Some Snapshots of India's Mathematical Past

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Outline

The aim of this talk will be to

- Give an exposition of select snapshots from the history of ancient Indian mathematics, covering various periods.
- Highlight the distinctive features and significance of the discoveries in their context.
- Discuss interrelations with other cultures and scope for follow up.

Pythagoras theorem and its converse

Sulbasutras which were composed in aid of the activity of construction of *vedis* and *chitis* for *yajnas* are the major source of ancient knowledge of geometry in India.

There are four major sulvasutras. The earliest of them, the Baudhayana sulvasutra, is believed to be from around 800 BCE.

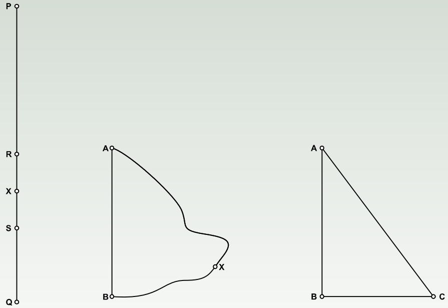
The Pythagoras theorem is stated explicitly in all the four.

In Baudhayana sulvasutra it is also used to produce squares whose area coincides with the sum (resp. difference) of two given squares.

The converse of the theorem was used extensively for producing right angles, via use of “Pythagorean triples”.

Method of application of the converse statement

The Pythagorean triples $(3, 4, 5)$ and $(5, 12, 13)$ were extensively used for the purpose. In the Apastamba sulbasutra there is a construction of Mahavedi illustrating also the use of the triples $(8, 15, 17)$ and $(12, 35, 37)$



Figures 3a, 3b and 3c: Construction of perpendicular by the *Nyanchana* method

Jaina formula for arc-length in a circle

Mathematics played a significant role in Jaina scriptures, from about 5th c. BCE and 3rd c. CE, inspired by their cosmological and geographical models.

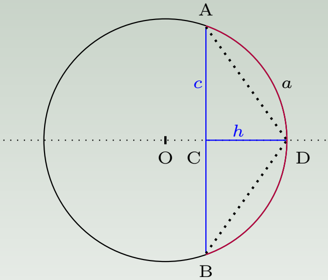
The terrestrial world was concieved to consist of the Jambudvipa (th earth) in the shape of a flat disc, surrounded alternately by rings of water and land, with the pattern extending indefinitely (to infinity).

Also, the region of Jambudvipa was divided by 7 mountain ranges, running parallel to each other, and there also a variety of similar details of geographical nature.

The models in particular inspired understanding geometry of the circle.

The arc-chord formula

Let c be a chord and h the corresponding “arrow”, in a circle, and let a be the corresponding minor arc.



The ancient Jainas had an intriguing approximate formula:

$$a = \sqrt{6h^2 + c^2}.$$

The formula is clearly an ad hoc and approximate one, and is inspired by the Pythagoras theorem.

The motivation for the formula seems to come as follows: The length of the straight-line path ADB is readily seen, using the Pythagoras theorem (which was known to the Jainas) to be $\sqrt{4h^2 + c^2}$. One may hence look for an expression for the arc in the form $\sqrt{xh^2 + c^2}$ for the arclength. Considering the diameter as a special case of a chord, we get the value of x to be 6, with the Jaina value for the ratio of circumference to diameter, which was $\sqrt{10}$.

Heron of Alexandria (2nd c) also had formulae for the arc as

$$\sqrt{4h^2 + c^2} + \frac{1}{4}h$$

and (a more refined)

$$\sqrt{4h^2 + c^2}\left(1 + \frac{h}{c}\right) - h \qquad .$$

Comparison with Heron's formula

Angle	arclength true value	Jaina value	Heron's 1st value	Heron's 2nd value
15°	0.5236	0.5243 (+0.0007)	0.5306 (+0.0070)	0.5224 (-0.0012)
30°	1.0472	1.0525 (+0.0053)	1.0688 (+0.0216)	1.0400 (-0.0072)
45°	1.5708	1.5858 (+0.0150)	1.6049 (+0.0341)	1.5549 (-0.0159)
60°	2.0944	2.1213 (+0.0269)	2.1250 (+0.0306)	2.0774 (-0.0170)
75°	2.6180	2.6511 (+0.0331)	2.6203 (+0.0023)	2.6281 (+0.0101)
90°	3.1416	3.1623	3.0784	3.2426

Aryabhata's trigonometry

Aryabhata gave a listing the successive differences of Rsines, where R=3438, of angles which are multiples of 24th part of the right angle, (viz $3^\circ 45'$).

मखि भखि फखि धखि णखि ञखि
ङखि हस्झ स्ककि किष्ग श्घकि किघ्व[1] ।
घ्लकि किग्र हक्य धकि किच[2]
स्ग श्झ[3] ङ्व क्ल प्त फ छ कलार्धज्याः ॥ १२ ॥

225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7— these are the Rsine-differences (at intervals of 225 minutes of arc) in terms of minutes of arc.

Computation of Rsines

They were aware of the formulae corresponding to $\sin^2\theta + \cos^2\theta = 1$, and $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$. The tables seem to have been computed using these and the obvious values for $\sin 30^\circ = \frac{1}{2}$ and $\sin 45^\circ = 1/\sqrt{2}$

$$8 \to 4 \to 2 \to 1;\ \ 12 \to 6 \to 3;\ \ 4 \to 20 \to 10 \to 5;$$

$$2 \to 22 \to 11;\ \ 6 \to 18 \to 9;\ \ 10 \to 14 \to 7.$$

Aryabhata also discusses another method for computations based on the observation: if $S_1, \dots, S_{24}$ are Rsine θ , with θ as successive multiples of $3^\circ 45'$ then

$$(S_{k+1} - S_k) - (S_k - S_{k-1}) \approx -S_k/S_1. \quad (*)$$

Then starting with S_1 the successive values can be computed. However this is not adopted, at least fully, as the errors accumulate.

Note that the relation $(*)$ recognized here is a finitary version differential equation $\dfrac{d^2 \sin t}{dt^2} = -\sin t.$

Bhaskara I's formula

There is also an interesting formula, attributed to Bhaskara I (ca. 600); in our notation it is: for a θ given in angle measure in the range $0 \leq \theta \leq 180$, to

$$\sin\theta \approx 4\frac{\theta(180-\theta)}{40500-\theta(180-\theta)}.$$

In the usual radian measure the formula is equivalent to

$$\sin\theta \approx 16\frac{\theta(\pi-\theta)}{5\pi^2-4\theta(\pi-\theta)}.$$

The formula is remarkably accurate, with less than 1 % error, except for very small values of θ.

Many astronomers of the time seem to have found it quite useful, and sometimes preferred it over usage of the table.

It is not clear how such a formula was arrived at.

One of the explanations goes as follows: a simplest expression that one would seek would be a quadratic rational function. Moreover it should be symmetric in θ and $180 - \theta$. This together with the stipulation that the values should be 0 and $\theta = 0^\circ$ and 180°, 1 at 90° and $\frac{1}{2}$ at 30° determines the (five free) coefficients of such an expression to be as chosen.

In Grahalaghava (1520 CE) of Ganesa Daivajna a modified constant of 40320 is adopted in place of 40500. In terms of trial and error, using modern equipements, roughly the optimal constant is in the range 40400 and 40500.

Tehre are also other modifications in the constant involved, concerned with different objectives, such as getting it to be asympottic to θ at 0.

Kuttaka

This is about finding integer solutions of equations for the form $ax + b = cy + d$, or equivalently $ax - by = c$, where a, b, c, d are positive integers.

Aryabhata introduced a method for solving the problem, which is known as kuttaka.

It is based on a process of simplifying equations, with regard to size of the coefficients, by a process of mutual division.

For the equation $16x - 487y = 138$, it goes as follows:

$7x_1 - 16y_1 = -138,$
$2x_2 - 7y_2 = 138$
... and back-substitution after some stage.

The mutual division runs as follows :

```
16) 487 (30
    480
    ---
     7) 16 (2
        14
        --
        2) 7 (3
           6
           -
           1×2+138          optional number=2
              1 4 0 (70
              1 4 0
              -----
                  0
```

"Pell's equation"

Finding integer solutions for the equation $x^2 - dy^2 = 1$, d a non-square integer, was posed as a challenge by Fermat, in the 17th century to his British contemporaries (for certain specific values of d). initial solutions came from Brouncker and Wallis.

The equation came to be called Pell's equation, a name introduced by Euler (on account of a misunderstanding). Lagrange completed the picture, using "continued fraction expansions".

In India study of the equation goes back to Brahmagupta in, the 7th century. He considered it in the more general form $x^2 - dy^2 = k$ with k a variable nonzero integer; it was called varga prakriti.

Brahmagupta's identity

Brahmagupta introduced the identity (now known after him) $(x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2$. This was treated as a way of composing of two solutions corresponding to possibly different values of k. Using this he solved various special cases, including $d = 92$ and 83.

For $d = 92$ he starts with the observation $10^2 - 92 \times 1^2 = 8$ (now $k = 8$). Composing the solution with itself one gets $192^2 - 92 \times 20^2 = 64$. This he reduces to $24^2 - (\frac{5}{2})^2 = 1$, giving a rational solution. Now composing the solution with itself gives the integer solution $1151^2 - 92 \times 120^2 = 1$.

Brahmagupta also developed some more tricks and shortcuts. He also proved if you solve the equation, for a d, with $k = \pm 4, \pm 2,$ or -1, then you can solve for $k = 1$.

Brahmagupta's identity also shows that if there is one solution then there are infinitely many solutions.

Chakravala

The issue was treated as a challenge problem by many mathematicians in India in the subsequent centuries. A method for solving it was introduced by Jayadeva (11th c.).

Nothing is known about Jayadeva; his solution is known from a later commentary by Udayadivakara. The method was later popularized by Bhaskaracharya (1150).

The method is based on using Brahmagupta's composition ("Bhavana"), systematically in a certain way.

The Indian practitioners were content with applying the method; a confirmation that the method will always lead to a solution, was given by Krishnaswamy Ayyangar in 1929-30, and improved understanding provided by Selenius (1975).

The Chakravala algorithm

The method goes as follows:

Assume we have a triple (a, b, k) that satisfies $x^2 - dy^2 = k$. We also have the obvious triple $(\xi, 1, \xi^2 - d)$ for any ξ.

Composing these we get a triple $(a\xi + db, a + b\xi, k(\xi^2 - d))$.

We now choose ξ so that $a + b\xi$ is divisible by k (using kuttaka). This forces also that $a\xi + db$ is divisible by k and we get

$$\left(\frac{a\xi + db}{k}\right)^2 - d\left(\frac{a + b\xi}{k}\right)^2 = \frac{\xi^2 - d}{k}.$$

We further choose ξ (among the admissible one's) to be such that $\xi^2 - d$ is minimum, and repeat the procedure with the new triple. This eventually leads to a solution of the "Pell equation".

Chakravala examples

i) Consider the equation $x^2 - 105y^2 = 1$. We have $10^2 - 105 = -5$. Applying the algorithm with $a = 10$, $b = 1$ and $k = -5$ the optimal value for ξ is 10 and the new triple corresponds to $41^2 - 105(4)^2 = 1$ thus solving the equation! [The continued fraction expansion of $\sqrt{105}$ is $1; \overline{1, 2, 2, 2, 1, 2}$ and it would take many steps to solve the equation by the CF method.] Not all examples are this dramatic however!]

ii) For $d = 67$, which was one of Fermat's challenge problems, starting with $8^2 - 67 = -3$ one successively arrives at the relations $41^2 - 67 \times 5^2 = 6$, $90^2 - 67 \times 11^2 = -7$, followed by $221^2 - 67 \times 27^2 = -2$; continuing with the algorithm takes another 7 iterations, but here one can compose the relation with itself and dividing the result throughout by 4 we get $48842^2 - 67 \cdot 5967^2 = 1$, thus solving the equation.

Kerala school of Madhava

The school, founded by Madhava lasted from the later half of the 14th century to the 17th century. Nilakantha, Jyesthadeva, Sankara Variar are some of the other figures from the school.

The Kerala school developed ideas that are germinal to calculus. In particular they had obtained series expansion

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots \quad \text{(Gregory series).}$$

As the series being unsuitable for computing π, they introduced “correction terms” (*antya samskara*) producing sequences converging rapidly.

They also had the Leibnitz series for the arctan function, and the Newton series for the sine and cosine functions.

The work on series was used also for improving the trigonometric tables.

* Thank you *

for your attention.