KPZ physics in integrable spin chains

Abhishek Dhar
International centre for theoretical sciences
TIFR, Bangalore

Dipankar Roy, Avijit Das, Manas Kulkarni, Konstantin Khanin, Herbert Spohn

Stability of Quantum Matter, ICTS
January 15-26, 2024

Outline

- KPZ Universality.
- KPZ signatures in classical integrable spin chains: dynamical correlations in equilibrium.
  Das, Kulkarni, Spohn, **AD**, PRE (2019)
  Roy, **AD**, Spohn, Kulkarni, PRB (2023)
- KPZ signatures in classical integrable spin chains: System size scaling of spin current in boundary driven non-equilibrium steady states.
  [Roy, **AD**, Spohn, Kulkarni, arXiv:2306.07864]
- Predictions from nonlinear fluctuating hydrodynamics.
  (special feature: degenerate flux jacobian)
  Roy, **AD**, Khanin, Kulkarni, Spohn (arXiv:2401.06399)

KPZ Universality

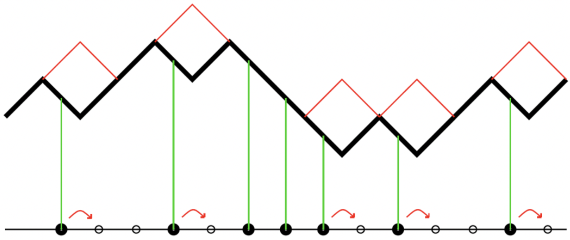
The KPZ equation describes the hydrodynamic (large length and time scales) behaviour of a growing 2D interface (Kardar, Parisi, Zhang, 1986).

$$\partial_t h(x,t) = D\partial_x^2 h(x,t) + \lambda(\partial_x h(x,t))^2 + \sqrt{2D\chi}\eta(x,t)$$

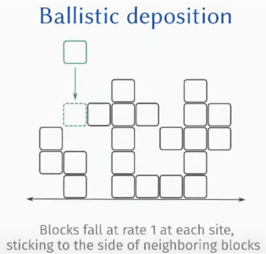
Making the transformation $\partial_x h = \rho$ leads to the noisy Burgers equation:

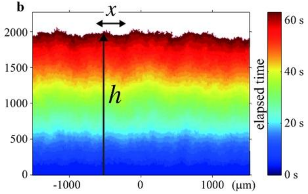
$$\partial_t \rho(x,t) = \partial_x \left[D\partial_x \rho(x,t) + \lambda\rho^2(x,t) + \sqrt{2D\chi}\eta(x,t)\right].$$

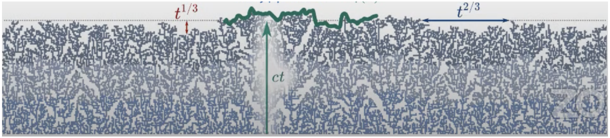
The above equations correspond to the two representations of the TASEP lattice gas model.



KPZ Universality -other examples







Two important length scales: correlations $x \sim t^{2/3}$ height fluctuations $\delta h \sim t^{1/3}$

- Observed in expriments on burning paper, growing bacterial colonies, turbulent liquid crystals [Takeuchi, Sano (2010)]
- TASEP, Polymers in random media, directed percolation, PNG model, random matrices.

KPZ Universality

Aatish Bhatia (NYT Science writer)

KPZ Universality

KPZ universality class is characterized by:

1. Space-time correlations: $\langle \partial_x h(x,t), \partial_x h(0,0)\rangle = \chi \frac{1}{(\Gamma_{\parallel} t)^{2/3}} f_{\mathrm{KPZ}}\left(\frac{x}{(\Gamma_{\parallel} t)^{2/3}}\right)$
2. Height fluctuations $\delta h = h(0,t) - vt$ scales as $t^{1/3}$. Universal distribution, depending on boundary conditions:

$$P\left[\zeta = \frac{\delta h}{(\Gamma_{\perp} t)^{1/3}}\right] = F(\zeta)$$

F is Tracy-Widom (Wedge, flat initial conditions), Baik-Rains (Stationary initial conditions)

Integrable quantum spin chain

Heisenberg spin ($s = 1/2$) chain:

$$H = -\sum_{i=1}^{N} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right]$$

Integrable model with very surprising transport properties. Defining $C(x,t) = \langle S_x^z(t) S_0^z(0) \rangle$ one finds:

- Gapless phase $\Delta < 1$: $C(x,t) = \frac{1}{t} f_1\left(\frac{x}{t}\right)$
- Gapped phase $\Delta > 1$: $C(x,t) = \frac{1}{t^{1/2}} f_2\left(\frac{x}{t^{1/2}}\right)$
- Isotropic point $\Delta = 1$: $C(x,t) = \frac{1}{t^{2/3}} f_3\left(\frac{x}{t^{2/3}}\right)$

Ljubotina, Znidaric, Prosen (2019) — found from numerics that $f_3 \equiv f_{KPZ}$.

Classical integrable spin chain - Ishimori-Haldane model

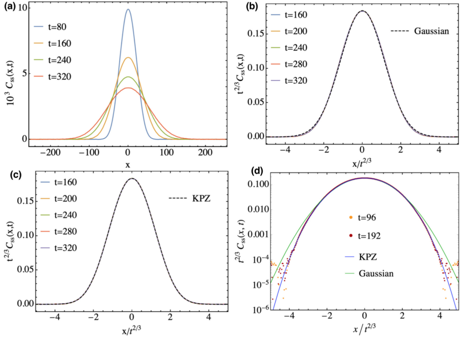
A. Das, M. Kulkarni, H. Spohn, **AD**, PRE (2019)

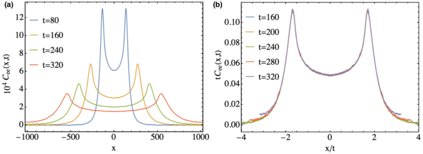
$\mathbf{S}_i$: Classical spins. Hamiltonian [Ishimori (1982), Haldane (1982)]

$$H = -\sum_{x=1}^{N} \ln\left(1 + \mathbf{S}_x.\mathbf{S}_{x+1}\right)$$

$$C_{ss}(x,t) = \langle \mathbf{S}_x(t)\mathbf{S}_0(0)\rangle.$$

Equilibrium canonical ensemble with $T = 1$, $\langle \mathbf{S}\rangle = 0$.





Energy correlations show ballistic scaling

Earlier work: KPZ scaling in current auto-correlations:
Prosen, Zunkovic (2013)- (Faddeev-Takhtajan/ Ishimori-Haldane)

Ishimori-Haldane chain + integrability breaking terms

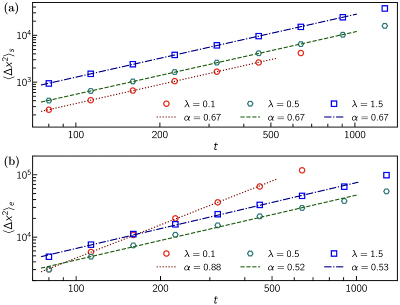
D. Roy, **AD**, H. Spohn, M. Kulkarni, PRB (2023)

Add nonintegrable part to IH model:

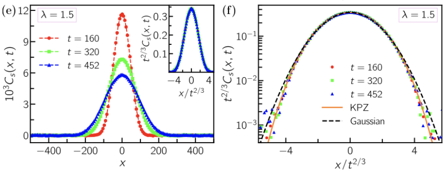
$$H = -\sum_{x=1}^{N} [\ln(1 + \mathbf{S}_x . \mathbf{S}_{x+1}) + \lambda \mathbf{S}_x . \mathbf{S}_{x+1}]$$

$$C_{ss}(x,t) = \langle \mathbf{S}_x(t) . \mathbf{S}_0(0) \rangle .$$

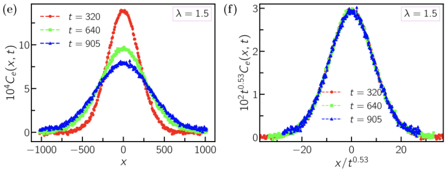
Equilibrium canonical ensemble with $T = 1$, $\langle \mathbf{S} \rangle = 0$.



Spin correlations



Energy correlations



KPZ scaling in nonequilibrium steady state (NESS): Quantum spins

Znidaric, PRL **106** (2011)
Lindblad baths/Linear response/infinite temperature:

$J \sim 1/N^{1/2}$.

J. Stat. Mech. (2011) P12008
Linear response / finite temperature

Prosen, PRL **107** (2011) — strong drive:

$J \sim 1/N^2$

Long range static correlations

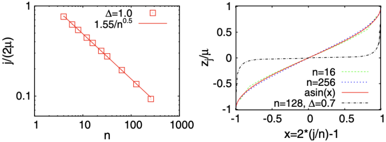


FIG. 2 (color online). Left frame: the scaling of the spin current on the system size n for $\Delta = 1$. The current decays only as $\sim 1/\sqrt{n}$ (solid line), indicating a superdiffusive transport. Right frame: Scaling of the magnetization profile at $\Delta = 1$ (two overlapping dashed curves) is very similar to arcsin$x$ (red solid curve). For $\Delta < 1$ the profile is flat (dot-dashed curve).

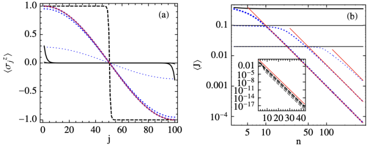


FIG. 1 (color online). Spin profiles $\langle\sigma_j^z\rangle$ at $n = 100$ (a) and spin currents $\langle J\rangle$ vs size n (b), for $\Delta = 3/2$ (dashed line), $\Delta = 1$ (dotted blue line), and $\Delta = 1/2$ (full black line), all for three different couplings $\varepsilon = 1, 1/5, 1/25$ using thick, medium, and thin curves, respectively. Red full curves show closed-form asymptotic results [see the text]: $\sigma_j^z = \cos\pi\frac{j-1}{n-1}$, $\langle J\rangle = \pi^2\varepsilon^{-1}n^{-2}$ for $\Delta = 1$ in the main panels (a),(b) and $\langle J\rangle \propto e^{-n\,\mathrm{arccosh}\Delta}$ in (b), inset.

NESS current scaling with system size

From Green-Kubo we expect [since $\langle J(t)J(0)\rangle \sim t^{-2/3}$]:

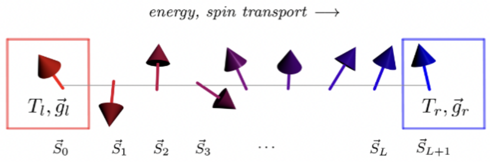
$$\sigma \sim \int_0^{L/v_c} dt \langle J(t)j(0)\rangle \sim L^{1/3} \quad \Rightarrow \quad J \sim \frac{1}{L^{2/3}}.$$

However, this has so far not been observed in quantum systems.

What happens in the classical case? — larger system sizes accessible.

[Roy, Dhar, Spohn, Kulkarni, arXiv:2306.07864] — Study of classical Ishimori-Haldane (IH) chain with boundaries at different temperatures β_ℓ, β_r and magnetic field, g_ℓ and g_r.

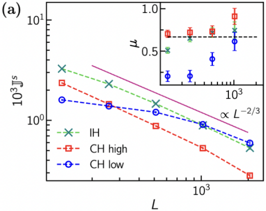
[Baths obtained by equilibrating pairs of boundary spins via Monte-Carlo — can verify thermalization.]

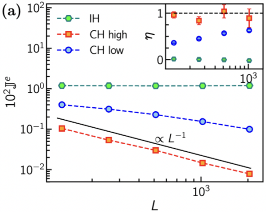


NESS spin and energy current scaling

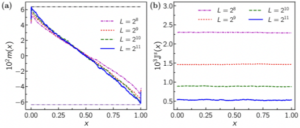
[Roy, Dhar, Spohn, Kulkarni, arXiv:2306.07864]

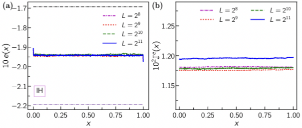
Spin current: $J_s \sim \frac{1}{L^{2/3}}$.
Energy current: $J_e \sim L^0$.



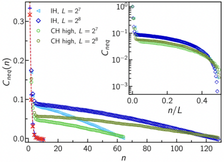


NESS Magnetization and Energy profile





NESS Long-range correlations
$C(n) = \langle S^z_{L/2} S^z_{L/2+n} \rangle$



Integrable spin chains: summary

KPZ universality class is characterized by:

1. Space-time correlations: $S(x,t) = \langle \partial_x h(x,t), \partial_x h(0,0)\rangle = \chi \frac{1}{(\Gamma_{\parallel} t)^{2/3}} f_{\mathrm{KPZ}} \left(\frac{x}{(\Gamma_{\parallel} t)^{2/3}}\right)$
2. Height fluctuations $\delta h = h(0,t) - vt$ scales as $t^{1/3}$. Universal distribution, depending on boundary conditions:

$$P\left[\zeta = \frac{\delta h}{(\Gamma_{\perp} t)^{1/3}}\right] = F(\zeta)$$

F is Tracy-Widom (Wedge, flat initial conditions), Baik-Rains (Stationary initial conditions)

Summary of results for integrable spin chains:

1. Space-time correlations of magnetization $\langle \mathbf{S}(x,t).\mathbf{S}(0,0)\rangle$ show both KPZ scaling and KPZ scaling functions (quantum and classical simulations)
2. Magnetization transfer fluctuations for stationary initial conditions: show KPZ scaling $t^{1/3}$ but the scaling function seems DIFFERENT from Baik-Rains —
[google experiment (Pedram Roushan's talk) and
classical simulations [Krajnik,Schmidt,Ilievski,Prosen, PRL (2024)]

A possible theory: Fluctuating Hydrodynamics?

Phenomenological theories for KPZ scaling: Vasseur/Gopalakrishnan/Ilievski/De Nardis (2018-); Bulchandani (2020)

De Nardis, Gopalakrishnan, Vasseur (PRL, 2023) [DGV] propose effective hydrodynamic equations for the fluctuations in magnetization $m(x,t)$ and of the "fluid velocity" field $\phi(x,t)$.

They obtain the coupled two component noisy Burgers equation:

$$\partial_t m(x,t) + \partial_x \left(m\phi - D_m \partial_x m - \sqrt{2D_m \chi}\xi_m \right) = 0,$$
$$\partial_t \phi(x,t) + \partial_x \left(\frac{\lambda_m}{2} m^2 + \frac{\lambda_\phi}{2} \phi^2 - D_\phi \partial_x \phi - \sqrt{2D_\phi \chi}\xi_\phi \right) = 0.$$

ξ_m, ξ_ϕ uncorrelated white noise.

DGV derivation is based on GHD equations and incorporates role of giant quasiparticles.

Main results:
(i) Spin correlations have KPZ form;
(ii) current fluctuations sum of two Baik-Rains distributions.

Similar approach used for energy current fluctuations in FPU chain: **AD**, Saito, Roy (PRL, 2018).

Degenrate flux Jacobian coupled noisy Burgers equation

Roy, Dhar, Khanin, Kulkarni, Spohn (arXiv:2401.06399)

- These class of equations were first studied by Ertas and Kardar (PRE, 1993)
- General structure: Euler equations for conserved fields u_1, u_2 have the form $\partial_t u_\alpha + \partial_x j_\alpha = 0$. Flux jacobian is given by $A_{\alpha\nu} = \partial j_\alpha / \partial_{u_\nu}$.
- Fluctuations around statinary state, ϕ_1, ϕ_2, satisfy the equations

$$\partial_t \phi_\alpha(x,t) + \partial_x \left(A_{\alpha\nu}\phi_\nu + G^\alpha_{\nu,\mu}\phi_\nu\phi_\mu - D_{\alpha\nu}\partial_x\phi_\nu - B_{\alpha\nu}\xi_\nu \right) = 0,$$

- Degenerate eigenvalues of matrix A implies strongly interacting modes. Usual predictions of nonlinear fluctuating hydrodynamics do not apply — this is the case of interest here.
- For certain choices of the G-matrices, the stationary state fluctuations are Gaussian — DGV study numerically this class.
- A two-lane TASEP type particle model (Popkov, Schutz 2012) leads to exactly the same two-component hydrodynamic equations.

Our work - Numerical study of the DGV Burgers system and the corresponding lattice model.

Coupled KPZ: $m - m$ correlations

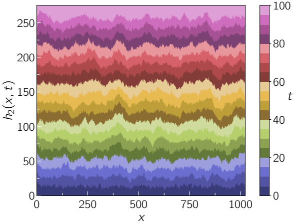
Roy, Dhar, Khanin, Kulkarni, Spohn (arXiv:2401.06399)

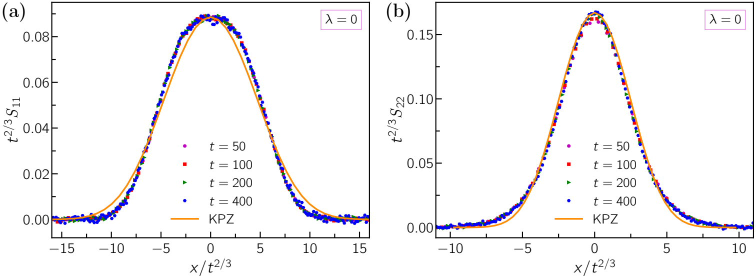
Coupled KPZ equations are:

$$\partial_t h_1 = \partial_x^2 h_1 + 2b\,\partial_x h_1\,\partial_x h_2 + \sqrt{2}\xi_1,$$

$$\partial_t h_2 = \partial_x^2 h_2 + b\left[(\partial_x h_1)^2 + \lambda(\partial_x h_2)^2\right] + \sqrt{2}\xi_2.$$

NOTE: $m = \partial_x h_1, \;\; \phi = \partial_x h_2$.

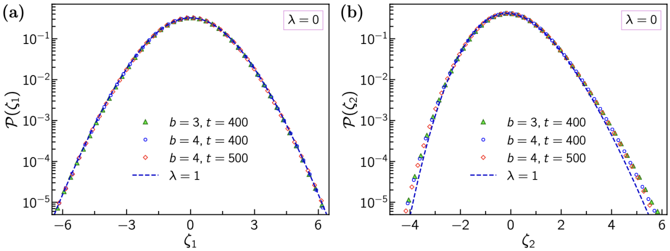




- KPZ $t^{2/3}$ scaling - YES
- KPZ scaling function - NO

Coupled KPZ: Current Fluctuations

Roy, Dhar, Khanin, Kulkarni, Spohn (arXiv:2401.06399)



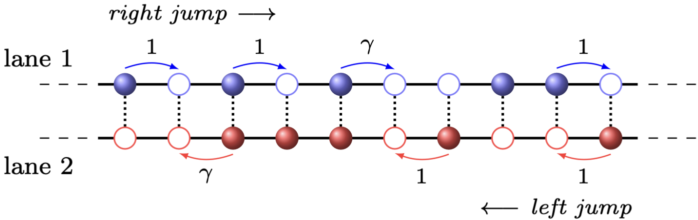
Distribution	$\mathbb{M}$	$\mathbb{V}$	$\mathbb{S}$	$\mathbb{K}$
Baik-Rains [41, 42]	0	1.150	0.359	0.289
ζ_1	0	1.545	0	0.152

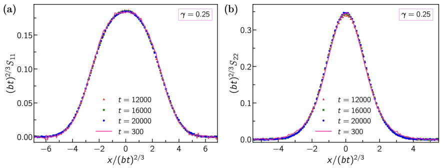
Current fluctuations can be interpreted as sum of two Baik-Rains distributions.

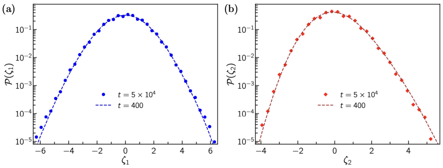
However — this is not what is observed for the actual spin model.

Coupled KPZ and lattice two-lane model

Roy, Dhar, Khanin, Kulkarni, Spohn (arXiv:2401.06399)







Data from the two models (continuum Burgers and two-lane TASEP) are compared after a rescaling of time — leads to perfect agreement of the scaling functions (Universality).

Summary

Anomalous spin transport in the isotropic Heisenberg spin chain.

- Results for Classical integrable spin chain — Ishimori Haldane model
  - Spin-spin dynamical correlations described by the KPZ scaling function. Energy transport ballistic.
  - KPZ correlations survive strong integrability breaking, but symmetry preserving perturbations. The same perturbation causes energy transport to become diffusive.
  - Studied boundary driven nonequilibrium steady state (NESS): Observed the expected system-size scaling $J \sim \frac{1}{L^{2/3}}$.
    Need to understand the difference from quantum case.
- Results for coupled KPZ equations with degenerate flux Jacobian.
  - Numerical results for KPZ equations and corresponding lattice model.
  - Spin correlations and current fluctuations show KPZ scaling ($x \sim t^{2/3}$).
  - Spin correlations do not follow KPZ scaling function
  - Established universality between KPZ equation and coupled TASEP model.

Open question: the recently proposed fluctuating hydrodynamic theory explains the observed KPZ scaling but do not seem to reproduce the observed scaling functions. Further numercial work needed to confirm this.