Surprises in Slow Spheroid Sedimentation

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Chajwa, Menon, SR, Govindarajan, Phys Rev X **10**, 041016 (2020)
Chajwa, Menon, SR PRL **122**, 224501 (2019)

OUTLINE

- Background
  - instability and fluctuations in slow sedimentation
- Two discs: Kepler orbits and more
  - inertia from gravity; gravity from fluid mechanics
- The delicate dynamics of disc arrays
  - phantom springs and “stable sedimentation
  - non-normal dynamics and transient algebraic growth
- Summary

BACKGROUND

13 August 1819 - 1 February 1903

Stoked about Stokes: Nat Rev Phys 2019

density ρ, viscosity μ, velocity U, size a, Re = ρUa/μ << 1

Ignore inertia: velocity field **u**(**x**) obeys the Stokes equation

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

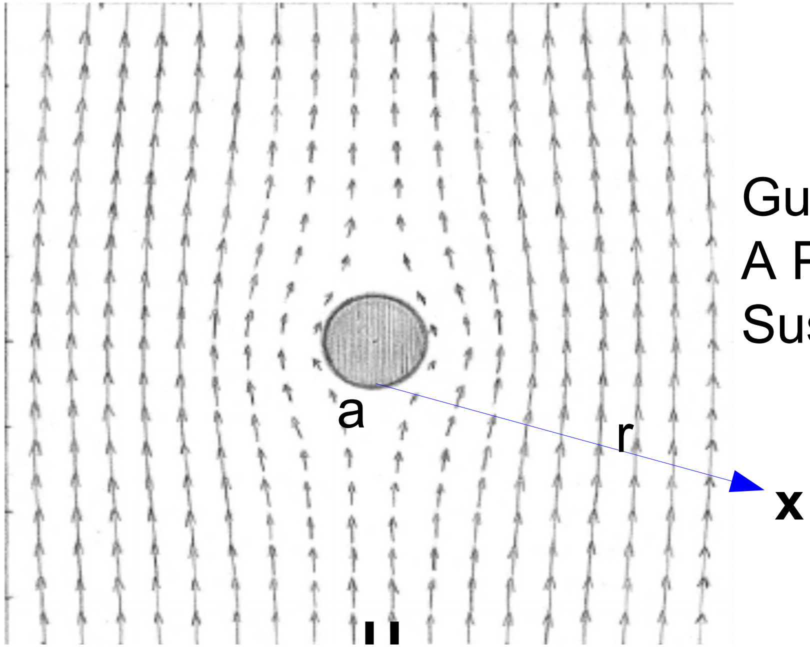
pressure forces

$$\nabla \cdot \mathbf{u} = 0$$

incompressibility

www.damtp.cam.ac.uk/user/hinch/teaching/CISMejh12handout.pdf

Flow around one sedimenting particle

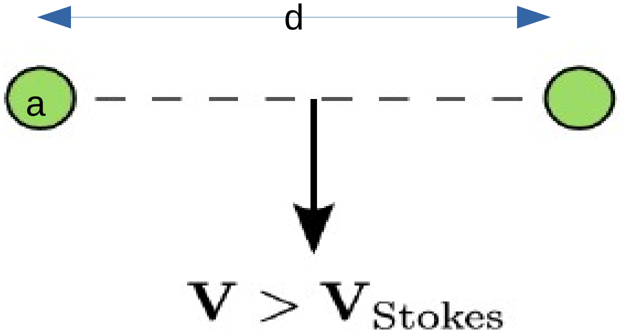


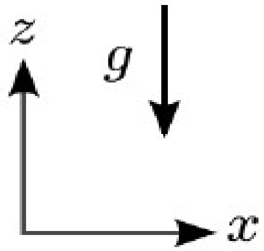
Guazzelli & Morris (& Pic)
A Physical Introduction to Suspension Dynamics

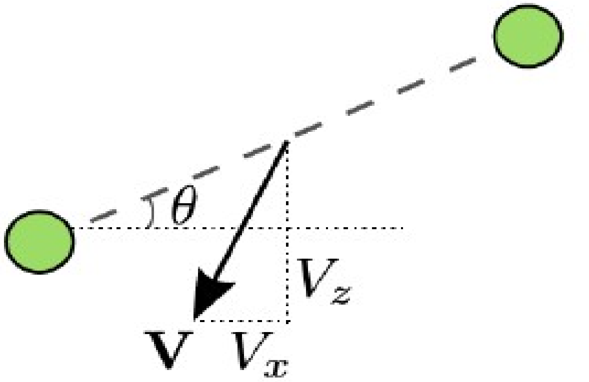
$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right)$$

www.damtp.cam.ac.uk/user/hinch/teaching/CISMejh12handout.pdf

Two settling spheres: the line–of–centres force







$$F_D = 6\pi a\mu U\left[1 - (3/4)(a/d)\right]$$

Mutual drag reduction

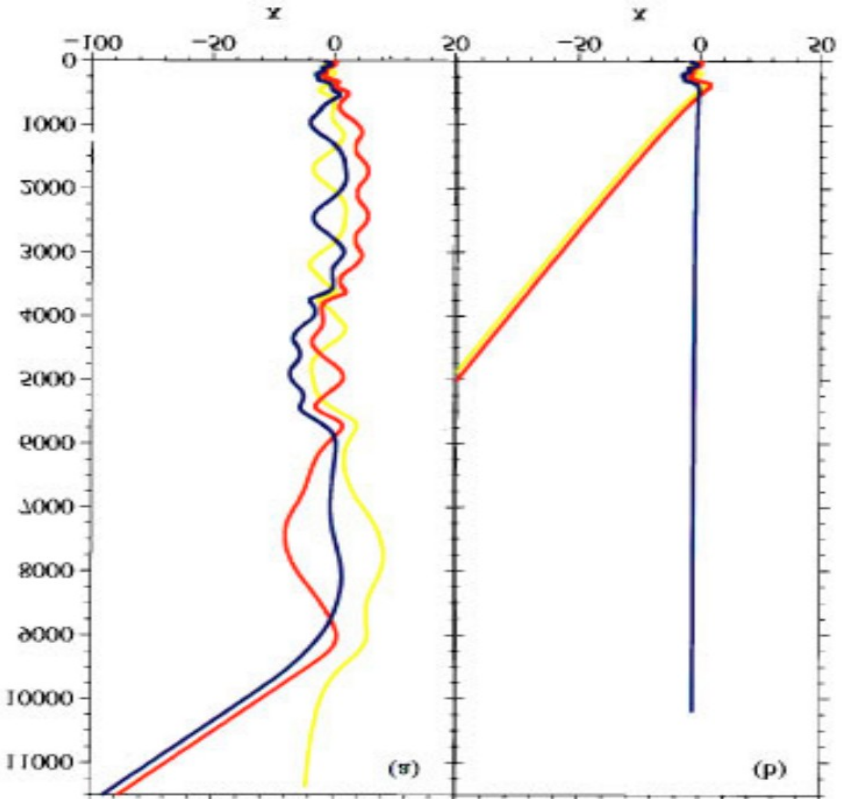
$$F_{LC} = 6\pi a\mu U(3/4)(a/d)\sin\theta$$

Line-of-centres force

Russel, Savile & Showalter 1989
K Vijay Kumar, IISc PhD thesis 2010

Three–particle Stokesian sedimentation is chaotic

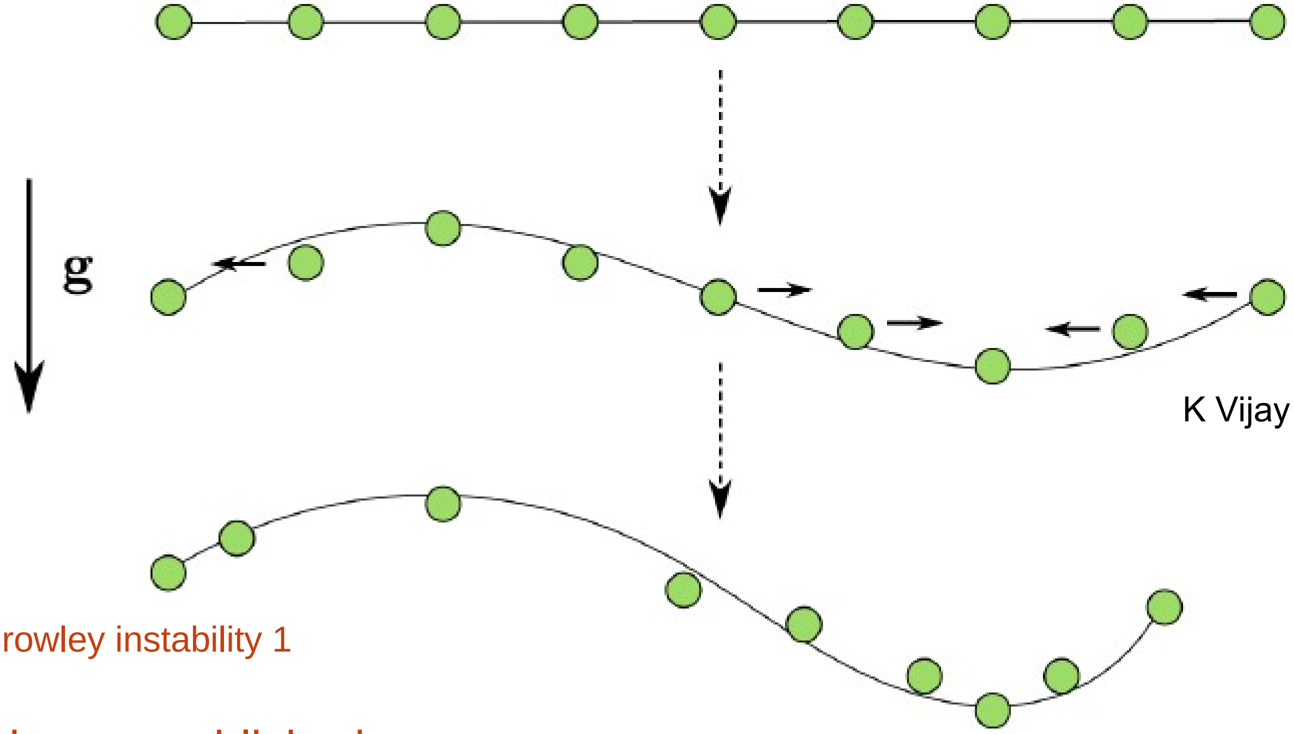
three discs



Janosi et al.
Phys Rev E 1997

CROWLEY'S INSTABILITY

J.M. Crowley, JFM **45**, 151-159 (1971); Phys Fluids **19**, 1296 (1976)



K Vijay Kumar, IISc PhD thesis 2010

Sphere array: Crowley instability 1

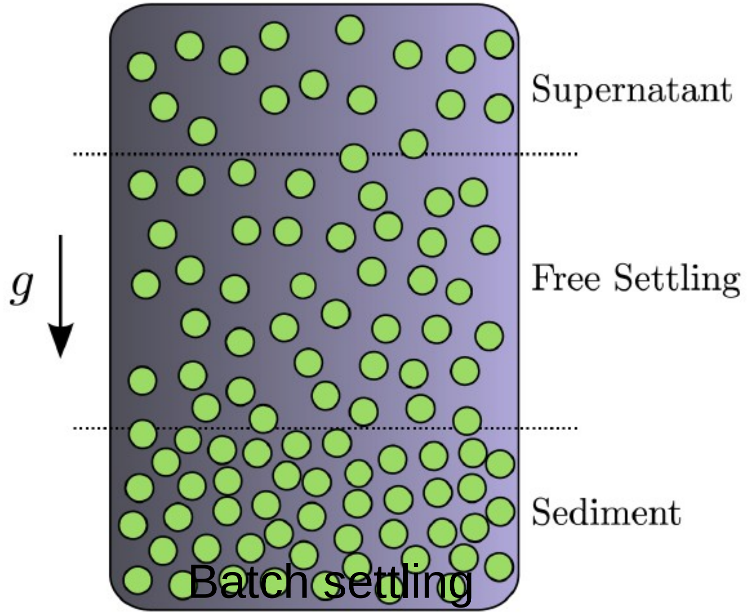
Rahul Chajwa, unpublished

Sphere array: Crowley instability 2

Pair dynamics is the building block

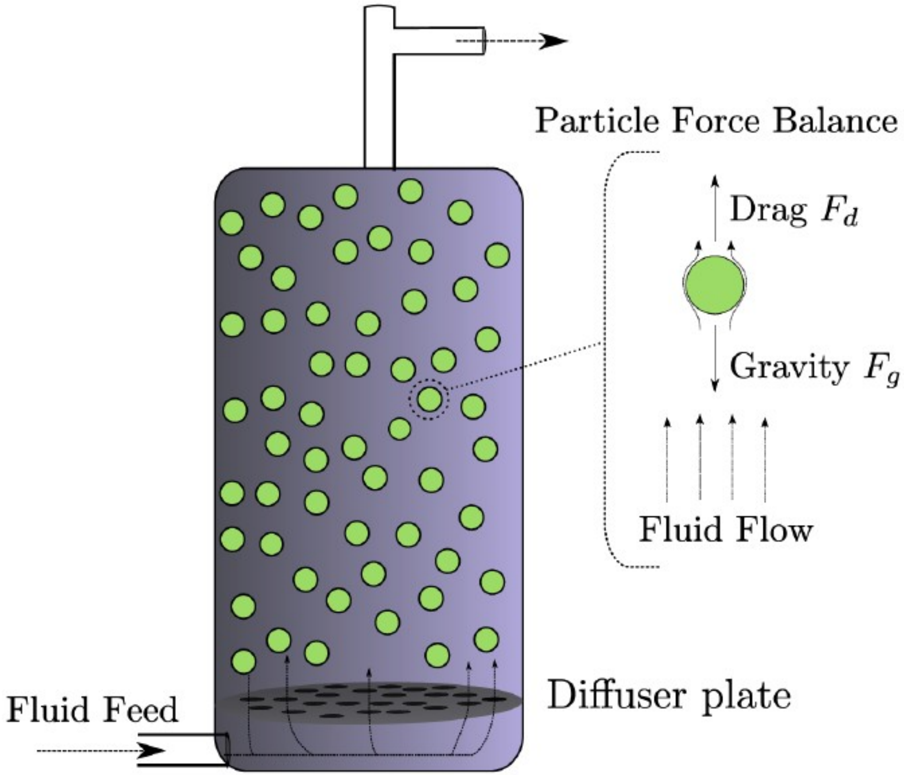
Strong phase separation: Lahiri, Barma, SR 1997-2000

Sedimentation
many-body long-range statistical mechanics



Batch settling

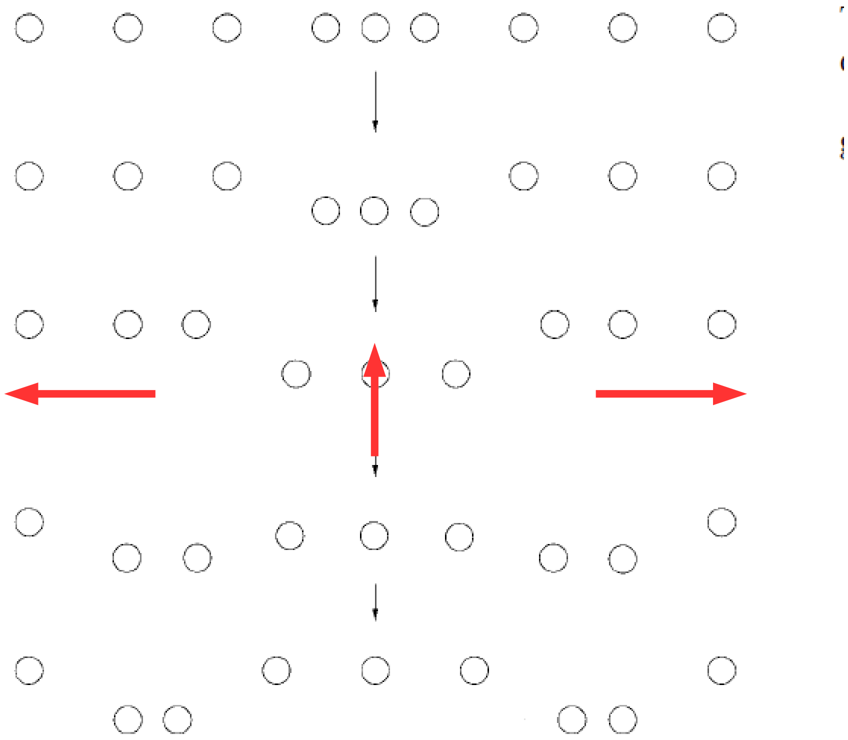
The velocity fluctuations problem
Caflisch-Luke 1985
Koch & Shaqfeh
Levine et al 1998; SR Adv Phys 2001
Ladd, Guazzelli, Hinch...



Fluidised bed

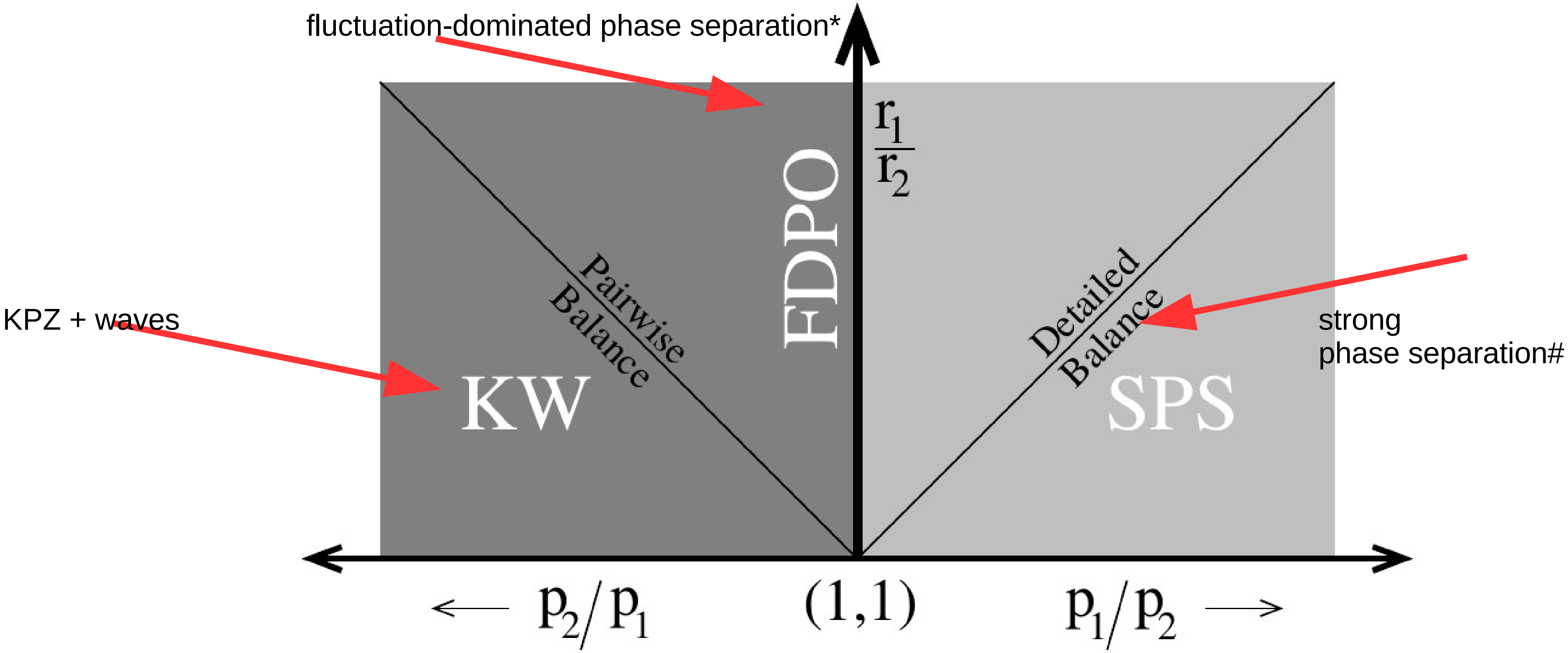
K Vijay Kumar, IISc PhD thesis 2010, modified from
https://upload.wikimedia.org/wikipedia/commons/9/96/FluidisedBed.svg

A stably settling array



Waves in a purely dissipative drifting flux lattice
Simha/SR PRL 1999

Waves vs phase separation: exceptional point



#Lahiri-SR 1997; Lahiri, Barma, SR 2000; Barma, Das, Basu, SR 2002
*Das & Barma PRL 2000; Das, Barma & Majumdar PRE 2001

TWO DISCS
Kepler orbits and more

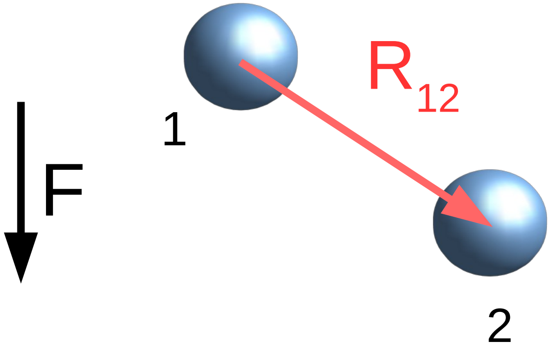
R. Chajwa et al. Phys. Rev. Lett. **122**, 224501 (2019)

Physical Review Lett
@PhysRevLett

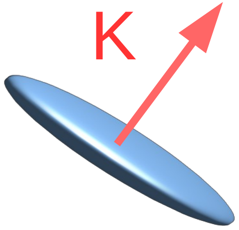
#CoolVideo
Two disks released in a viscous fluid perform graceful dances
reminiscent of the motion of planets in Keplerian orbits.

Stokesian settling

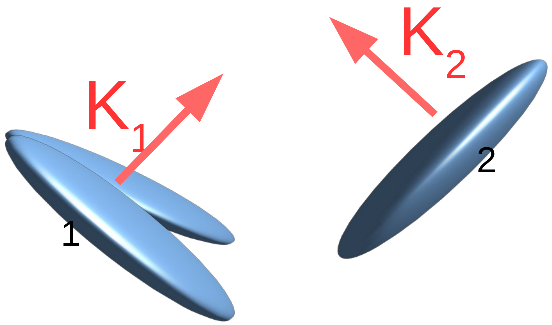
Isotropic pair



Oriented apolar



Oriented apolar pair



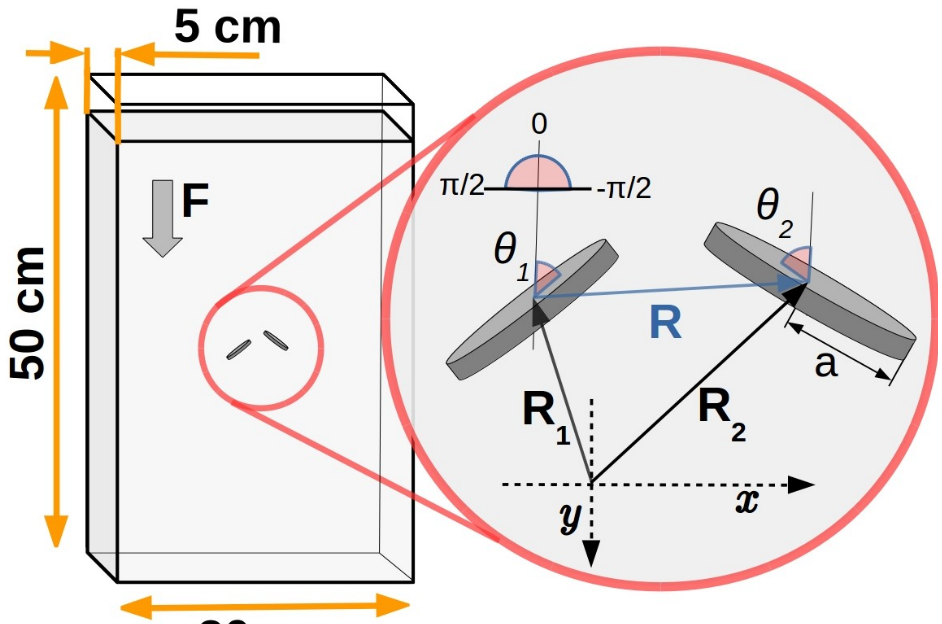
Time reversal symmetry ensures conservation of R_{12} and K

?

Much richer possibilities!

Role of shape: Witten & Diamant arXiv:2003.03698

Settling pair of disks



Control

a

Initial separation **d** and Orientation **θ**

cf. S. Jung et al. PRE (2006)

Experiments

Fluid : silicone oil
60,000 cSt, 0.97 gm/cc

Disk: aluminium
Diameter 2a=12mm
Thickness t=1mm
Density 2.7m/cc

Re ~ 10^{-4}

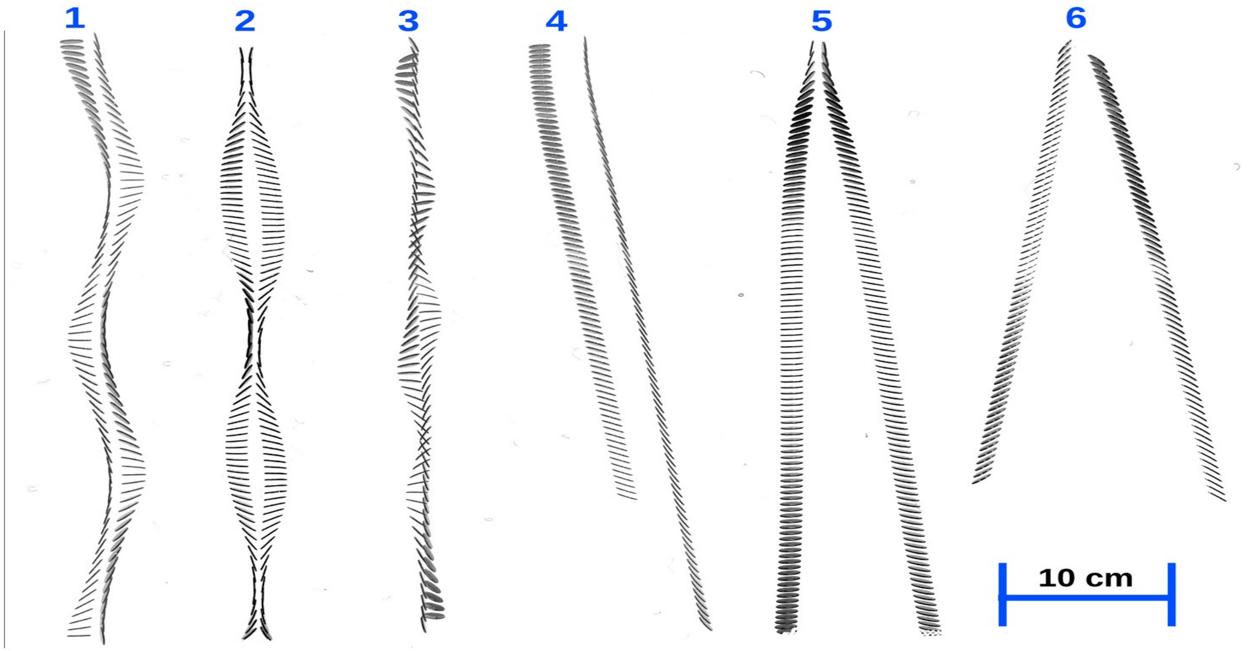
System size:
Height 50 cm ~ 80a
Width 30 cm ~ 48 a
Thickness 5 cm ~ 7a

symmetric movie

Rich behaviour of pairs

We observe

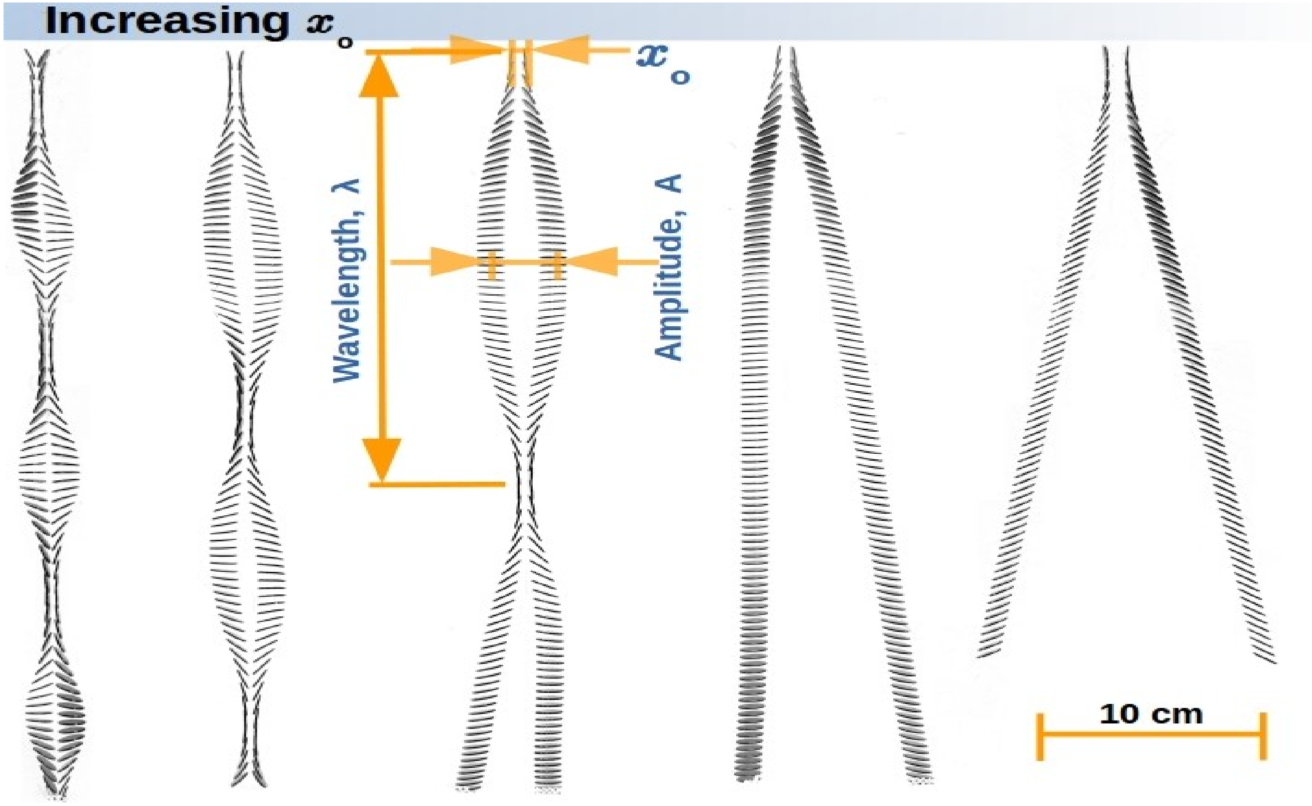
Bound period orbits (1-3) and scattering (4-6)



Our focus

- Quantify the transition from bound to scattering
- Demarcate the phase boundary in (R_i, θ_i) plane, i = 1,2
- A theory based on hydrodynamic interactions

Symmetric settling

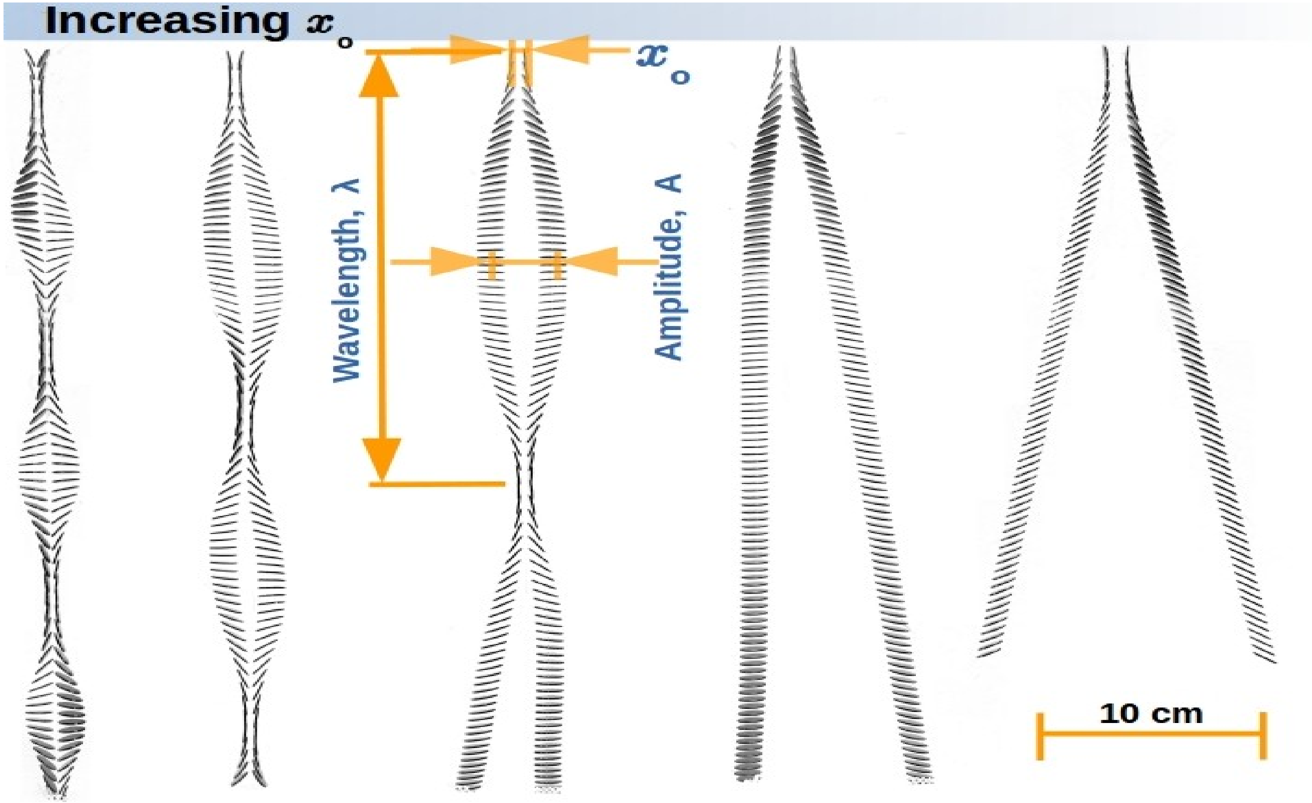


symmetric bound to scattering movie

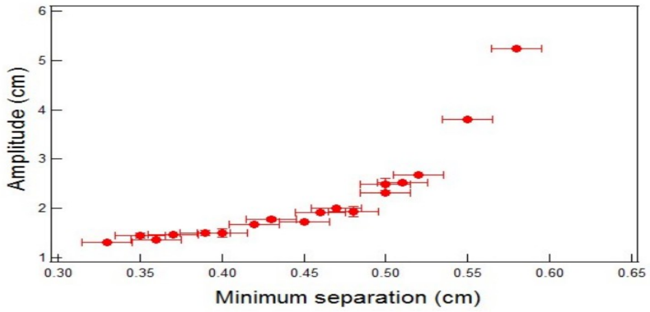
Perp: bound to scattering movie

Rocking movie

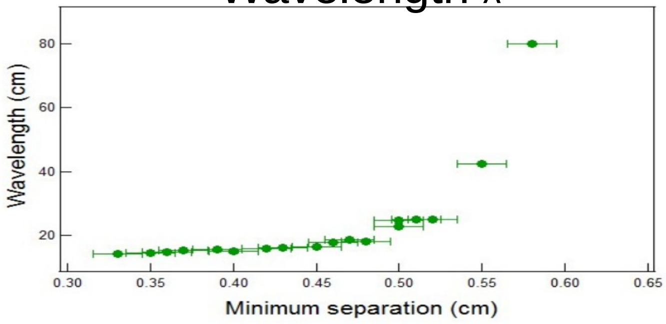
Symmetric settling



Amplitude A



Wavelength λ



symmetric bound to scattering movie

Perp: bound to scattering movie

Rocking movie

Bound-to-scattering transition?
A, λ diverge at threshold min-sep?

Mapping to Kepler orbits: effective Hamiltonian

$$\theta^- \equiv \theta_2 - \theta_1 \qquad x \equiv x_2 - x_1$$

$$\mathcal{H} \equiv 4F\alpha \ \sin^2 \frac{\theta^-}{2} + 2F\gamma/x$$

$$\dot{x} = \partial_{\theta^-}\mathcal{H}$$

$$\dot{\theta^-} = -\partial_x \mathcal{H}$$

$$\frac{1}{x} - \frac{1}{x_o} = \frac{\alpha}{\gamma}(\cos\theta^- - \cos\theta^-_o)$$

Far-field equations

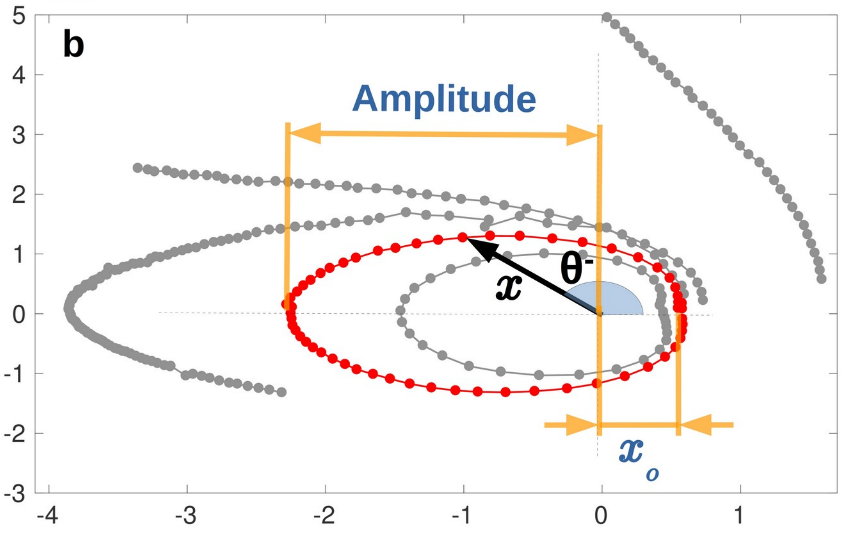
Mapping to Kepler orbits

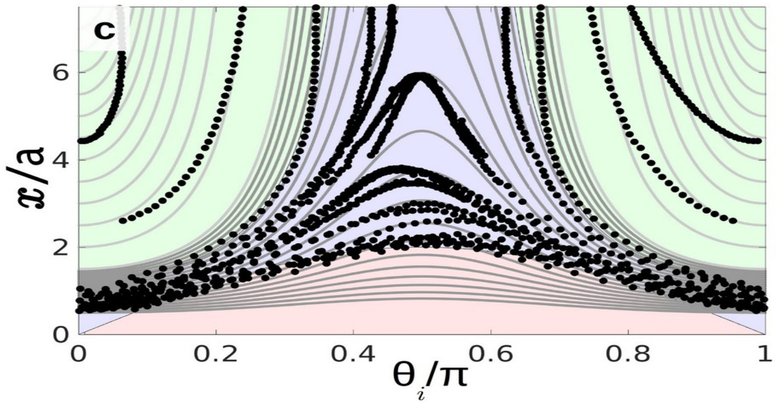
Effective Hamiltonian

$$\mathcal{H} \equiv 4F\alpha\ \sin^2 \frac{\theta^-}{2} + 2F\gamma/x$$

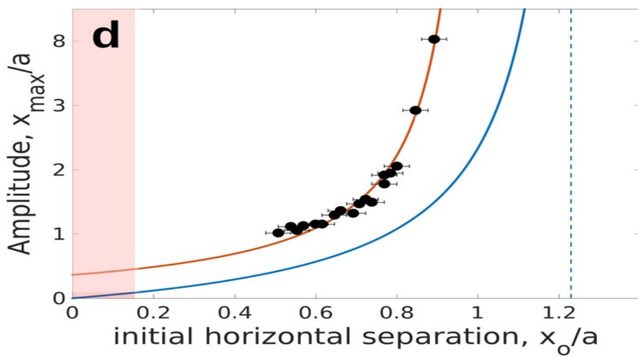
Kepler orbits

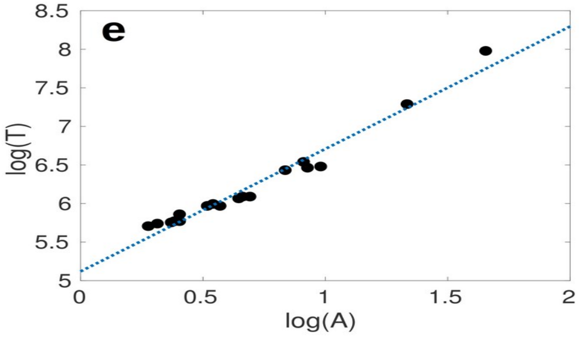
$$\frac{1}{x} - \frac{1}{x_o} = \frac{\alpha}{\gamma}(\cos\theta^- - \cos\theta_o^-)$$





Kepler's 3rd Law



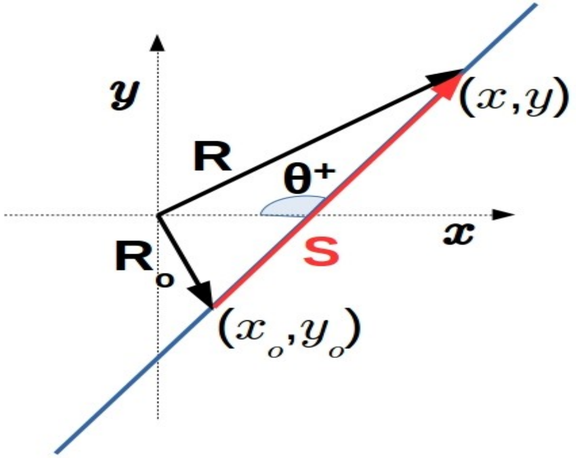


Effective Hamiltonian for tilted pairs too

$$x \equiv x_2 - x_1,\; y \equiv y_2 - y_1,$$

$$\theta^- \equiv \theta_2 - \theta_1 \text{ and } \theta^+ \equiv \theta_1 + \theta_2$$

$$\dot{S} = \partial_{\theta^-}\mathcal{H}. \qquad \dot{\theta^-} = -\partial_S \mathcal{H}$$



Effective Hamiltonian in (S, θ⁻) plane

$$\mathcal{H} \equiv 4F\alpha\ \sin^2\frac{\theta^-}{2} + 2F\frac{\bar{\gamma}(S)}{R(S)}$$

$$\bar{\gamma}(S) \equiv \gamma\,(y_o - S\sin\theta^+)\,/\,(y_o\cos\theta^+ + x_o\sin\theta^+)$$

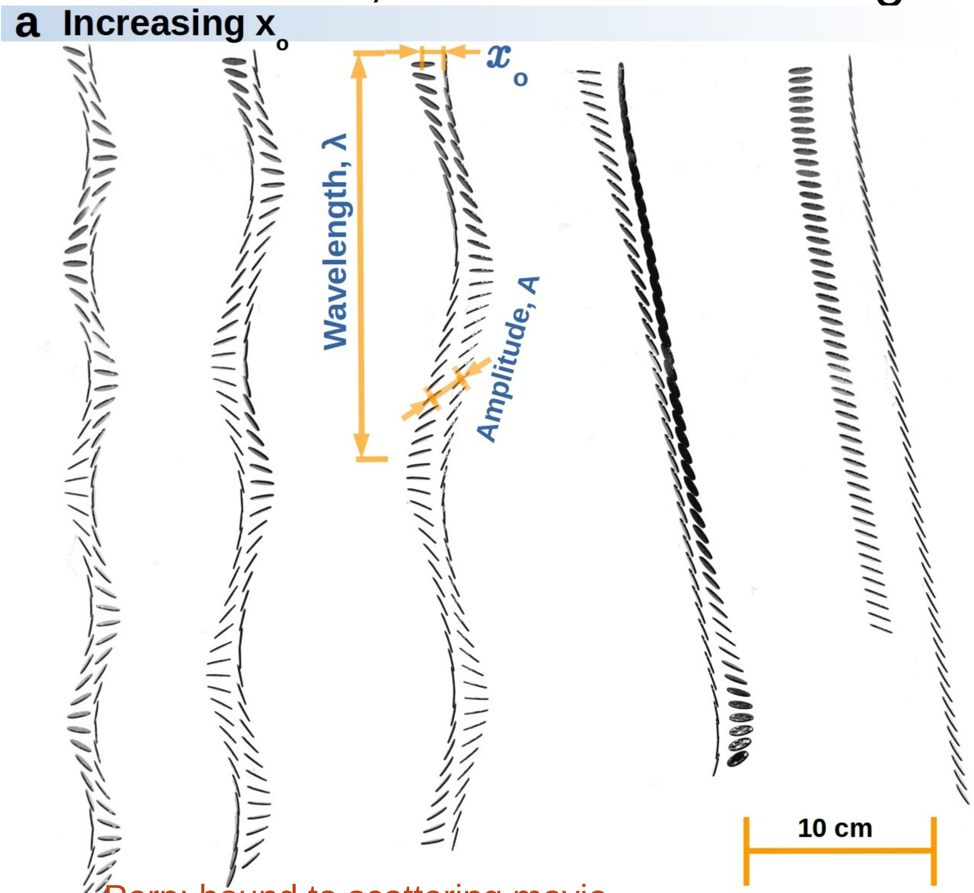
$$R(S) = (S^2 + {R_o}^2 + 2Sx_o\cos\theta^+ + 2Sy_o\sin\theta^+)^{1/2}$$

Keplerian limit: $\theta^+ \rightarrow 0, \pi$

Orientation θ^- mimics momentum

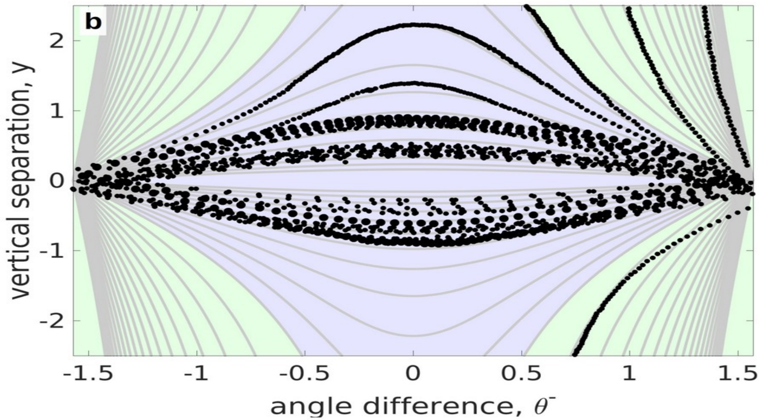
Richer than Kepler

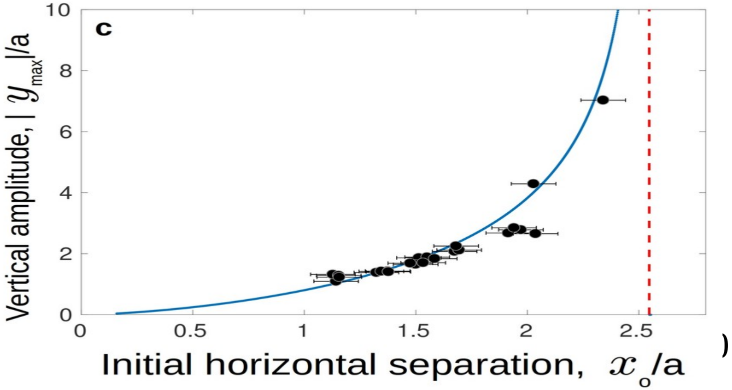
$\theta^+ \longrightarrow \pi/2$, Bound to scattering



Perp: bound to scattering movie

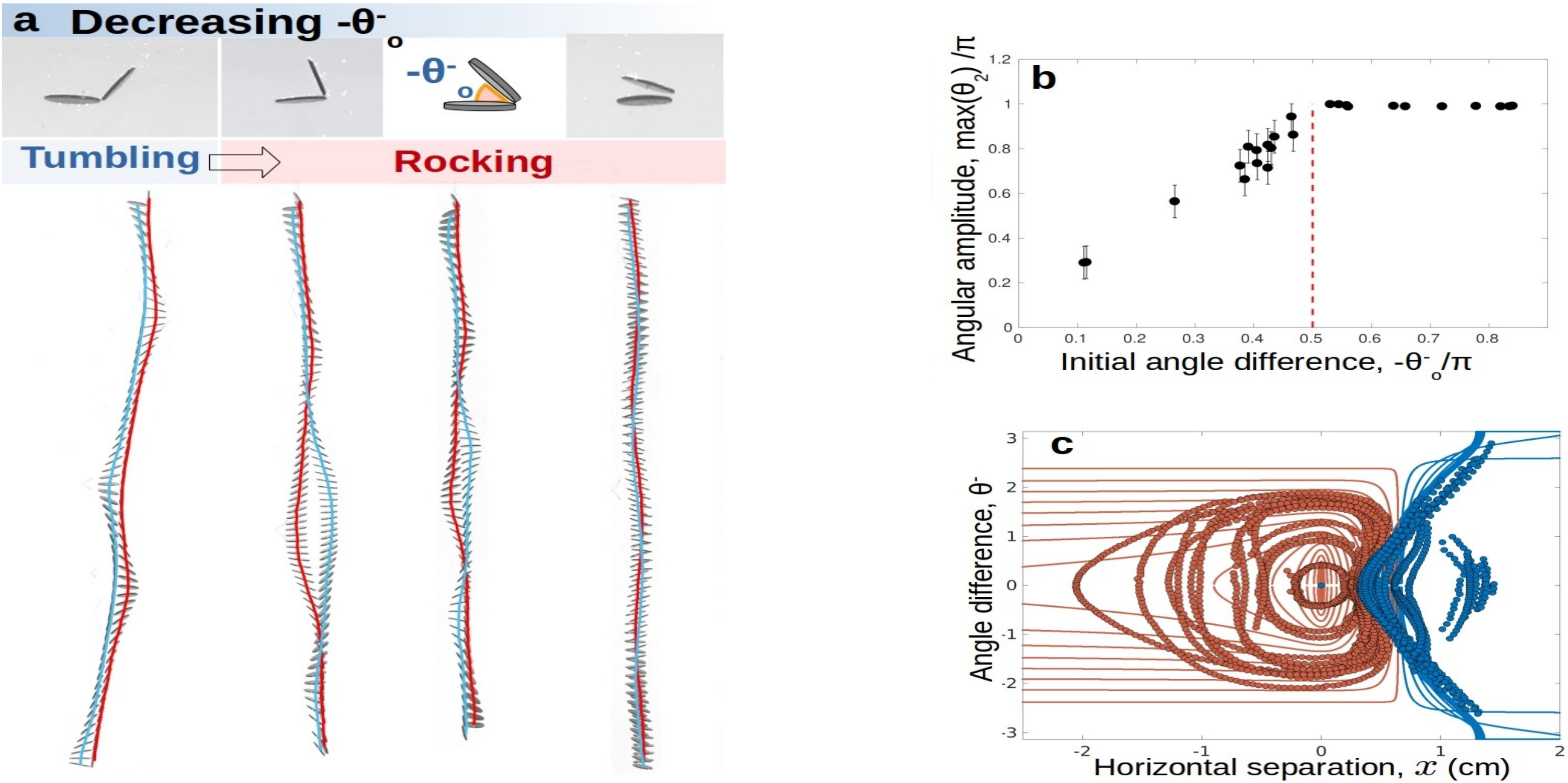
$$y = \pm x_o \cos\theta^- / \sqrt{(8a/\pi x_o)^2 - \cos^2\theta^-}$$





period ~ amplitude^3

Rocking dynamics

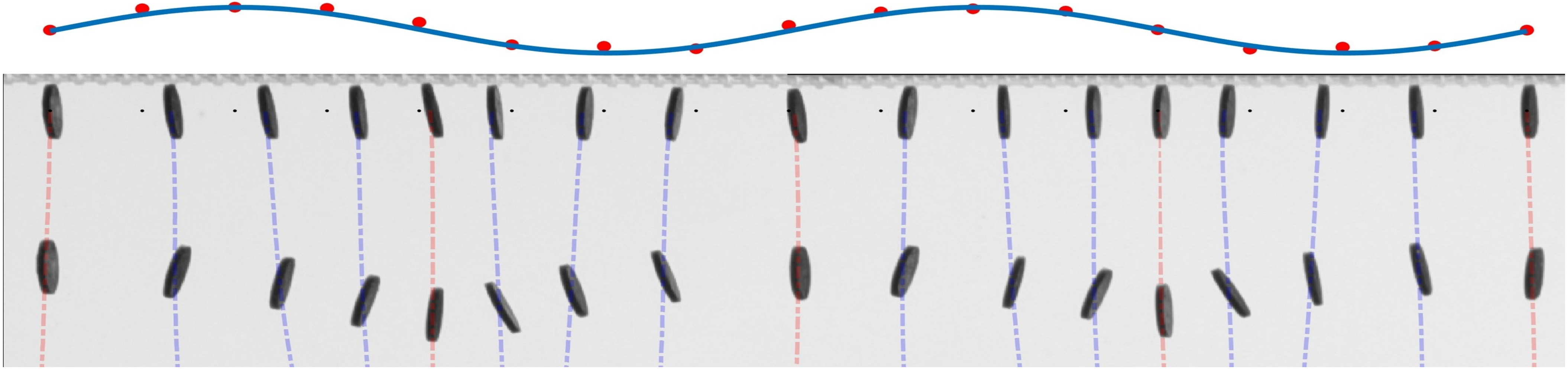


Rocking movie

Summary of two-disc problem

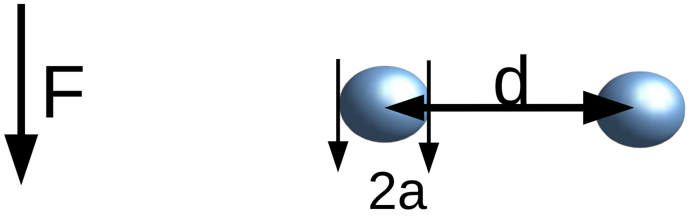
- Under gravity, tilt ~ horizontal motility: “active”
- Emergent inertia and Hamiltonian
- Kepler and non-Kepler orbits
- Many discs, uniform: noisy

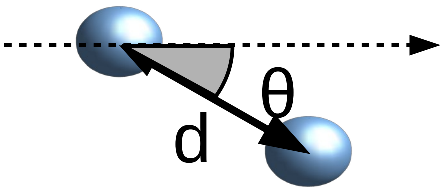
II. Sedimenting disk lattices



R Chajwa, N Menon, SR, R Govindarajan, Phys Rev X **10**, 041016 (2020)

Recall Crowley's sphere array instability



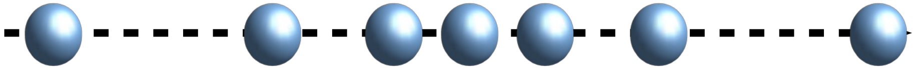


Reduced drag

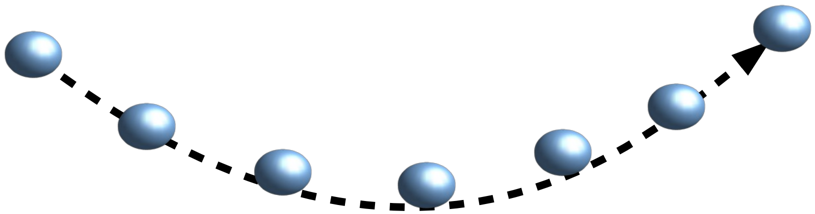
$$U_{||} = \frac{F}{6\pi\mu a}\left(1 + \frac{3a}{4d}\right)$$

Lateral drift

$$U_{\perp} = \frac{F}{12\pi\mu a}\left(\frac{3a}{4d}\right)\sin 2\theta$$



Locally dense region fall faster due to reduced drag, making a valley

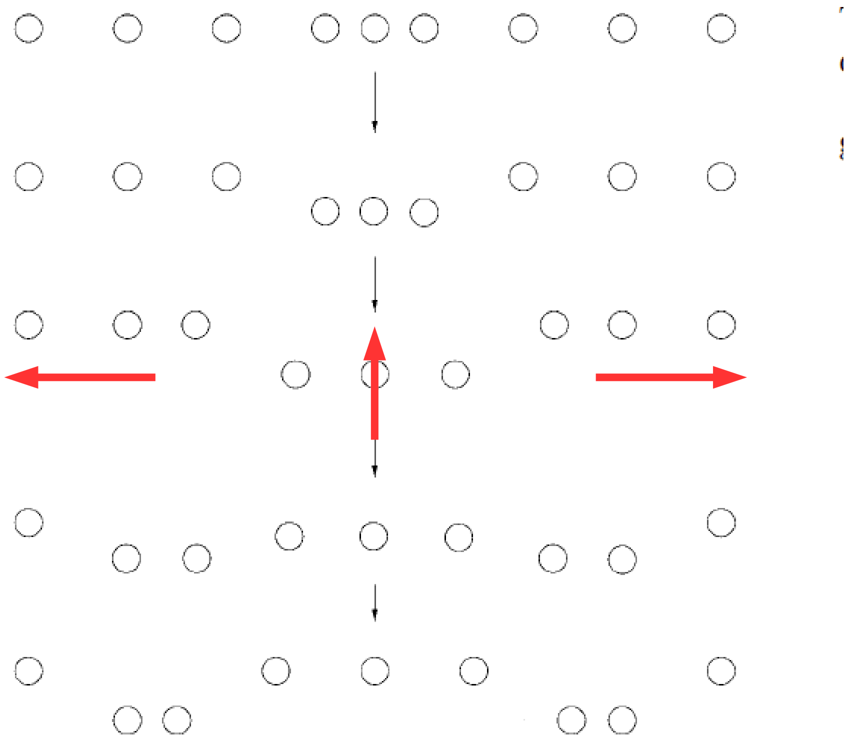


Local valley compresses the lattice, via lateral drift

Crowley_mechanism

J.M. Crowley, JFM **45**, 151-159 (1971); Phys Fluids **19**, 1296 (1976)

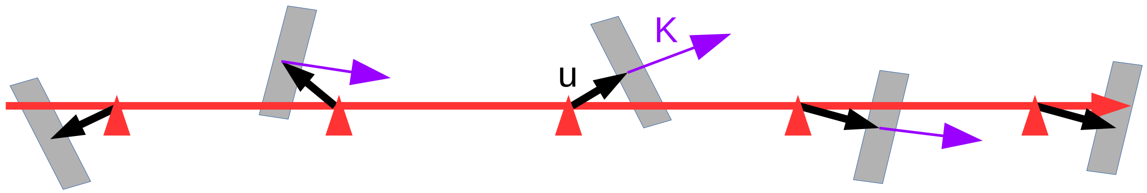
Challenge: stably sedimenting array



Change sign of line-of-centres drift? Compete with it? Discs instead of spheres?

Disc-array hydrodynamics from symmetry

Displacement field **u**, orientation field **K**
Only hydrodynamic interactions.
No pair potential, no elasticity



- Stokesian time reversal: velocities & forces
- Translational invariance
- Rotational invariance in perp subspace
- Symmetry under inversion of orientations

$$\frac{\partial u_x}{\partial t} = \lambda_1 \frac{\partial u_z}{\partial x} + \alpha K_x K_z,$$

$$\frac{\partial u_z}{\partial t} = \lambda_2 \frac{\partial u_x}{\partial x} + \beta K_z^2,$$

$$\frac{\partial K_z}{\partial t} = \gamma K_x \frac{\partial^2 u_x}{\partial x^2}.$$

Linearize about standing discs

$$\omega_\pm = \pm k_x \sqrt{\lambda_1 \lambda_2 + \alpha\gamma}$$

Crowley: $\lambda_1 \lambda_2 < 0$; disc drift $\alpha\gamma$ can compete

Disc-array hydrodynamics from symmetry

Displacement field **u**, orientation field **K**
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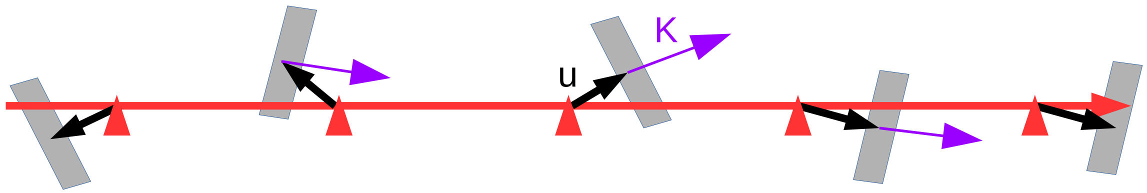

- Stokesian time reversal: velocities & forces
- Translational invariance
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$$\frac{\partial u_x}{\partial t} = \lambda_1 \frac{\partial u_z}{\partial x} + \alpha K_x K_z,$$

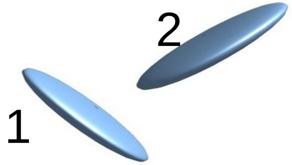
$$\frac{\partial u_z}{\partial t} = \lambda_2 \frac{\partial u_x}{\partial x} + \beta K_z^2,$$

$$\frac{\partial K_z}{\partial t} = \gamma K_x \frac{\partial^2 u_x}{\partial x^2}.$$

Linearize about standing discs

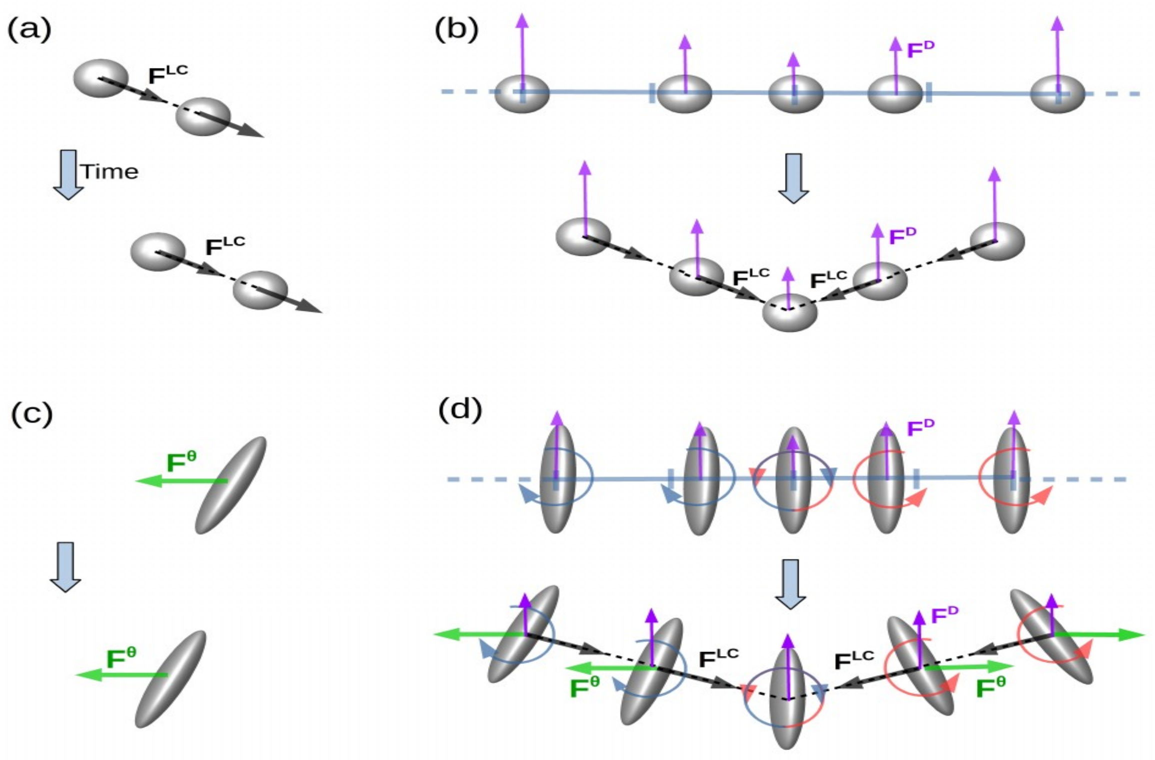
$$\omega_\pm = \pm k_x \sqrt{\lambda_1 \lambda_2 + \alpha\gamma}$$

Stokes time-reversibility + apolar:
no restoring torque in K equation for uniform or nonuniform K

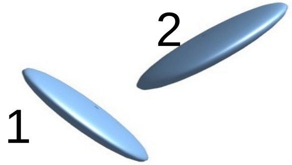


Ingredients of lattice dynamics

Competing mechanism



- S. Wakiya, J. Phys. Soc. Jpn. 20, 1502 (1965)
- S. Kim, Int. J. Multiphase Flow 11, 699 (1985)
- S. Jung et. al. , Phys. Rev. E 74, 035302(R) (2006).
- R. Chajwa et. al. , PRL, 122, 224501 (2019)



Ingredients of lattice dynamics

depends on shape

- Orientational drift: $U_x^0 = \frac{F\alpha(e)}{12\pi\mu a}\sin 2\theta$
- Reduced drag:

$$U_z^1 = -\frac{F}{6\pi\mu a}\left(\frac{3a}{4r}\right)(1 + \frac{(y_1 - y_2)^2}{r^2})$$

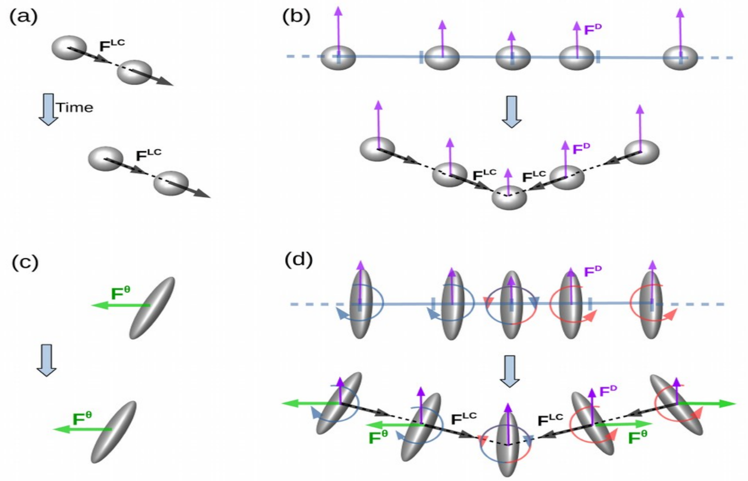
- Line of centres drift:

$$U_x^1 = -\frac{F}{6\pi\mu a}\left(\frac{3a}{4r^3}\right)(x_1 - x_2)(y_1 - y_2)$$

$$\dot{\theta}_1 = \frac{F(x_1 - x_2)}{8\pi\mu r^3}$$

- Mutual rotation:

Competing mechanism



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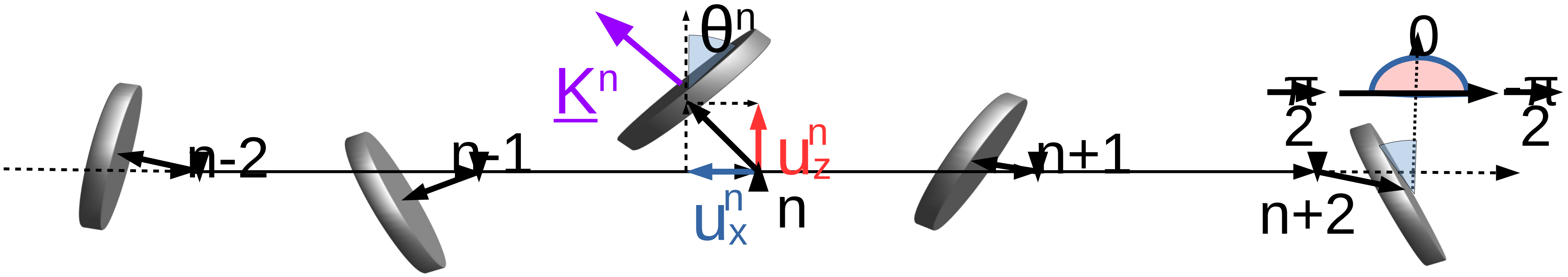
- R. Chajwa et. al. , PRL 122, 224501 (2019)

From pair to collective dynamics



Tilt

Orientation glide

$$\frac{d{u_x}^n}{dt} = -\frac{3a^2}{4d^2}\sum_{l=1}^{\infty}\frac{{u_z}^{n+l} - {u_z}^{n-l}}{l^2} + \frac{\alpha(e)}{2}\sin 2\theta^n$$

Reduced drag

$$\frac{d{u_z}^n}{dt} = +\frac{3a^2}{4d^2}\sum_{l=1}^{\infty}\frac{{u_x}^{n+l} - {u_x}^{n-l}}{l^2} + \alpha(e)\sin^2\theta^n$$

Rotational coupling

$$\frac{d\theta^n}{dt} = \frac{3a^3}{2d^3}\sum_{l=1}^{\infty}\frac{{u_x}^{n+l} + {u_x}^{n-l} - 2{u_x}^n}{l^3}$$

Lattice of masses and springs

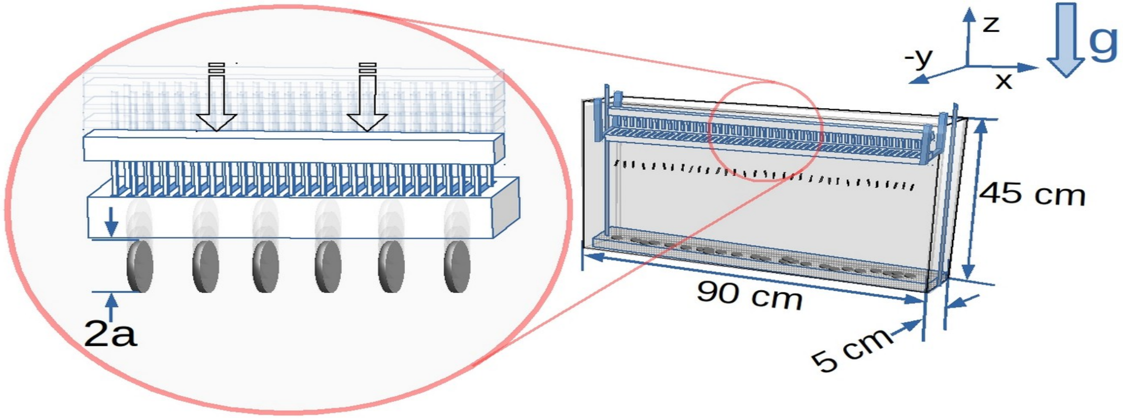
No damping of “momentum” at zero or nonzero wavenumber (apolar + Stokes T-reversibility)

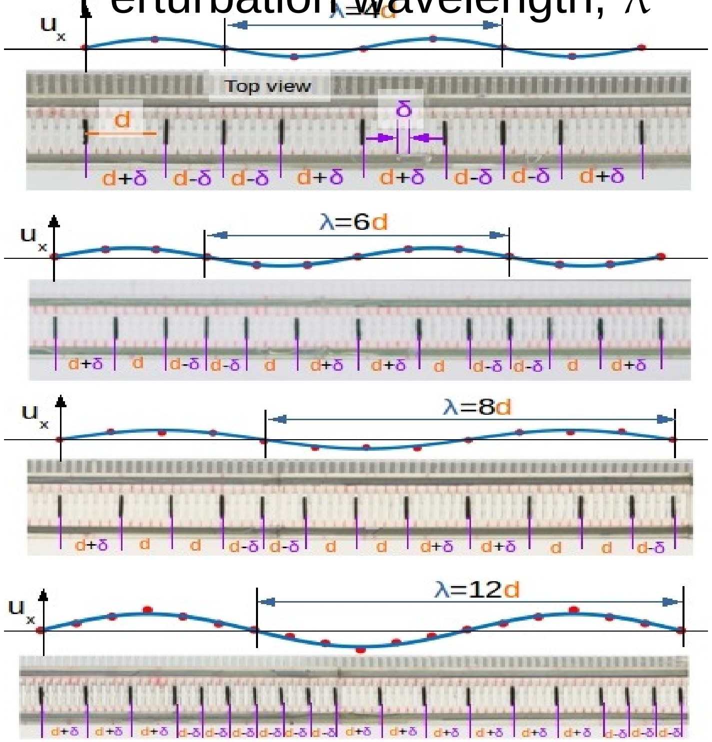
Experiments

Control:

Lattice spacing, d

Perturbation wavelength, λ





Fluid: silicone oil, 5,000 cSt, 0.97 gm/cc

Disc: 3D printed with resin
Diameter 2a = 8 mm
Thickness t = 1mm
Density 1.12 gm/cc

Typical Re ~ 10^{-4}

System size: Height 45 cm = 112.5 a ,
Width 90 cm = 225 a Depth 5cm = 12.5 a

Two dynamical regimes in (q,d) plane

Wavelike modes

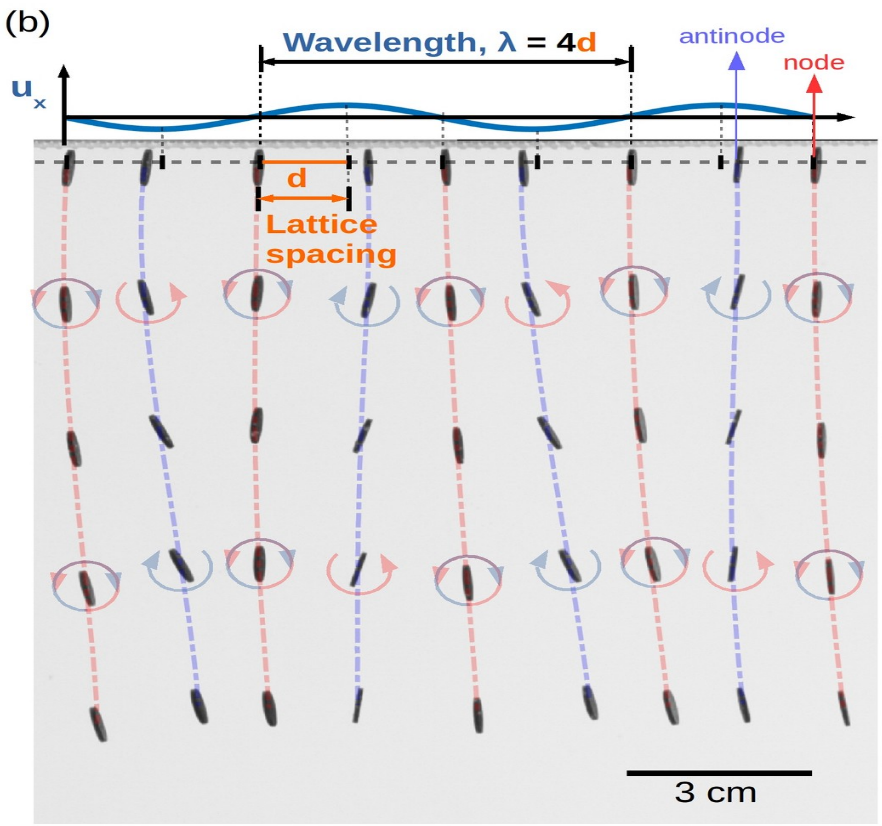
wavelike mode movie

Clumping instability

Unstable mode movie

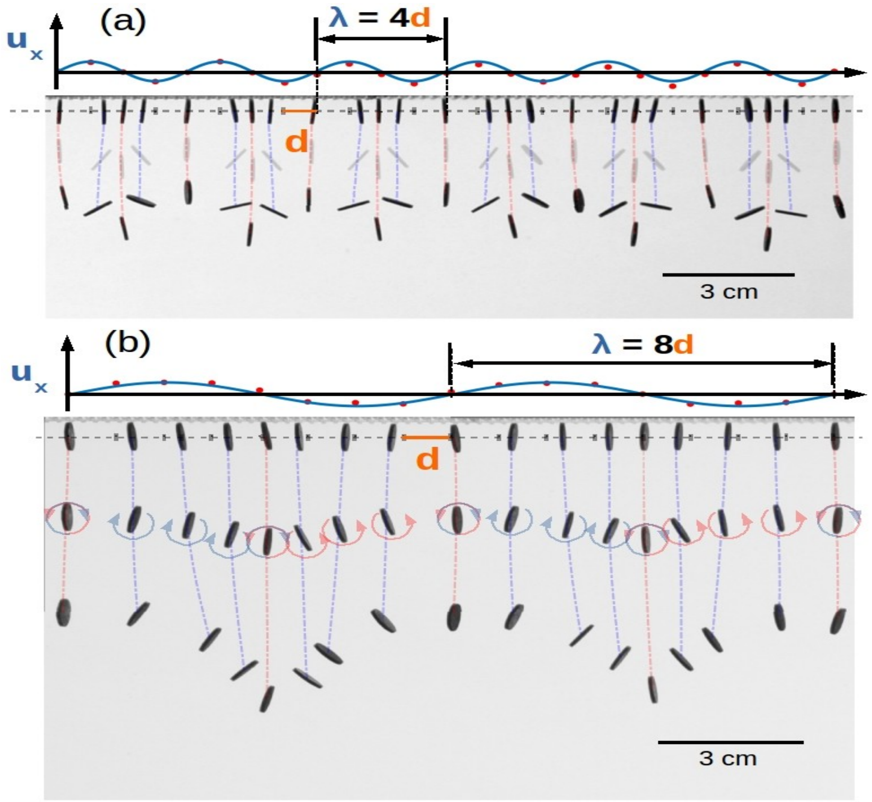
Two dynamical regimes in (q,d) plane

Wavelike modes



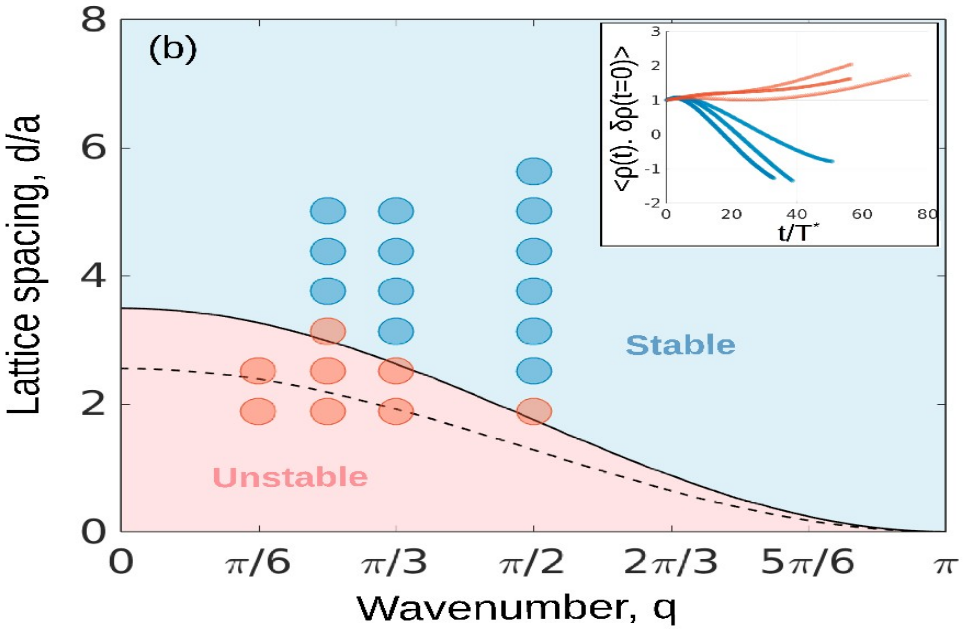
wavelike mode movie

Clumping instability



Unstable mode movie

A universal instability boundary

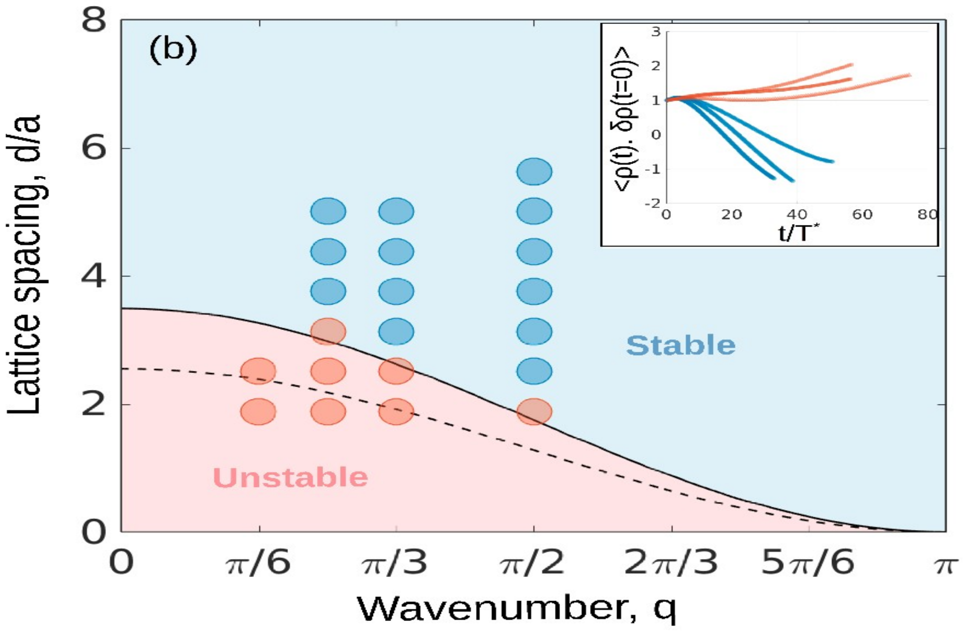


Inset: overlap of time-displaced concentration fields

Dynamical matrix in fourier space

$$\begin{pmatrix} 0 & -(3ia^2/2d^2)\sin q & \alpha(e) \\ +(3ia^2/2d^2)\sin q & 0 & 0 \\ -(6a^3/d^3)\sin^2 q/2 & 0 & 0 \end{pmatrix}$$

A universal instability boundary



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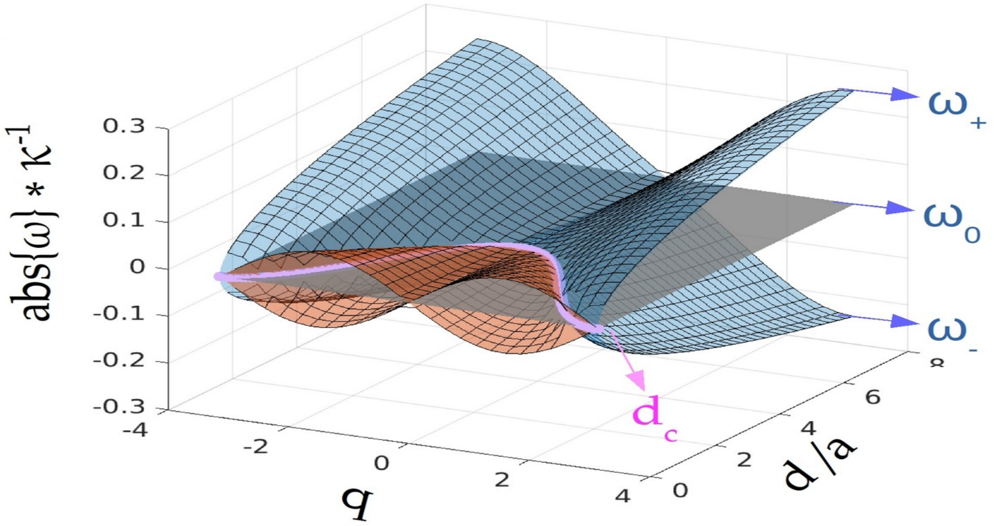
Rescaling lattice spacing, $\tilde{d} = 2d\alpha(e)/3a$

Universal linear stability condition

$$\tilde{d} \geq \cos^2 \frac{q}{2}$$

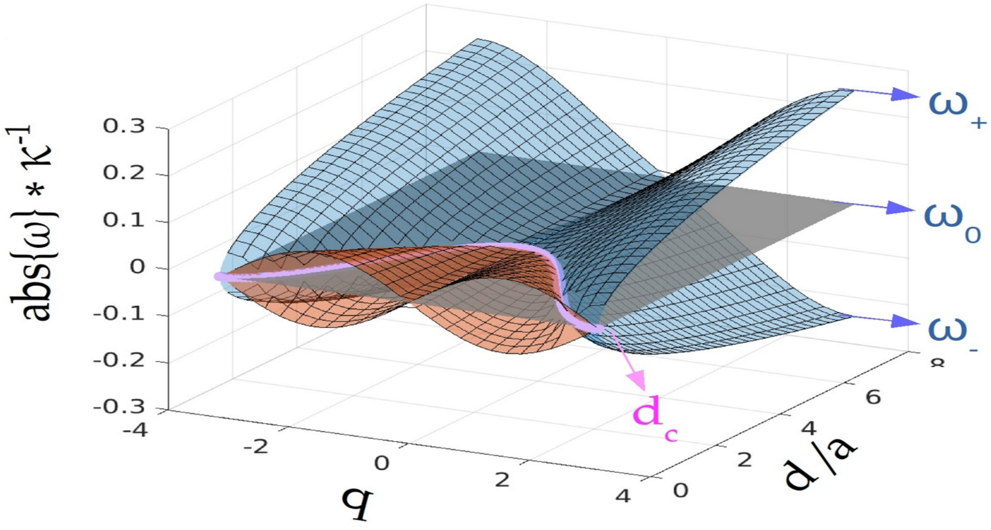
- α depends on geometry of apolar shape
- Known for oblate and prolate spheroids



$$i\omega_{\pm}(q) = \pm\frac{3a^2}{2d^2}\left|\sin\frac{q}{2}\right|\sqrt{\left(-\frac{d\pi}{2a}+4\cos^2\frac{q}{2}\right)}$$

u_x: “broken-symmetry” mode

$\kappa = 6\pi a^2/d^2$



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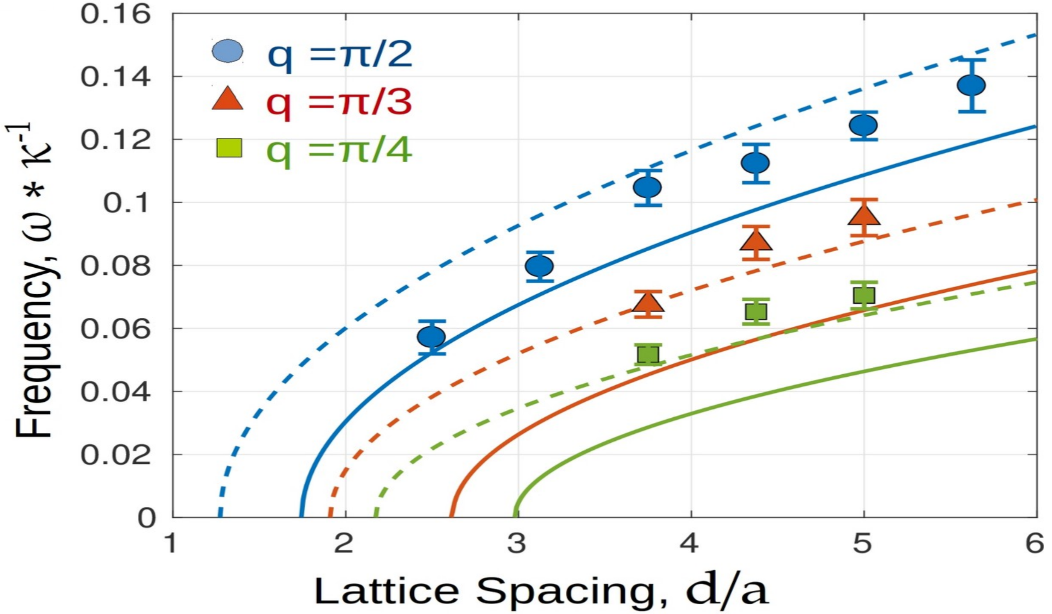
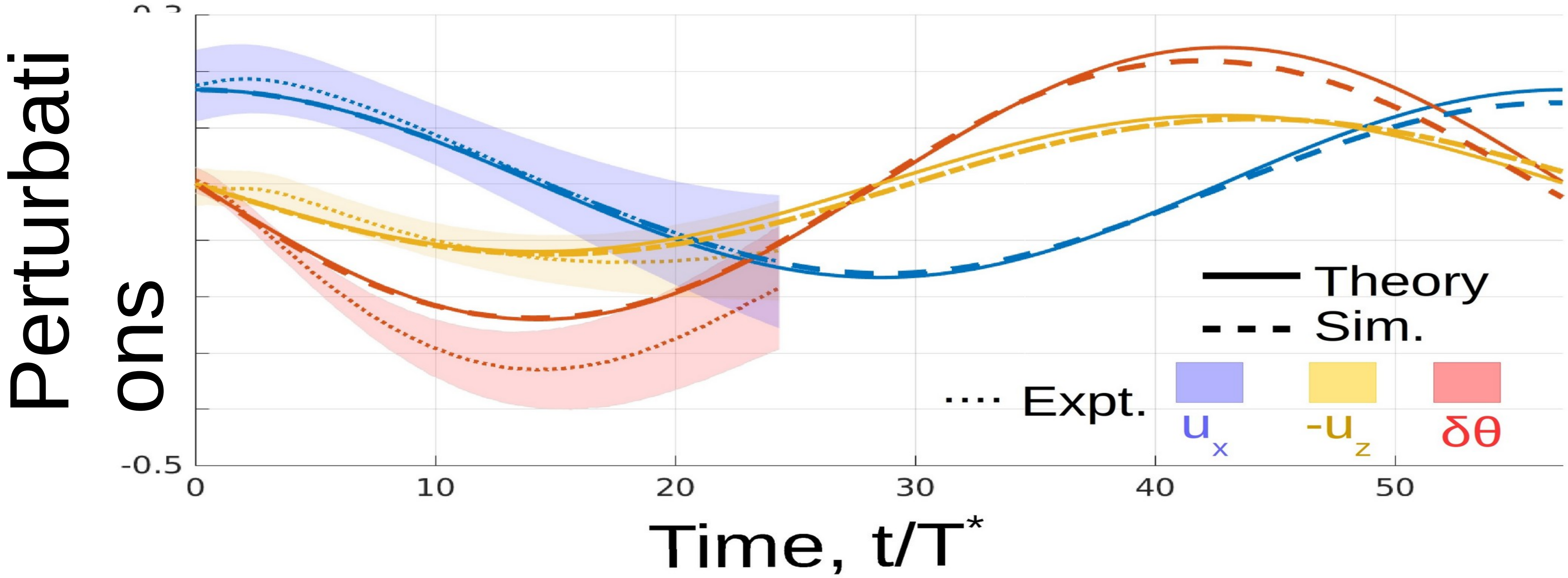


u_x: “broken-symmetry” mode

So: large-spacing disc arrays are stable??

Non-linearly unstable wave



Late times movie

Sim wave movie

$u_z=0$ sector: emergent Hamiltonian dynamics

$$\dot{u}_x^n = \frac{\partial H}{\partial \theta^n}, \quad \dot{\theta}^n = -\frac{\partial H}{\partial u_x^n}$$

$$H = \frac{\alpha(e)}{4}\sum_m (1 - \cos 2\theta^m) + \frac{3a^3}{4d^3}\sum_{l,m}\frac{(u_x^m - u_x^{m+l})^2}{l^3}$$

Mass = $1/\alpha(\mathrm{e})$

Conserved momentum = $\sum_n \theta^n$

Hookean spring stiffness = $3a^3/2d^3$

Rescaling to obtain a natural energy norm

$$\tilde{\mathbf{X}}_q \equiv [U_q, W_q, \Theta_q]^T$$

$$[\sqrt{3a^3/d^3}\sin(q/2)u_q^x, \sqrt{3a^3/d^3}\sin(q/2)u_q^z, \sqrt{\alpha(e)/2}\,\theta_q]^T$$

$$|U_q|^2 + |\Theta_q|^2 = H$$

Crowley

elastic modes

$$\tilde{\mathbf{A}}(q) = \begin{bmatrix} 0 & -i\lambda & 1 \\ i\lambda & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\lambda & 0 \\ i\lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Hermitian + real antisymmetric
Each is *normal*

Rescaling to obtain a natural energy norm

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Crowley elastic modes

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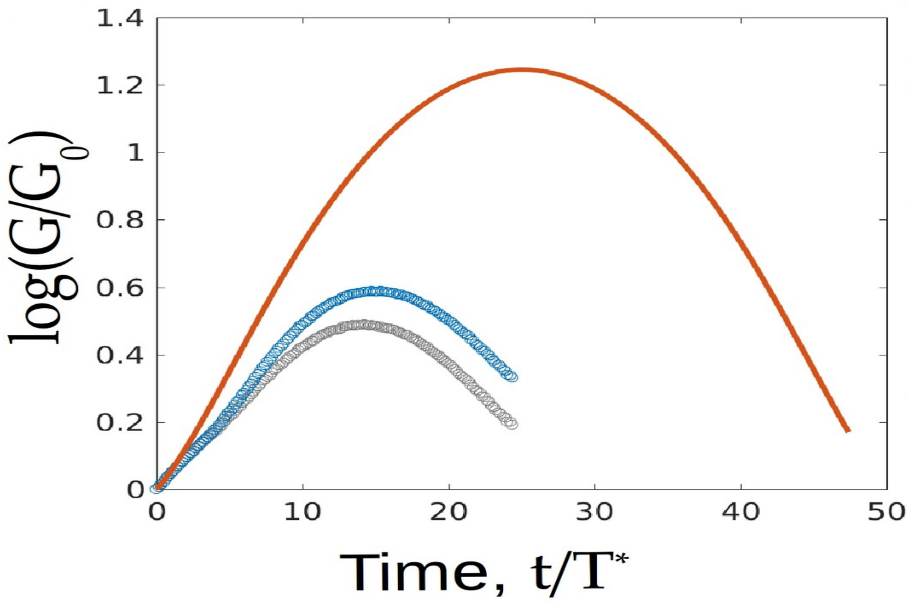
non-normal

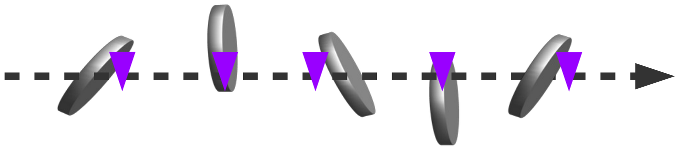
$$[\tilde{\mathbf{A}}(q), \tilde{\mathbf{A}}^\dagger(q)] \neq 0.$$

- Perturbation energy not conserved
- Modal analysis insufficient
- Expect algebraic growth

Algebraic growth



Maximally growing mode
(red curve)

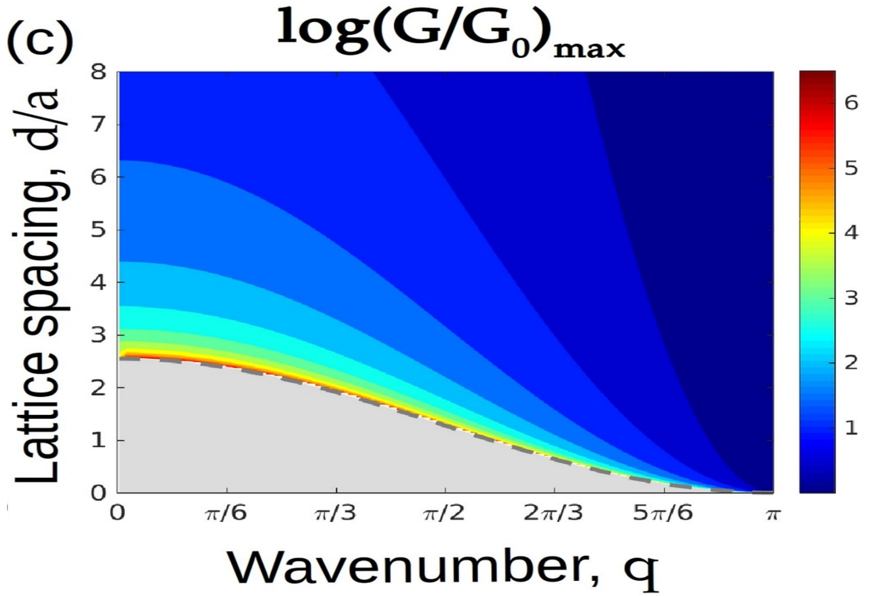


$$\frac{d}{dt}\langle\tilde{\mathbf{X}}_q|\tilde{\mathbf{X}}_q\rangle = 6\frac{a^2}{d^2}\sin q\,\mathrm{Im}(U_q^* W_q)$$

P J Schmid, Annu Rev Fluid Mech 2007
Bale & Govindarajan Resonance 2010

Non-normal dynamics

- Amplitude grows algebraically even in the stable regime.
- Transient growth pushes the system to non-linear regime.
- Noise amplification.
- Numerics with tiny amplitude: algebraic growth without nonlinearity.



Late times movie

Clumping at late times movie

Noisy movie

Summary

- Collective sedimentation: shape matters
- Emergent inertia and Hamiltonian
- Disc arrays: “energy conservation” suppresses linear instability
  - elasticity from viscous hydro, waves in theory and expt
- Beware non-normality
  - eigenvalues are deceptive, transient algebraic growth beats linear stability

Challenge: in-plane polar particles, effectively motile system?