Hitchin–Kobayashi correspondence for bundles with parabolic structure group

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(Based on joint work with Oscar García-Prada)

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Theorem (Donaldson, 1985–1987; Uhlenbeck–Yau, 1986)

There exists a Hermitian metric h solving the *Hermitian–Einstein equation*

$$\Lambda F_h = -i\mu \mathbf{I}_E$$

if and only if $(E, \overline{\partial}_E) \to X$ is *slope polystable.*

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Question

What if the structure group is **non-reductive** (e.g., a **parabolic subgroup**)?

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$$P = \left\{ \begin{pmatrix} L_1 & W \\ 0 & L_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} \mathbf{I}_1 & W \\ 0 & \mathbf{I}_2 \end{pmatrix} \right\} \leq \mathrm{GL}_n(\mathbb{C})$$

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$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E_1} & \Phi \\ 0 & \bar{\partial}_{E_2} \end{pmatrix}$$

  — where $\Phi \in \Omega^{0,1}(X, \mathrm{Hom}(E_2, E_1))$ — such that

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- **Gauge group:**

$$\mathscr{G}_P = (\mathscr{G}_{L_1} \times \mathscr{G}_{L_2}) \ltimes \Omega^0(X, \mathrm{Hom}(E_2, E_1)).$$

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or equivalently (for $\tau := \tau_2 - \tau_1$)

$$\begin{cases} \Lambda F_h - i\tau\pi_h = -i\left(\dfrac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)}\tau + \mu\right)\mathbf{I}_E \\ \\ \pi_h^{\perp}\overline{\partial}_E\pi_h = 0 \end{cases} \iff \begin{cases} \Lambda(F_{h_1} - \Phi \wedge \Phi^*) = -i\tau_1 \mathbf{I}_1 \\ \\ \Lambda(F_{h_2} - \Phi^* \wedge \Phi) = -i\tau_2 \mathbf{I}_2 \\ \\ \overline{\partial}^*_{\mathrm{Hom}(E_2,E_1)}\Phi = 0 \end{cases}$$

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- There is a moment map interpretation.
- Bradlow–García-Prada (1995) and Daskalopoulos–Uhlenbeck–Wentworth (1995) propose some stability conditions (depending on a parameter α) and prove a Hitchin–Kobayashi correspondence when $\alpha = \tau = \tau_2 - \tau_1 > 0$.

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Definition (Almost holomorphic structures on a principal bundle)

Let $\pi : E_H \to X$ a smooth principal H-bundle. An ***(almost) holomorphic structure*** on $E_H \to X$ is an ***(almost) complex structure*** J on E_H which makes the projection π and the H-action ***(pseudo)holomorphic***.

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Note that $E_L \cong E_H/U$ and that U is contractible. Hence

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Proposition

$$\mathscr{G}_H \cong \mathscr{G}_L \ltimes \Gamma(E_L(U))$$

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Theorem (García-Prada–RC, 2026)

For all $(e, u) \in E_L \times U(\cong E_H)$,

$$J_{(e,u)} = \begin{pmatrix} \hat{J}_e & 0 \\ -2J_{U,u}(r_u)_{*,1}\Phi_e & J_{U,u} \end{pmatrix},$$

where

- $\Phi \in \Omega^{0,1}_{\mathrm{Ad}}(E_L, \mathfrak{u}) \cong \Omega^{0,1}(X, E_L(\mathfrak{u}))$,
- J_U complex structure in U,
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Moreover, J is integrable if and only if $\hat{J}$ is integrable and

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Remark (Deformation of complex structures)

J can be understood as a deformation of the complex structure $\hat{J} \oplus J_U$ in $E_L \times U$.

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- an action of $\mathscr{G}_H$ on holomorphic structure in E_L by
$$g \cdot \hat{J} := \hat{g} \cdot \hat{J},$$
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Using Grothendieck's ***non abelian cohomology***, the fibre of the map

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is identified with

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- Alternatively, this also follows from the previous proposition using a non-abelian Dolbeault resolution due to Onishchik (1967).

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Definition

A ***connection*** in E_K is a smooth K-equivariant splitting of

$$0 \longrightarrow VE_K \longrightarrow TE_K \longrightarrow \pi_K^* TX \longrightarrow 0.$$

It can be identified with a smooth K-equivariant 1-form $B \in \Omega^1(E_K, \mathfrak{k})$.

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Chern–Singer correspondence (Singer, 1959)

There is a one-to-one correspondence between holomorphic structures in E_G and

$$\mathscr{A}_K^{1,1} := \{B \in \mathscr{A}_K : F_B^{0,2} = 0\}.$$

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Let P be a parabolic subgroup of G, and assume E_G is a smooth G-extension of structure group of a P-bundle E_P (with $\xi \in \Gamma(E_G/P)$ the extension).

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- We can use the characterization of holomorphic structures for bundles with an L-reduction to give another generalization of the Chern–Singer correspondence.

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- By Chern–Weil theory, the values of the parameter c are restricted (but not fixed, we have ***free parameters***).

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Theorem (Atiyah–Bott, 1982; Donaldson, 1985-1987)

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- On some instances, it may include more "exotic" solutions (work in progress!).

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Hence the K-action on $(G/P, \omega_\eta)$ is Hamiltonian, with moment map

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This is a particular instance of a very general equation for pairs introduced by Mundet i Riera (2000).

We want to address the following

Question (Existence problem)

For which pairs (B, ξ) as above is there a gauge transformation $g \in \mathscr{G}_G$ such that

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- Mundet i Riera (2000) also gives a symplectic interpretation of the equations.

Equivalence with non-central Hermitian Yang–Mills equation

Given $c \in \mathfrak{z}(\mathfrak{q})$, we can always write

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Assume that $iH_\eta = i\sum_{\alpha \in A} \tau_\alpha H_\alpha$ with $\tau_\alpha > 0$ for all $\alpha \in A \subseteq \Delta$. Under the inclusion map $\Omega^0(X, E_Q(\mathfrak{q})) \subseteq \Omega^0(X, E_K(\mathfrak{k}))$, $iH_\eta \mapsto -\mu_\eta(\xi)$.

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Proposition (García-Prada–RC, 2026)

The following statements are equivalent.

(1) There exists $g' \in \mathscr{G}_P$ such that

$$\Lambda F_{g'\cdot B} = c_\xi.$$

(2) There exists $g \in \mathscr{G}_G$ such that

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Moreover, the solution to (1) is unique if and only if the solution to (2) is unique.

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UPSHOT: Existence (and uniqueness) is equivalent for both equations!

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Coupled Hermitian Yang–Mills–Hodge equations

$$\begin{cases} \Lambda \left(F_A + \pi_{\mathfrak{l}}[\Phi, \tau(\Phi)] \right) = c \\ \overline{\partial}_A^* \Phi + i\Lambda \pi_{\mathfrak{u}}[\Phi, \tau(\Phi)] = 0. \end{cases}$$

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Question (Existence problem)

For which pairs (A, Φ) as above is there a pair $(\hat{g}, f) \in \mathscr{G}_L \ltimes \Gamma(E_L(U))$ such that

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- By Chern–Weil theory, the value of the parameter c is restricted, but not fixed (we have ***free parameters***).

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Proposition (García-Prada–RC, 2026)

The following statements are equivalent.

(1) There exists $g \in \mathscr{G}_P$ such that

$$\Lambda F_{g \cdot B} = c_\xi .$$

(2) There exists $(\hat{g}, f) \in \mathscr{G}_L \ltimes \Gamma(E_L(U)) \cong \mathscr{G}_P$ such that

$$\begin{cases} \Lambda \left(F_{\hat{g} \cdot A} + \pi_{\mathfrak{l}}[\hat{g} \cdot (f \cdot \Phi), \tau(\hat{g} \cdot (f \cdot \Phi))] \right) = c \\ \overline{\partial}^*_{\hat{g} \cdot A} \left(\hat{g} \cdot (f \cdot \Phi) \right) + i \Lambda \pi_{\mathfrak{u}}[\hat{g} \cdot (f \cdot \Phi), \tau(\hat{g} \cdot (f \cdot \Phi))] = 0. \end{cases}$$

Moreover, the solution to (1) is unique if and only if the solution to (2) is unique.

Let

$$\mathscr{A}_Q^{1,1} Z := \left\{ (A, \Phi) : F_A^{0,2} = 0 \text{ and } \overline{\partial}_A \Phi - \frac{1}{2}[\Phi, \Phi] = 0 \right\}.$$

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Theorem (García-Prada–RC, 2026)

The holomorphic action of $\mathscr{G}_P \cong \mathscr{G}_L \ltimes \Gamma(E_L(U))$ on $\mathscr{A}_Q^{1,1}Z$ is Hamiltonian, with an Ω-moment map given by

$$\begin{aligned} \mathfrak{m} : \Omega \times \mathscr{A}_Q^{1,1}Z &\longrightarrow \mathrm{Lie}(\mathscr{G}_L) \oplus \mathrm{Lie}(\Gamma(E_L(U))) \\ (\mathscr{G}_K, \omega_{\mathscr{A}}, A, \Phi) &\longmapsto (\Lambda \left(F_A + \pi_{\mathfrak{l}}[\Phi, \tau(\Phi)] \right), \overline{\partial}_A^*\Phi + i\Lambda\pi_{\mathfrak{u}}[\Phi,\tau(\Phi)]). \end{aligned}$$

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- We thus have a “moment map” interpretation of the equations (in particular, of the generalization of the harmonicity equation).

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- We thus have a “moment map” interpretation of the equations (in particular, of the generalization of the harmonicity equation).
- This gives a natural non-reductive generalization of the moment map of Atiyah–Bott–Donaldson.

Meromorphic reductions and degree of principal bundles

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Alternatively,

$$\deg(E_G)(\sigma, \chi) = \frac{i}{2\pi} \int_{X_0} \chi \circ \pi_{\mathfrak{z}(\mathfrak{q})}(\Lambda F_{A_\sigma}) \omega^{[n]}.$$

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- (ξ, c)-*stable* (resp., (ξ, c)-*semistable*) if for every parabolic subgroup P' of G such that $P' \subseteq P$, every meromorphic reduction σ and every antidominant character χ in $\mathfrak{p}'$ we have

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Remarks

- Lübke–Teleman (2006) propose equivalent stability conditions for (E_G, ξ).
- The stability conditions can be restated in terms of the P-bundle E_P directly.

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- Lübke–Teleman (2006) prove a Hitchin–Kobayashi correspondence for (E_G, ξ) and the Hermitian Yang–Mills–Higgs equation (cf. Mundet i Riera (2000)).

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As a consequence,

Hitchin–Kobayashi correspondence I (García-Prada–RC, 2026)

Let $B \in \mathscr{A}_K$ such that $F_B^{0,2} = 0$ and $\overline{\partial}_B \xi = 0$. There exists a gauge transformation $g \in \mathscr{G}_P$ such that

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Hitchin–Kobayashi correspondence II (García-Prada–RC, 2026)

Let (A, Φ) be a Chern–Singer pair. There exists a pair $(\hat{g}, f) \in \mathscr{G}_L \ltimes \Gamma(E_G/P)$ such that

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if and only if E_P is c-polystable. Furthermore, if E_P is c-stable then the solution is ***unique***.

Thank you for your attention!