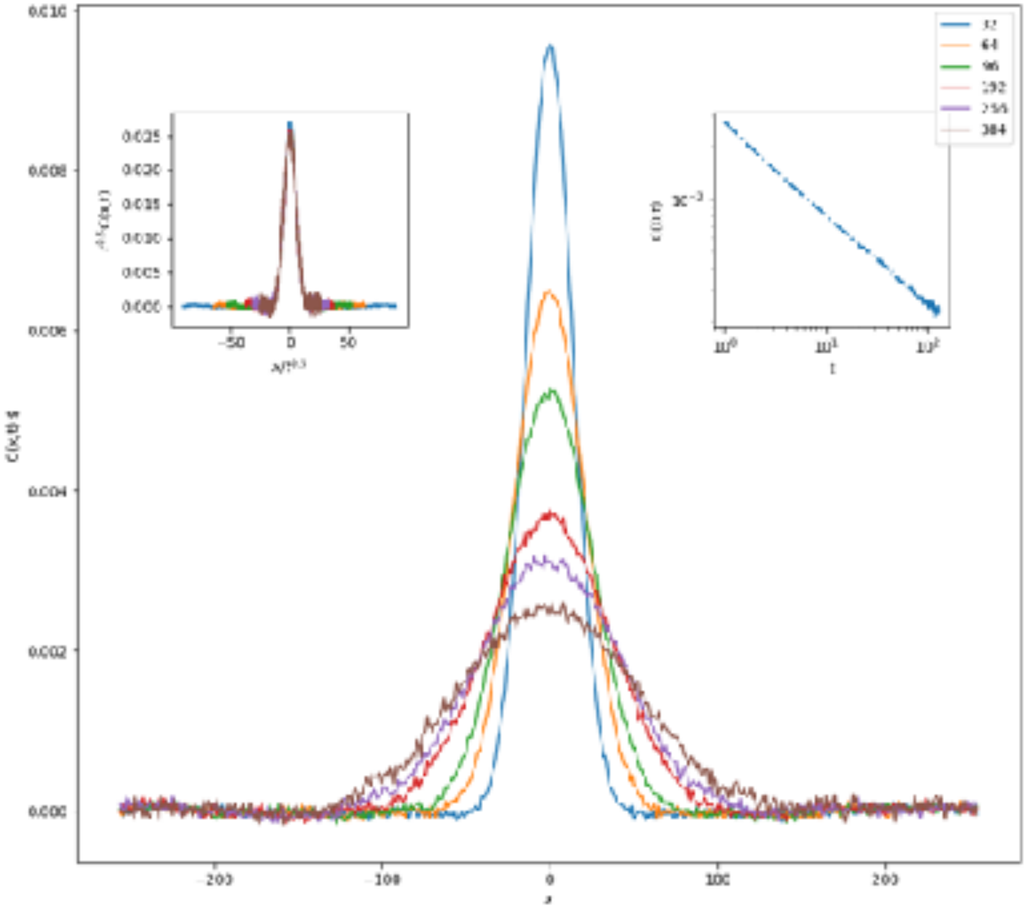
Magnetisation relaxes to zero



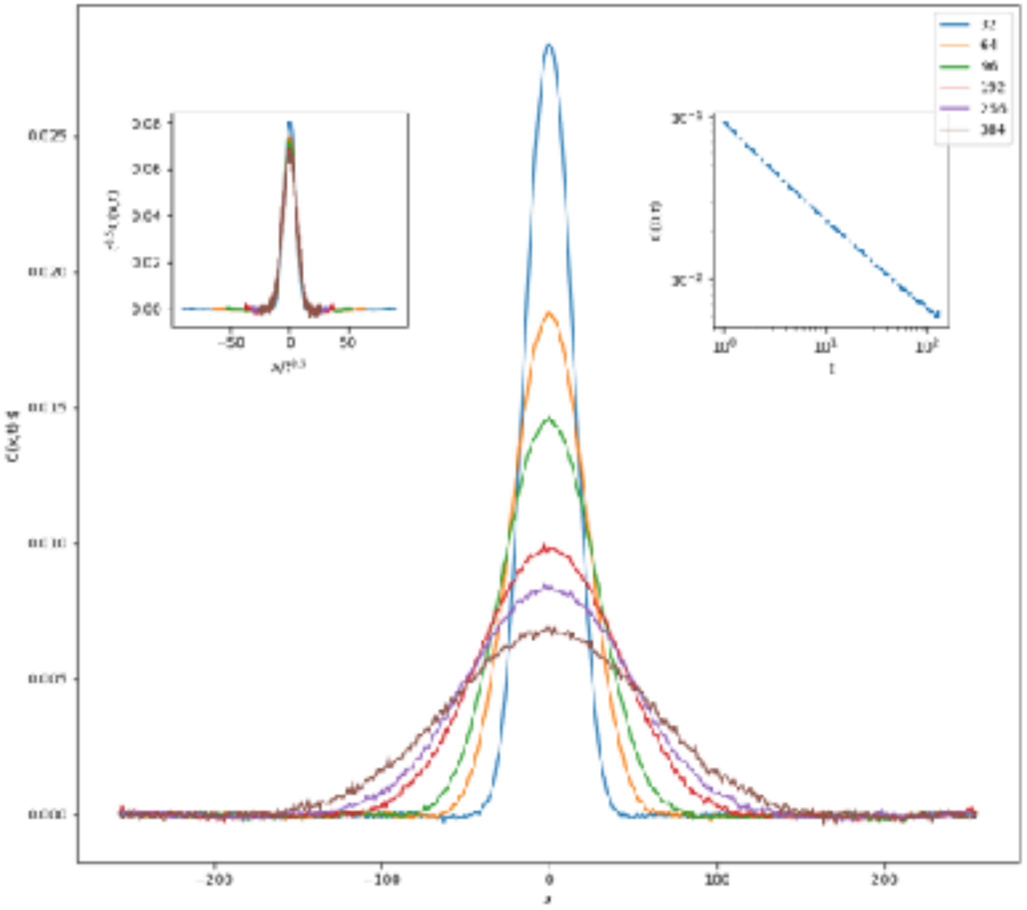
The magnetisation relaxes to zero at long times starting from different initial conditions

Heisenberg model (Fully reciprocal model)

Energy correlations



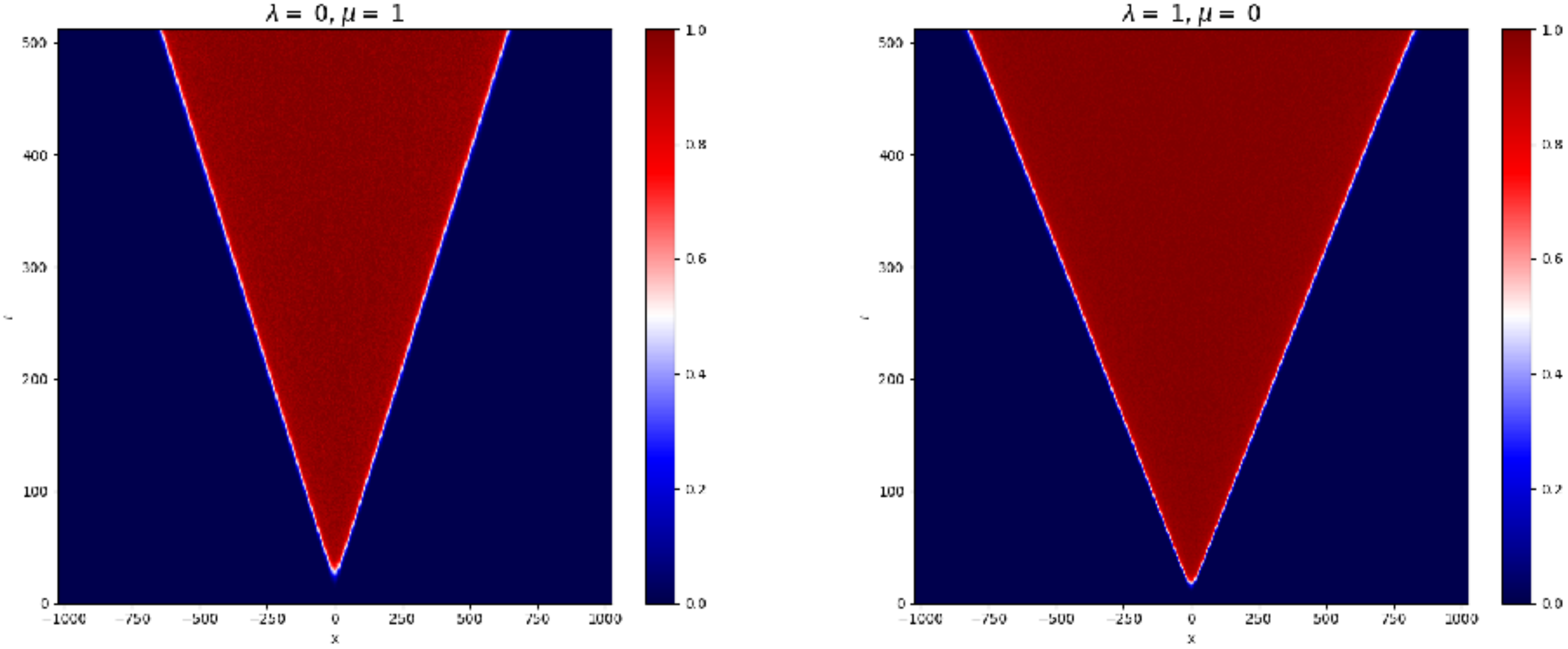
Magnetization correlations



Both energy and magnetisation display diffusion

Decorrelator (Non-reciprocal and Heisenberg)

$$D(x,t) = 1 - \langle \mathbf{S}_a(x,t).\mathbf{S}_b(x,t)\rangle$$



Non-reciprocal model

Heisenberg model

Light cone spreading of perturbations in both systems

$D(x,t) \sim \exp\left[\alpha t\left(1-(x/v_B t)^2\right)\right]$ **close to the front**

Different butterfly velocities v_B and Lyapunov exponents α for the two models

Quantum model

$$\frac{d\hat{\mathbf{S}}_i}{dt} = \hat{\mathbf{S}}_i \times \left(\hat{\mathbf{S}}_{i+1} - \hat{\mathbf{S}}_{i-1} \right)$$

Also, non-Hamiltonian

Consider two sites 1 and 2 $\frac{d\hat{\mathbf{S}}_1}{dt} = \hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2$ $\frac{d\hat{\mathbf{S}}_2}{dt} = \hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2$

Initialize $\hat{S}_1^\alpha(t=0) = \hat{\sigma}_1^\alpha \otimes \hat{I}_2$ $\hat{S}_2^\alpha(t=0) = \hat{I}_1 \otimes \hat{\sigma}_2^\alpha$

$$\left[\hat{S}_i^\alpha(t=0)\right]^\dagger = \left[\hat{S}_i^\alpha(t=0)\right] \qquad \left[\hat{S}_j^\alpha(t=0), \hat{S}_k^\beta(t=0)\right] = i\delta_{jk}\epsilon^{\alpha\beta\gamma}\hat{S}_k^\gamma(t=0)$$

$$\left[\hat{S}_i^\alpha(t\neq 0)\right]^\dagger \neq \left[\hat{S}_i^\alpha(t\neq 0)\right] \qquad \left[\hat{S}_j^\alpha(t\neq 0), \hat{S}_k^\beta(t\neq 0)\right] \neq i\delta_{jk}\epsilon^{\alpha\beta\gamma}\hat{S}_k^\gamma(t\neq 0)$$

Non-unitary evolution

Can the evolution be described in terms of a non-Hermitian Hamiltonian?

Summary

- We have studied the dynamics of a non-reciprocal classical spin chain.
- The dynamics conserves the staggered magnetisation and a pseudo-energy.
- A hydrodynamic treatment can self-consistently yield diffusion for the staggered magnetisation and relaxation for the magnetisation.
- We have numerically confirmed the presence of diffusion for the conserved quantities and relaxation for the magnetisation.
- The model also exhibits the spreading of chaos like its reciprocal counterpart in the form of a ballistic spread of an appropriately defined decorrelator.
- The reciprocal model thus exhibits an analogue of the thermal behaviour of its reciprocal counterpart.

Thanks for your attention