# Demazure crystal structure for flagged reverse plane partitions

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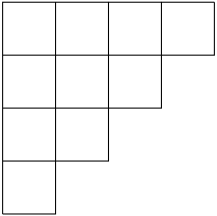
# Overview

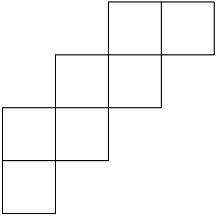
# Reverse plane partition

## Partition and skew shapes

- A **partition** $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a weakly decreasing finite sequence of non-negative integers.
- A partition is visualized as its Young diagram.
- A skew shape $\lambda/\mu$ is the set-theoretic difference $\lambda - \mu$ of the Young diagrams.
- We consider partitions with at most *n* parts ($n \in \mathbb{N}$ is fixed).



$(4, 3, 2, 1)$

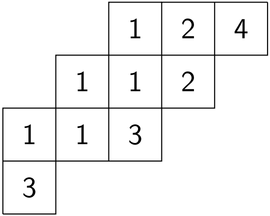


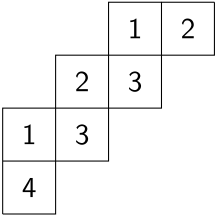
$(4, 3, 2, 1)/(2, 1)$

# Reverse Plane Partition

## Definition

A **reverse plane partition** of skew shape $\lambda/\mu$ is a filling of the skew diagram $\lambda/\mu$ with positive integers which is weakly increasing along both rows and columns.



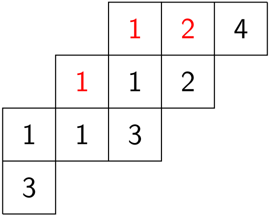




## Semi-standard tableau

A **semi-standard tableau** of shape $\lambda/\mu$ is a reverse plane partition of the same shape such that the entries are strictly increasing along columns.

# Row reading word, weight, height of a RPP $T$

- The **weight** of $T$ is defined as $wt(T) = (t_1, t_2, \cdots)$, where $t_i$ is the number of columns of $T$ that contain an $i$.
- The **row reading word** $r_T$ is defined as follows: omit all entries from $T$ which are equal to the entry immediately below it; then read $T$ from left to right and bottom to top.
- The **height** $h(T)$ of $T$ is defined as the sequence of positive integers whose $i^{th}$ part (from the left) is the row number of the $i^{th}$ letter (from the left) in $r_T$.




row reading word 3113124, weight $(3, 1, 2, 1)$, height 4333221

# Flagged reverse plane partition

## Flagged reverse plane partition

- A **flag** $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_n)$ is defined as a weakly increasing sequence of positive integers with $\Phi_n = n$.
- A reverse plane partition respects flag $\Phi$ if every entry in the $k^{th}$ row is at most $\Phi_k$ for all $k$.
- $R$ respects the flag $(1, 2, 3, 4)$ whereas $S$ does not.

$R =$

| | | | |
|---|---|---|---|
| | | 1 | 1 |
| | 1 | 2 | |
| 1 | 3 | | |
| 4 | | | |

$S =$

| | | | |
|---|---|---|---|
| | | 1 | 2 |
| | 2 | 3 | |
| 1 | 3 | | |
| 4 | | | |

# Flagged skew Schur polynomial

- Tab$(\lambda/\mu, \Phi)$ is the set of all semi-standard tableaux of shape $\lambda/\mu$ which respect flag $\Phi$.

## Definition

The **flagged skew Schur polynomial** is defined as

$$s_{\lambda/\mu}(X_\Phi) = \sum_T \mathbf{x}^{wt(T)},$$

where $T$ varies over Tab$(\lambda/\mu, \Phi)$ and for $\alpha \in \mathbb{Z}_+^n, \mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

## Special case

- If $\Phi = (n, n, \cdots, n)$ then $s_{\lambda/\mu}(X_\Phi) = s_{\lambda/\mu}(x_1, x_2, \cdots, x_n)$ which is skew Schur polynomial.

# Flagged skew dual stable Grothendieck polynomial

- $\mathrm{RPP}(\lambda/\mu, \Phi)$ is the set of all reverse plane partitions of shape $\lambda/\mu$ which respect flag $\Phi$.

**Definition ([Kim20], [Hwa+21])**

The **flagged skew dual stable Grothendieck polynomial** $g_{\lambda/\mu}(X_\Phi)$ is defined by

$$g_{\lambda/\mu}(X_\Phi) = \sum_R \mathbf{x}^{wt(R)},$$

where $R$ varies over $\mathrm{RPP}(\lambda/\mu, \Phi)$.

- $g_{\lambda/\mu}(X_\Phi) = s_{\lambda/\mu}(X_\Phi) +$ lower degree terms.
- If $\Phi = (n, n, \cdots, n)$ then $g_{\lambda/\mu}(X_\Phi) = g_{\lambda/\mu}(x_1, x_2, \cdots, x_n)$ which is the skew dual stable Grothendieck polynomial.
- $g_{\lambda/\mu}(x_1, x_2, \cdots, x_n) = s_{\lambda/\mu}(x_1, x_2, \cdots, x_n) +$ lower degree terms.

# Key Polynomial

## Demazure crystal

For $\sigma \in \mathfrak{S}_n, \lambda \in \mathcal{P}[n]$, the **Demazure crystal** $\mathcal{B}_\sigma(\lambda)$ is defined as:

$$\mathcal{B}_\sigma(\lambda) := \{f_{i_1}^{k_1} f_{i_2}^{k_2} \cdots f_{i_p}^{k_p} T_\lambda : k_j \geq 0\} \setminus \{0\} \subset \mathrm{Tab}(\lambda),$$

where $s_{i_1} s_{i_2} \cdots s_{i_p}$ is any reduced expression of $\sigma$ and $T_\lambda$ is the unique semi-standard tableau of shape and weight both equal to $\lambda$.

## Key polynomial

For $\sigma \in \mathfrak{S}_n, \lambda \in \mathcal{P}[n]$, the **key polynomial** is $\kappa_\alpha := \sum_{T \in \mathcal{B}_\sigma(\lambda)} \mathbf{x}^{wt(T)}$

$(\sigma.\lambda = \alpha)$.

## Theorem (Reiner-Shimozono)

The key polynomials $\{\kappa_\alpha\}$ as $\alpha$ varies over all compositions form an $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \cdots]$.

# Previous results

- A polynomial $f \in \mathbb{Z}[x_1, x_2, \cdots]$ is called **key positive** if it is a sum of key polynomials.

**Theorem (Reiner-Shimozono)**

$s_{\lambda/\mu}(X_\Phi)$ is key positive. Explicitly,

$$s_{\lambda/\mu}(X_\Phi) = \sum_{Q} \kappa_{\widehat{\beta(Q)}},$$

where $Q$ varies over all $(\lambda/\mu, \Phi)$-compatible tableaux and $\beta(Q)$ is the weight of the left-key tableau $K\_(Q)$ of $Q$.

The crystal-theoretic version of the above result is as follows:

**Theorem (K-Raghavan-Sathish-Viswanath)**

$\mathrm{Tab}\,(\lambda/\mu, \Phi)$ is a disjoint union of Demazure crystals.

# New Results

## Main Theorem (K)

$\mathrm{RPP}(\lambda/\mu, \Phi)$ is a disjoint union of Demazure crystals (up to isomorphism). More precisely,

$$\mathrm{RPP}(\lambda/\mu, \Phi) \cong \bigsqcup_{Q} \mathcal{B}_{\tau}(\widehat{\beta(Q)}^{\dagger}),$$

where $Q$ varies over all $(\lambda/\mu, \Phi)$-compatible tableaux for RPP and $\tau$ is any permutation such that $\tau.\widehat{\beta(Q)}^{\dagger} = \widehat{\beta(Q)}$.

## Corollary

$g_{\lambda/\mu}(X_{\Phi})$ is key positive. Explicitly,

$$g_{\lambda/\mu}(X_{\Phi}) = \sum_{Q} \kappa_{\widehat{\beta(Q)}},$$

where $Q$ varies over all $(\lambda/\mu, \Phi)$-compatible tableaux for RPP.

# A sketch of proof

Fix $Q$, a $(\lambda/\mu, \Phi)$-compatible tableau for RPP.

$$\mathcal{C}(\lambda/\mu, Q, \Phi) := \{T \in \mathrm{RPP}(\lambda/\mu, \Phi) : \begin{bmatrix} h(T) \\ r_T \end{bmatrix} \xleftrightarrow{\text{Burge}} (-, Q)\}.$$

Thus $\mathrm{RPP}(\lambda/\mu, \Phi) = \bigsqcup_{Q} \mathcal{C}(\lambda/\mu, Q, \Phi)$, where $Q$ runs over all $(\lambda/\mu, \Phi)$-compatible tableaux for RPP.

## Proposition

$\Psi : \mathcal{C}(Q, \lambda/\mu, \Phi) \to \mathcal{B}_\tau(\widehat{\beta(Q)}^{\dagger})$ via $T \mapsto P(r_T)$ is a weight-preserving bijection which commutes the crystal raising and lowering operators. ($P(r_T)$ is the unique semi-standard tableau Knuth equivalent to $r_T$.)

# References


Siddheswar Kundu, K.N. Raghavan, V. Sathish Kumar, Sankaran Viswanath
Saturation for Flagged Skew Littlewood-Richardson coefficients
arXiv:2305.05195.

Siddheswar Kundu
Demazure crystal structure for flagged reverse plane partitions
arXiv:2305.05195.

Victor Reiner, Mark Shimozono
Key polynomials and a flagged Littlewood-Richardson rule
*J. Combin. Theory Ser. A*
https://doi.org/10.1016/0097-3165(95)90083-7

Pavel Galashin
A Littlewood-Richardson rule for dual stable Grothendieck polynomials
*J. Combin. Theory Ser. A*
https://doi.org/10.1016/j.jcta.2017.04.001

# References


Jang Soo Kim
Jacobi–Trudi formulas for flagged refined dual stable Grothendieck polynomials
*Algebraic Combinatorics*
https://doi.org/10.5802/alco.203

Byung-Hak Hwang, Jihyeug Jang, Jang Soo Kim, Minho Song and U-Keun Song
Refined canonical stable Grothendieck polynomials and their duals
arXiv:2104.04251

Kohei Motegi, Travis Scrimshaw
Refined dual Grothendieck polynomials, integrability, and the Schur measure
arXiv:2012.15011

Thank You