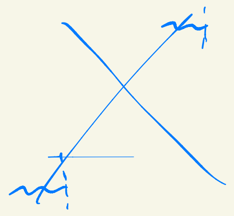
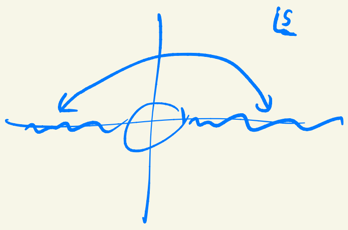
Last time

Causality $\to$ $S(\omega)$ analytic in upper-half-plane

+unitary $|S(\omega)| \leq 1$ " " "

$M_{2\to 2}$





$$M_\psi = \int_0^M dp\, \psi(p) M(s, -p^2) \Rightarrow |M_\psi(s)| \leq \text{const} \cdot |s|^{J_0}$$

compact in momentum, normalizable in b

$J_0 \leq 1$

Lecture 2: EFT bootstrap

1) sum rules
2) low energies
3) high energies
4) Linear programming
   [+some results]

refs: - SCH + van Rong 2011.02957
- Arkani-Hamed + Huang 1709.04891
  + Huang 2012.15849
- Caron-Huot + Seven + Zhiboedov 2006.08221

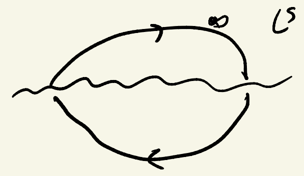
earlier:

Adams + Arkani-Hamed + Dubovsky + Nicolis + Rattazzi th/0602178

Lecture 3: primal vs dual, gravity / CFTs, open questions.

$\left(\int_0^M dp\, \psi(p)\right)$ $\underbrace{\oint_\infty \frac{M(s', -p^2)\, ds'}{(s' - p^2/2)^{1+k}}}_{B_k(p^2)} = 0$ (Fixed-$t = p^2$)

$\frac{s'-u'}{2}$: nice for crossing

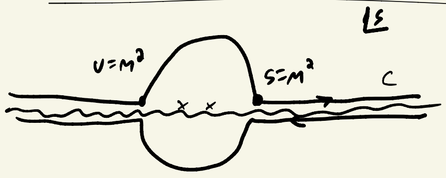


Similar to a k-subtracted dispersion relation for M (or derivative) around $s = p^2/2$

converge if $k > J_0$ $\quad (J_0 \leq 1)$

Note: these vanish for k=odd if amp is signature even.

Note: we also use in papers: $\frac{1}{[su]^{\frac{k+1}{2}}}$ (k odd) or $\frac{s-u}{[su]^{\frac{k+2}{2}}}$ (k even)



$$B_k(p^2) = \oint_C (\dots) = 0$$

$$-B_k(p)\Big|_{low} = +B_k(p^2)\Big|_{high}$$

$|s| \sim M^2$.

discontinuity at energies $> M$

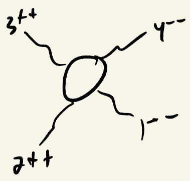
dispersive sum rules relate low and high energies

(Kramers-Kronig)

---

with spin: $M = \sum_i \begin{pmatrix}\text{polarizations} \\ \text{dotted into moms...}\end{pmatrix}_i \cdot M_i(s,t)$

Ex: massless gravitons in $D=4$



$M = \underbrace{[14]^4 \langle 23 \rangle^4}_{\sim t^4,} F(s,t)$ — vanishes like s^{1-4}

Fixed $t\ (= -(p_1+p_3)^2)$

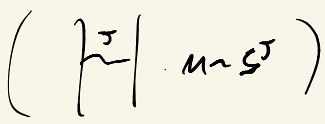
$$\int \frac{ds'}{s'^{1+k}} F(s', -p^2)$$

Fixed-u

$$\int \frac{ds'}{s'^{k-3}} F(s', t = p^2 - s)$$

both converge when $\boxed{k > J_0}$

we say that B_k is a spin-k sum rule if it converges on exchanges of a particle of spin $< k$

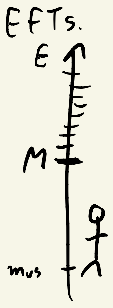


$M \sim s^J$

"super convergence" : no denominator   aka "antisubtraction"

happens when $J_1 + J_2 - 1 \geq J_0$

EFTs.



Let's discuss single real scalar in detail

EFT is useful when $m_{us} \ll M$ $\Rightarrow$ $m_{us} = 0$

Assume that EFT is weakly coupled at $\lesssim M$ $\Rightarrow$ $\mathcal{L}$

$$\mathcal{L} = -\frac{(\partial\phi)^2}{2} \left(-\frac{m^2\phi^2}{2}\right) - \frac{g\phi^3}{3!} - \frac{\lambda\phi^4}{4!}$$

$$+ \# \cancel{\frac{(\partial^2\phi)^2}{M^2}} + \# \frac{\phi(\partial\phi)^2}{M} + \dots \frac{g_2}{2}(\partial\phi)^4 + \dots$$

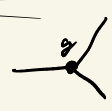
Wait! experiments don't measure $\mathcal{L}$! Nor ϕ!

remove by $\phi \to \phi + \frac{\#}{M^2}\partial^2\phi + \frac{\#}{M^4}\partial^4\phi + \dots$

anything $\sim$ leading Eoms is removable

~~$\phi\, \partial_\mu \phi\, \partial^\mu \phi = \phi^2 \partial^2 \phi \simeq 0$~~

what survives $\simeq$ on-shell amplitudes M_3



$$M_4 = \overbrace{-g^2\left(\tfrac{1}{s} + \tfrac{1}{t} + \tfrac{1}{u}\right) - \lambda}^{\text{relevant}} + \underset{(\text{polynomials})}{\text{higher-derivatives}} + O(\text{loops}) \quad \text{in } s, t\, (u).$$

---

symmetrical polynomials: ~~linear:~~ ~~$s + t + u = 0$~~   ~~$(\partial^4\phi)^2\phi^2$~~

quadratic : $s^2 + t^2 + u^2$

cubic : stu

all symmetrical polys are products of these two!

(sH) $M_{low} = -g^2\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right) - \lambda$

$+ g_2(s^2+t^2+u^2) + g_3\, stu + g_4 (s^2+t^2+u^2)^2 + g_5\, stu(s^2+t^2+u^2)$

$+ g_6 (s^2+t^2+u^2)^3 + g_6' (stu)^2 + \ldots$

$+ O(\text{loops})$

---

what we constrain: observables linear in $M_{2\to 2}$ at $|s| \sim M^2$, $|t| < M^2$

$\updownarrow$ = nonlinear fcts of Lagrangian parameters.

with Spin:

in $D=4$ massless

$M^{++--++} = [14]^4 \langle 23\rangle^4 f(s,t)$ → = polynomial

$[g_4 + \cdots\cdots]$

"Riem^4"

$M^{++++--} = [13]^2[23]^2 \; [1\;2\;4]^2\langle 34\rangle^2$

$[\text{const} + \cdots]$

$D^4 \text{Riem}^4$

$M^{++++++} = \langle 12\rangle^4 \langle 34\rangle^4$ or $\langle 13\rangle^2\langle 24\rangle^2$

$= \frac{\langle 12\rangle^2\langle 34\rangle^2}{[12]^2[34]^2}\, h(s,t)$ ✓ Fully sym

Riem$^4 \to (s^2+t^2+u^2)^2$

Dutta + Godde + Gopalka 1910.14392 + Halden + Janagal + Minwalla

Ourieux + Kitahara + Machado + Shadmi + Weiss 2008.09652

← for massive amps in 4d.

Key concept: local module

= basis of module (over s,t) for polynomials in pols and moms

⇒ contact terms

⇒ polynomials multiplying these!

$-B_2(-p^2)\big|_{low} = g_2 + p^2 g_3 + p^4 g_4 + \cdots\cdots$ (grow like s^2)

$s^2 t^2$, "s", m^4

$-B_4(-p^2)\big|_{low} = g_4 + p^2 g_5 + \cdots$

relevant part is killed!

At high-energies: Symmetries ↳ partial wave analysis.

$\text{Im}\, M\big|_{high} = \sum_J \boxed{\text{Im}\, a_J(s)}\, n_J\, \underbrace{P_J(\cos\theta)}_{\text{spherical harmonics}}$, $1+\frac{2t}{s}$

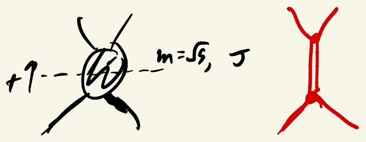
↑ dynamics.

$S = 1 + iM$

$S|s,J\rangle_{in} = (1 + i a_J(s))\, |s,J\rangle_{in}$

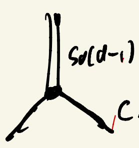
$|S|^2 \le 1$ ⇒ $0 \le \text{Im}\, a_J \le 2$

all we know: these unknowns are positive and bounded!

Spinning case: 

partial waves ⇔ couplings between a massive particle and two external ones

in 4d: nice in terms of helicity ↳ Wigner d-functions

in higher-d: 

$SO(d-1)$ Young Tableau

Chakraborty + Dutta + Gopalka + Kundu + Minwalla 2001.07117

$-B_2\big|_{low} = B_2\big|_{high}$

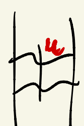
$g_2 + g_3 p^2 + g_4 p^4 + \cdots = \sum_J \int_{M^2}^{\infty} \frac{dm^2}{m^2}\, \text{Im}\, a_J(m^2)\, \frac{P_J\left(1-\frac{2p^2}{m^2}\right)}{\left(m^2(m^2-p^2)\right)^{\frac{k+2}{2}}}$

$= \left\langle \frac{P_J\left(1-\frac{2p^2}{m^2}\right)}{\left(m^2(m^2-p^2)\right)^{\#}} \right\rangle$ heavy states $m > M$, J even

---

correct logic: this should be smeared against normalizable $\psi(p)$ ⇒ converge at large J.

Simpler logic: do Taylor expansions around forward (ok if we neglect loops in EFT (?))

 ⇒ $M \sim e^{-2ub}$ at large b

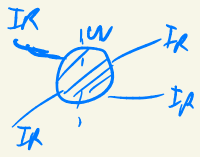
$\sim \frac{1}{u^{D-2}}$ at $p=0$

mass singularities spoil $\frac{1}{M}$ power counting if one tries forward limits at loop level.

$g_2 = \left\langle \frac{1}{m^4} \right\rangle$, $g_3 = \left\langle \frac{3 - \# J(J+D-3)}{m^6} \right\rangle$

$g_4 = \left\langle \frac{1}{2m^8} \right\rangle_{m > M}$, $0 = \left\langle \frac{J^2(J^2-\#)}{m^8} \right\rangle_{UV} = 0$

$0 = \langle \ldots \rangle$ ← null constraints from IR crossing symmetry.



How do we use this?

0) Find clever, ad hoc inequalities — Cauchy-Schwarz

1) Numerics (linear programming)

2) (double) Moment problem ↳ analytically list bdys

3) geometric fct thy

$\frac{1}{m^8} \le \frac{1}{m^4 M^4}$ for $m > M$

⇒ $g_4 \le \frac{g_2}{2M^4}$

$? \le g_3 \le \frac{3g_2}{M^2}$

$\frac{g_2(\partial)^4}{M^4} + \frac{g_3 \partial^6 \phi^4}{M^6}$

$0 = \left\langle \frac{J^2(J^2-\#)}{m^8} \right\rangle_{UV}$ let $b = \frac{2J}{m}$.

+ Cauchy Schwarz $\left\langle \frac{J^2}{m^6} \right\rangle^2 \le \left\langle \frac{1}{m^4} \right\rangle \left\langle \frac{J^4}{m^8} \right\rangle$

$\left\langle \frac{b^2}{m^4} \right\rangle^{\cancel{2}} \le \left\langle \frac{1}{m^4} \right\rangle \left\langle \frac{b^4}{m^4} \right\rangle$

$\le \left\langle \frac{1}{m^4} \right\rangle \# \cancel{\left\langle \frac{b^2}{m^4} \right\rangle}$

$$\boxed{\left\langle \frac{b^2}{m^4} \right\rangle \le \# \left\langle \frac{1}{m^4} \right\rangle}$$

As far as EFT coefficients g_0, states with $s > M^2$ have size $b \le \frac{1}{M}$.

extended states: strings, black holes ... can never dominate dispersive sum rules