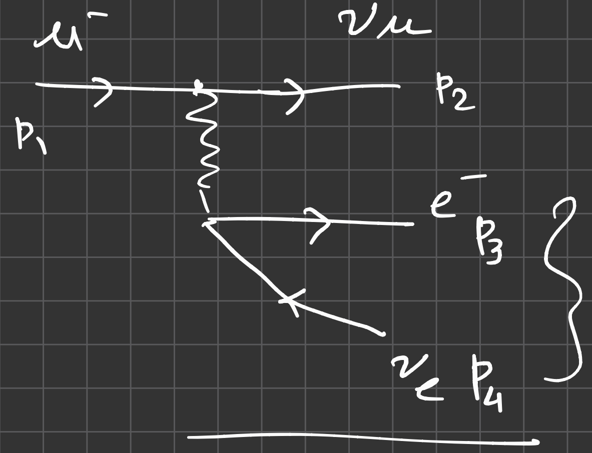
(1a) Muon lifetime (~ order of magnitude)

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \quad \Big| \quad \mathcal{L} \ni \frac{g_2}{\sqrt{2}} \bar{\nu}_\alpha \gamma^\mu P_L l_\alpha W_\mu^+ + \text{h.c.}$$



$$\mathcal{M} = \bar{u}_2 \frac{g_2}{\sqrt{2}} \gamma^\mu P_L u_1 \frac{g_{\mu\nu}}{q^2 - M_W^2} \bar{u}_3 \gamma^\nu P_L v_4 \frac{g_2}{\sqrt{2}}$$

$$P_{L,R} = \frac{1 \mp \gamma_5}{2}$$

$$\simeq \frac{g_2^2}{2 \times 4} \frac{(4)}{M_W^2} \left[\bar{u}_2 \gamma^\mu (1-\gamma_5) u_1 \, \bar{u}_3 \gamma^\nu (1-\gamma_5) v_4 \right]$$

$M_W \gg m_\mu$

$$\frac{g_2^2}{8 M_W^2} \sim \frac{G_F}{\sqrt{2}}$$

$$\simeq \cancel{4} \frac{G_F}{\sqrt{2}} \left(\frac{m_\mu}{2} \right)^2 \simeq \frac{G_F}{\sqrt{2}} m_\mu^2$$

$$d\Gamma = \frac{1}{2} \frac{1}{2m_\mu} \sum_s |\mathcal{M}|^2 \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}\left(p_1 - (p_2 + p_3 + p_4)\right)$$

$$\Gamma \simeq \frac{1}{2} \frac{1}{2m_\mu} \frac{G_F^2}{2} m_\mu^4 \frac{(4\pi)^2 (2\pi)^4}{(2\pi)^9} \frac{1}{8} m_\mu^2$$

$$\simeq \frac{1}{128\pi^3} G_F^2 m_\mu^5$$

$$\Gamma \simeq \frac{G_F^2 m_\mu^5}{192\pi^3}$$

$$\simeq 3\times10^{-19}\ \text{GeV}$$

$m_\mu = 0.106$ GeV

$G_F = 1.16\times10^{-5}\ \text{GeV}^{-2}$

$1\ \text{GeV} = 1.5\times10^{24}\ s^{-1}$

$\tau^{exp} = 2.2\times10^{-6}\ s$

$$\tau = \frac{1}{\Gamma} = 2.2\times10^{-6}\ s \simeq 2.2\ \mu s.$$

$$\boxed{\tau_\mu \simeq 2.2\ \mu s}$$

Tau

$$\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau \quad (1)$$

$$\tau^- \to e^- + \bar{\nu}_e + \nu_\tau \quad (1)$$

$$\tau^- \to \bar{u} + d + \nu_\tau \quad (3 \text{ colors} \times |V_{ud}|^2)$$

$$\tau^- \to \bar{u} + s + \nu_\tau \quad (3 \text{ colors} \times |V_{us}|^2)$$

$m_\tau \simeq 1.8$ GeV

$$\Gamma_\tau = \frac{G_F^2 m_\tau^5}{192\pi^3}\left(2 + 3\underbrace{|V_{ud}|^2}_{O(1)} + 3\underbrace{|V_{us}|^2}_{O(\lambda^2)}\right)$$

$$\simeq \frac{5\, G_F^2\, m_\tau^5}{192\,\pi^3} \simeq 5\,\frac{m_\tau^5}{m_\mu^5}\,\Gamma_\mu$$

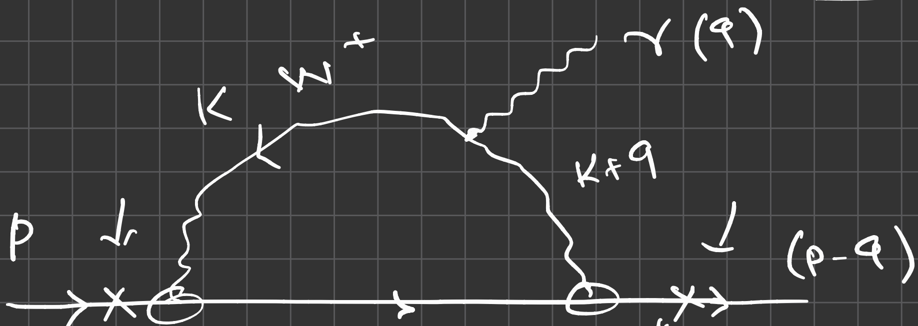
$$\simeq 7\times 10^{6}\,\Gamma_\mu$$

$\tau_\tau \simeq 3\times 10^{-13}$ Sec. | $\tau^{exp} \simeq 2.9\times 10^{-13}$ s.

—x—

(1b) $l_\alpha \to l_\beta + \gamma$; $\alpha \neq \beta$ $\quad m_\alpha > m_\beta$

$l_\alpha = \{e, \mu, \tau\}$



$\frac{g}{\sqrt{2}}\gamma^\nu P_L U_{\alpha i}$ $\quad$ $\frac{g}{\sqrt{2}}\gamma^\mu P_L U^*_{\beta i}$

$\bar{u}Uv$

$$\mathcal{L} \simeq \frac{g_2}{\sqrt{2}}\,\bar{\nu}_i\,\gamma^\mu P_L\, U_{\alpha i}\, l_\alpha\, W_\mu + h.c.$$

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4}\sum_i \left[\bar{u}_\beta\left(\frac{g}{\sqrt{2}}\gamma^\mu P_L U^*_{\beta i}\right)\frac{\not{p}+\not{k}+\boxed{m_{\nu_i}}}{(p+k)^2-m_{\nu_i}^2}\left(\frac{g}{\sqrt{2}}\gamma^\nu P_L U_{\alpha i}\right)u_\alpha\right]\Delta^{\nu\sigma}(k)\,\Delta^{\mu\rho}(k+q)$$

0

$-ie\,\underline{\Gamma_{\beta\sigma\gamma}}\,\epsilon^{\gamma}$ (Cheng & Li)

$$\sum_i \frac{1}{(p+k)^2 - m_{\nu_i}^2} U_{\beta i}^* U_{\alpha i} \equiv C_{\beta\alpha}$$

$\underline{m_{\nu_i} \ll m_\mu}$

$$C_{\alpha\beta} = \sum_i U_{\beta i}^* U_{\alpha i} \frac{1}{(p+k)^2}\left(1 - \frac{m_{\nu_i}^2}{(p+k)^2}\right)^{-1}$$

$$\simeq \sum_i U_{\beta i}^* U_{\alpha i} \frac{1}{(p+k)^2}\left(1 + \frac{m_{\nu_i}^2}{(p+k)^2}\right)$$

$$\simeq \underbrace{\frac{1}{(p+k)^2} \sum_i U_{\beta i}^* U_{\alpha i}}_{I} + \frac{1}{(p+k)^4} \sum_i U_{\beta i}^* U_{\alpha i}\, m_{\nu_i}^2$$

$$I \sim \sum_i U_{\beta i}^* U_{\alpha i} = \sum_i \left(U_{\alpha i} (U^\dagger)_{i\beta} \right)$$

$$= (UU^\dagger)_{\alpha\beta} = \delta_{\alpha\beta}$$

$$\alpha \neq \beta \quad = 0$$

$\Rightarrow$ GIM Cancellation

$$C_{\alpha\beta} \equiv \sum_i U^*_{\beta i} U_{\alpha i} m^2_{\nu_i}$$

$$\mathcal{M} = \frac{g_2^2 e}{8} C_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \boxed{\bar{u}_\beta} \gamma_\mu (1-\gamma_5) \frac{\not{p}+\not{k}}{(p+k)^4}$$

$$\gamma_\nu (1-\gamma_5) \boxed{u_\alpha} \times \Delta^{\nu\sigma}(k) \, \Delta^{\mu\rho}(k+q) \, \Gamma_{\gamma\sigma\rho} \, \epsilon^\gamma$$

$$\left\{ \begin{array}{ll} \bar{l}_\beta \, \sigma^{\mu\nu} \, l_\alpha F_{\mu\nu} & [\gamma^\mu, \gamma^\nu] \\ \Downarrow & \\ \bar{l}_{\beta R} \, \sigma^{\mu\nu} \, l_{\alpha L} \, F_{\mu\nu} & \end{array} \right\}$$

$$\simeq \frac{g_2^2 e}{8} C_{\alpha\beta} \quad m_\alpha \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{k \cdot k}{k^4 \cdot k^4}}_{\text{Convergent integral.}}$$

$$\simeq \frac{g_2^2 e}{8 M_W^4} \overset{[2]}{C_{\alpha\beta}} \overset{[1]}{m_\alpha} \simeq \frac{g_2^2}{8 M_W^2} e \underbrace{\sum_i U_{\beta i}^* U_{\alpha i} \frac{m_{\nu_i}^2}{M_W^2}}_{\tilde{C}_{\alpha\beta}}$$

$[4]$

$m_\alpha \quad m_\alpha^2$

$$\simeq \frac{G_F}{\sqrt{2}} e \, \hat{C}_{\alpha\beta} \frac{m_\alpha^3}{4\pi^2}$$

with $$\tilde{C}_{\alpha\beta} = \sum_i U_{\beta i}^* U_{\alpha i} \frac{m_{\nu_i}^2}{M_W^2}$$

$$\Gamma = \frac{1}{16\pi m_\alpha} |\mu|^2$$

$$= \frac{\alpha G_F^2}{128\pi^4} m_\alpha^5 |\tilde{C}_{\alpha\beta}|^2 \leftarrow$$

$$BR(l_\alpha \to l_\beta + \gamma) = \frac{\Gamma(l_\alpha \to l_\beta + \gamma)}{\Gamma(l_\alpha)}$$

$$\Gamma(\mu \to e\gamma) = \frac{3\alpha}{2\pi} \left| \hat{c}_{\alpha\beta} \right|^2 \underbrace{\frac{G_F^2 m_\mu^5}{192 \pi^3}}$$

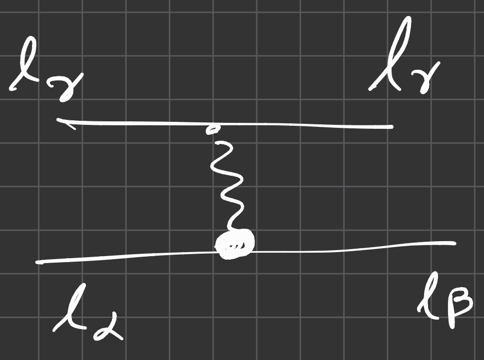
$$BR(\mu \to e\gamma) = \frac{3\alpha}{2\pi} \left| \hat{c}_{\alpha\beta} \right|^2$$

$$\hat{c}_{\alpha\beta} = \sum_i \underbrace{U_{\beta i}^*}_{\simeq\, O(1)} \underbrace{U_{\alpha i}}_{\simeq\, O(1)} \frac{m_{\nu_i}^2}{M_W^2} \sim \frac{10^{-3}}{10^{22}} \sim 10^{-25}$$

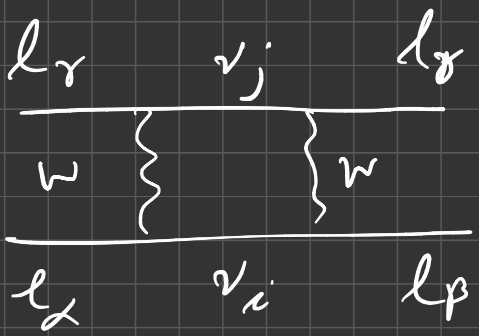
$$BR(\mu \to e\gamma) \simeq 10^{-2}\, 10^{-50} \sim 10^{-52}$$

$$BR(\mu \to e\gamma) < 4 \times 10^{-13}$$

$l_\alpha \to 3 l_\beta \qquad m_\alpha > m_\beta$



$$\sim \Gamma(l_\alpha \to 3 l_\beta) \simeq \alpha\, \Gamma(l_\alpha \to l_\beta \gamma)$$



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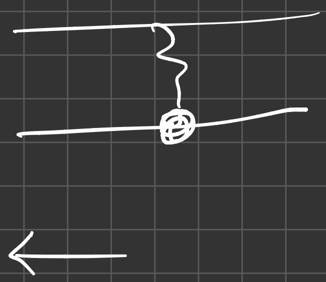
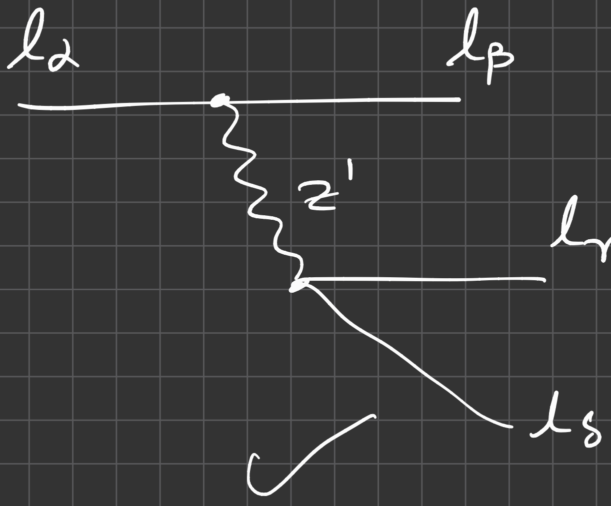


(1c) Covered!

(24) $M_{Z'} \gg m_\alpha$

$$\mathcal{L} \supset g_{ij}\, \bar{l}_i \gamma^\mu l_j Z'_\mu + h.c.$$

$\tau \to 3\mu$ $\quad (l_\alpha \to l_\beta l_\gamma l_\delta)$ $\quad m_\alpha \gg \ldots$



$$\mathcal{M} \simeq \frac{g_{\alpha\beta}\, g_{\gamma\delta}}{M_{Z'}^2}\, m_\alpha^2$$

$$\Gamma(l_\alpha \to l_\beta l_\gamma l_\delta) \simeq |\mathcal{M}|^2 \times \text{phase space}$$

$$\simeq \left| \frac{g_{\alpha\beta}\, g_{\gamma\delta}}{M_{Z'}^2} \right|^2 \frac{m_\alpha^5}{192\pi^3}$$

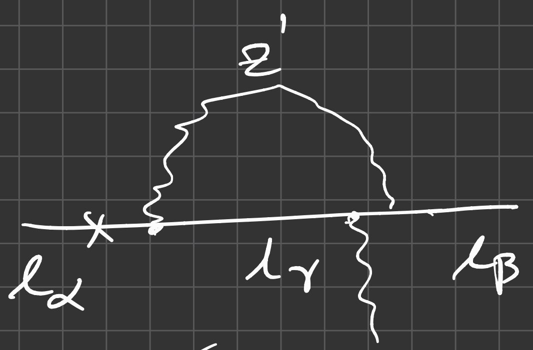
$$\Gamma(\mu \to 3e) = \frac{|g_{12}\, g_{11}|^2}{M_{Z'}^4} \frac{m_\mu^5}{192\pi^3}$$

$$BR(\mu \to 3e) \simeq \frac{|g_{12}\, g_{11}|^2}{M_{Z'}^4\, G_F^2}$$

$$BR(\mu \to 3e) < 10^{-12} \quad \text{(SINDRUM 1988)}$$

$$\frac{M_{Z'}}{\sqrt{|g_{11} g_{12}|}} > 10^{5.5}\ \text{GeV}$$

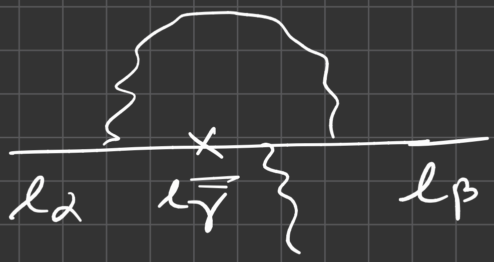
(2b) $l_\alpha \to l_\beta \gamma$



$$\mathcal{M} \simeq \frac{1}{4\pi^2} \frac{g_{\alpha\gamma}\, g_{\gamma\beta}}{M_{Z'}^2}\, e\, m_\alpha^3$$

$$\Gamma(l_\alpha \to l_\beta \gamma) = \frac{1}{16\pi m_\alpha} |\mathcal{M}|^2$$

$$BR(\mu \to e\gamma) \simeq \frac{3\alpha}{2\pi} \sum_\gamma \frac{|g_{1\gamma}\, g_{\gamma 2}|^2}{M_{Z'}^4\, G_F^2}$$

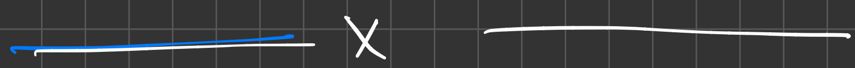


$$M \sim \sum_\gamma \underline{m_\gamma} \, \frac{g_{1\gamma}\, g_{2\gamma}}{\underline{M_{Z'}^2}}$$

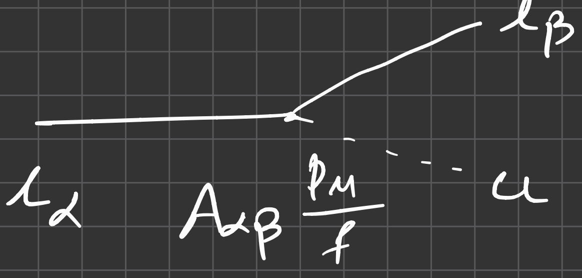
$$BR(\mu \to e\gamma) \simeq \frac{m_t^2}{\underline{m_\mu^2}} \, \frac{3\alpha}{2\pi} \sum_\gamma \frac{|g_{1\gamma}\, g_{\gamma 2}|^2}{M_{Z'}^4\, G_F^2}$$

$$BR(\mu \to e\gamma) < 4 \times 10^{-13}$$

$$\frac{M_{Z'}}{\sqrt{g_{13}\, g_{32}}} \gtrsim 10^6 \ \text{GeV}$$



(3) ALP $\quad m_a \ll m_\alpha$



$$l_\alpha \to l_\beta\, a$$

$$A_{ij} \boxed{\frac{\partial_\mu a}{f}} \bar{l}_i \gamma^\mu \gamma^5 l_j + h.c.$$

$$|\mu| \simeq \frac{|A_{\alpha\beta}|}{\underline{\underline{f}}} m_\alpha^2$$

$$\Gamma(\ell_\alpha \to \ell_\beta a) = \frac{1}{16\pi m_\alpha} |\mu|^2$$

$$\Gamma(\mu \to e a) \simeq \frac{1}{16\pi} \boxed{\frac{(A_{12})^2}{f^2}} m_\mu^3$$

$$BR(\mu \to e a) = \frac{\Gamma(\mu \to e a)}{\Gamma(\mu \to \dots)}$$

$$BR(\mu \to e a) \lesssim 10^{-6} \leftarrow$$

$$BR(\tau \to e a) < 10^{-6} \quad (\text{Belle-II})$$

$$BR(\tau \to \mu a) < 10^{-5} \quad (\text{Belle-II})$$

$$\boxed{\frac{f}{|A_{12}|} > 10^{10}\ \text{GeV}}$$

(4) Consider local $U(1)$ Symmetry under which $L_{Li} \equiv \begin{pmatrix} \nu_{Li} \\ l_{Li} \end{pmatrix}$ have charges q_{Li} and l_{Ri} have Charges q_{Ri}. The gauge interaction and mass term can be written as

$$\mathcal{L} \supset q_{Li} \overline{L}_{Li} \gamma^{\mu} L_{Li} Z'_{\mu} + q_{Ri} \overline{l}_{Ri} \gamma^{\mu} l_{Ri} Z'_{\mu}$$
$$+ \{ Y_{ij} \overline{L}_{i} H l_{Rj} + h.c. \} \quad \text{---} (1)$$

$\Downarrow$ Electroweak Symmetry breaking

$$\mathcal{L} \supset (q_{Li} \overline{l}_{Li} \gamma^{\mu} l_{Li} + q_{Ri} \overline{l}_{Ri} \gamma^{\mu} l_{Ri}) Z'_{\mu}$$
$$+ M_{ij} \overline{l}_{Li} l_{Rj} + \text{terms involving neutrinos} \quad (2)$$

Let $(l_{L,R})_i = (U_{L,R})_{ij} (l'_{L,R})_j$ where prime denotes (3) the charged lepton mass matrix is diagonal. $U_{L,R}$ can be determined as

$$M_{ij} \overline{l}_{Li} l_{Rj} = M_{ij} (U_L^*)_{ik} (U_R)_{jl} \overline{l}'_{Lk} l'_{Rl}$$
$$= (U_L^{\dagger})_{ki} M_{ij} (U_R)_{jl} \overline{l}'_{Lk} l_{Rl}$$
$$\equiv D_{kl} \overline{l}_{Lk} l_{Rl}$$

Such that $D_{kl} = m_k \delta_{kl}$

$$U_L^{\dagger} M U_R = D = \text{Diag.}(m_1, m_2, \ldots)$$

Now Consider the gauge interaction in the Charged lepton mass basis. Using (3) in (2)

$$\mathcal{L} \supset \Big\{ (U_L^\dagger)_{ki}\, q_{Li}\, (U_L)_{il}\; \bar{l}'_{Lk} \gamma^\mu l'_{Ll}\, Z'_\mu$$
$$+ \; L \rightarrow R \Big\}$$

In 4 component notation i.e. $l = \begin{pmatrix} l_L \\ l_R \end{pmatrix}$, the above can be written as

$$\mathcal{L} \supset (U_L^\dagger q_L U_L)_{ij} P_L + (U_R^\dagger q_R U_R)_{ij} P_R \;\; \bar{l}'_i \gamma^\mu l'_j Z'_\mu$$

$$\equiv g_{ij}\, \bar{l}'_i \gamma^\mu l'_j\, Z'_\mu$$

with

$$\boxed{g_{ij} = (Q_L)_{ij} P_L + (Q_R)_{ij} P_R}$$

$$(Q_{L,R})_{ij} = (U_{L,R}^\dagger q_{L,R} U_{L,R})_{ij}$$

If q_{Li} and q_{Ri} are universal then $q_{L,R} \propto \mathbb{I}$

$$\Rightarrow \quad Q_{L,R} \propto \mathbb{I} \quad \Rightarrow \quad \boxed{g_{ij} \propto \delta_{ij}}$$

Special structure of M, $q_{L,R}$ can lead to $Q_{L,R}$ such that $(Q_{L,R})_{ii} = 0$ and $(Q_{L,R})_{ij} \neq 0$

For example, Consider two generation for simplicity.

Let $M \sim \begin{pmatrix} A & B \\ B & A \end{pmatrix} \Rightarrow U_{2,R} \sim \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

And $q_{L,R} \sim \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$

Substitution of $q_{L,R}$ and $U_{L,R}$ in $Q_{L,R}$ imply

$$\boxed{Q_{2,R} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$