Towards a zero-one law for improvements to Dirichlet's approximation theorem

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Dirichlet's Theorem on simultaneous Diophantine approximation

Theorem (Dirichlet)

Fix $m, n \in \mathbb{N}$. For any $A \in M_{m,n}(\mathbb{R})$ and $t > 1$, there exists $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$ satisfying the following system of inequalities:

$$\|A\mathbf{q} - \mathbf{p}\|^m < \frac{1}{t} \quad \text{and} \quad \|\mathbf{q}\|^n \leq t.$$

- Here $\|(x_1, \ldots, x_k)\| = \max\{|x_i| : 1 \leq i \leq k\}$.
- $m = n = 1$, $|x - \frac{p}{q}| < \frac{1}{qt}$ and $1 \leq q \leq t$.

Corollary (Dirichlet)

For any $A \in M_{m,n}(\mathbb{R})$, there exist infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$ satisfying

$$\|A\mathbf{q} - \mathbf{p}\|^m < \|\mathbf{q}\|^{-n}.$$

Question

Can we improve Dirichlet's theorem?

Asymptotic approximation: Khintchine-Groshev Theorem

Let $\psi : [1, \infty) \to (0, \infty)$ be continuous, decreasing and $\lim_{t \to \infty} \psi(t) = 0$.

Definition

$A \in M_{m,n}(\mathbb{R})$ is ψ-approximable if $\exists$ ∞ many $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\})$ satisfying

$$\|A\boldsymbol{q} - \boldsymbol{p}\|^m < \psi(\|\boldsymbol{q}\|^n).$$

- A is ψ-approximable if and only if $A + A'$ is ψ-approximable for any $A \in M_{m,n}(\mathbb{Z})$.
- $W(\psi) \subset M_{m,n}(\mathbb{R}/\mathbb{Z})$ the set of ψ-approximable real matrices.

Theorem (Khintchine-Groshev)

$$Leb(W(\psi)) = \begin{cases} 1 & \text{if } \sum_k \psi(k) = \infty, \\ 0 & \text{if } \sum_k \psi(k) < \infty. \end{cases}$$

Uniform approximation

Definition (Kleinbock-Wadleigh, 2018)

$A \in M_{m,n}(\mathbb{R})$ is ψ-Dirichlet if for all sufficiently large t, there exists $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\})$ satisfying the following system of inequalities:

$$\|A\boldsymbol{q} - \boldsymbol{p}\|^m < \psi(t) \quad \text{and} \quad \|\boldsymbol{q}\|^n < t.$$

- A is ψ-Dirichlet if and only if $A + A'$ is ψ-Dirichlet for any $A' \in M_{m,n}(\mathbb{Z})$.
- Denote by $\mathrm{DI}(\psi) \subseteq M_{m,n}(\mathbb{R}/\mathbb{Z})$ the set of ψ-Dirichlet real matrices.

Question

Is there a zero-one law for Leb($DI(\psi)$) analogous to the Khintchine-Groshev theorem?

- Let $\psi_1(t) = \frac{1}{t}$. $\mathrm{DI}(\psi_1) = M_{m,n}(\mathbb{R}/\mathbb{Z})$ (Dirichlet).
- A is Dirichlet improvable $\Leftrightarrow$ A is $c\psi_1$-Dirichlet for some $0 < c < 1$.
- $\mathrm{Leb}(\mathrm{DI}(c\psi_1)) = 0$ (Davenport-Schmidt, 1969).
- $\mathrm{Leb}(W(\psi)) = \mathrm{Leb}(W(c\psi))$ for any $c > 0$.

Interesting cases for ψ

In view of results of Dirichlet and Davenport-Schmidt, to get interesting results for Leb(DI(ψ)), we need for any $0 < c < 1$

$$c\psi_1(t) < \psi(t) < \psi_1(t) \quad \text{for all sufficiently large } t.$$

That is to say we need

$$\psi(t) = \frac{1 - a(t)}{t},$$

where $a(t) : [1, \infty) \to (0, 1)$ is some function with $a(t) \to 0$ as $t \to \infty$.

Some heuristics:

- If $a(t)$ decays fast (so that ψ is close to ψ_1), then we expect DI(ψ) to be close to DI(ψ_1), thus having large Lebesgue measure.
- If $a(t)$ decays slow (so that ψ is close to $c\psi_1$ for some $0 < c < 1$), then we expect DI(ψ) to be close to DI($c\psi_1$), thus having small Lebesgue measure.

A zero-one law for Leb(DI(ψ)) when $m = n = 1$

Theorem (Kleinbock-Wadleigh, 2018)

Assume $m = n = 1$. Let $\psi(t) = \frac{1-a(t)}{t} : [1, \infty) \to (0, \infty)$ be continuous and decreasing with also $a(t) : [1, \infty) \to (0, 1)$ also decreasing. Then

$$Leb(DI(\psi)) = \begin{cases} 1 & \text{if } \sum_{k\in\mathbb{N}} k^{-1} a(k) \log\left(\frac{1}{a(k)}\right) < \infty, \\ 0 & \text{if } \sum_{k\in\mathbb{N}} k^{-1} a(k) \log\left(\frac{1}{a(k)}\right) = \infty. \end{cases}$$

- If $a(t) = (\log t)^{-c}$ ($\Leftrightarrow \psi(t) = \frac{1-(\log t)^{-c}}{t}$), then

$$\mathrm{Leb}(\mathrm{DI}(\psi)) = \begin{cases} 1 & \text{if } c > 1, \\ 0 & \text{if } 0 < c \le 1. \end{cases}$$

- Proof uses continued fractions, not applicable to higher dimensions.

Main result: a partial zero-one law on Leb(DI(ψ))

Theorem (Kleinbock-Strömbergsson-Y. 2021)

Fix $m, n \in \mathbb{N}$. Set $d = m + n$, $\alpha = \frac{d^2+d-4}{2}$ and $\beta = \frac{d^2-d}{2}$. Let $\psi(t) = \frac{1-a(t)}{t} : [1, \infty) \to (0, \infty)$ be continuous and decreasing with also $a(t) : [1, \infty) \to (0, 1)$ also decreasing. Then

$$Leb(DI(\psi)) = \begin{cases} 1 & \text{if } \sum_{k\in\mathbb{N}} k^{-1} a(k)^\alpha \log^\beta\left(\frac{1}{a(k)}\right) < \infty, \\ 0 & \text{if } \sum_{k\in\mathbb{N}} k^{-1} a(k)^\alpha \log^\beta\left(\frac{1}{a(k)}\right) = \infty \text{ and } (\star). \end{cases}$$

Here

$$\liminf_{t\to\infty} \frac{\sum_{1\le k\le t} k^{-1} a(k)^\alpha \log^{\beta+1}\left(\frac{1}{a(k)}\right)}{\left(\sum_{1\le k\le t} k^{-1} a(k)^\alpha \log^\beta\left(\frac{1}{a(k)}\right)\right)^2} = 0. \qquad (\star)$$

- When $m = n = 1$, ($\star$) is not needed (Kleinbock-Wadleigh).
- ($\star$) says that when the series $\sum_{k\in\mathbb{N}} k^{-1} a(k)^\alpha \log^\beta(\frac{1}{a(k)})$ diverges, the rate of divergence can not be too slow.

Examples

- $a(t) = (\log t)^{-c}$ ($\Leftrightarrow \psi(t) = \frac{1-(\log t)^{-c}}{t}$) then

$$\mathrm{Leb}(\mathrm{DI}(\psi)) = \begin{cases} 1 & \text{if } c > 1/\alpha, \\ 0 & \text{if } 0 < c \leq 1/\alpha. \end{cases}$$

- (zoom in at the critical case) $a(t) = (\log t)^{-1/\alpha}(\log\log t)^{-c}$ then

$$\mathrm{Leb}(\mathrm{DI}(\psi)) = \begin{cases} 1 & \text{if } c > \frac{\beta+1}{\alpha}, \\ 0 & \text{if } 0 < c < \frac{\beta}{\alpha}. \end{cases}$$

When $\frac{\beta}{\alpha} \leq c \leq \frac{\beta+1}{\alpha}$ our theorem does not give any information on $\mathrm{Leb}(\mathrm{DI}(\psi))$.
(When $m = n = 1$, Kleinbock-Wadleigh implies $\mathrm{Leb}(\mathrm{DI}(\psi)) = 0$).

Homogeneous dynamics

- Let $d = m + n$. $X = G/\Gamma = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ parameterizes the space of covolume 1 lattices in $\mathbb{R}^d$ via $g\Gamma \leftrightarrow g\mathbb{Z}^d$.
- μ the unique G-invariant probability measure on X.
- G acts on X via left multiplication.
- The matrix space $M_{m,n}(\mathbb{R}/\mathbb{Z})$ naturally embeds in X via

$$A \in M_{m,n}(\mathbb{R}/\mathbb{Z}) \mapsto \Lambda_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^d \in X.$$

It thus gets identified with the sub-manifold

$$Y = \{\Lambda_A : A \in M_{m,n}(\mathbb{R}/\mathbb{Z})\} \subseteq X$$

endowed Lebesgue measure.

Dani correspondence

Proposition (Dani Correspondence)

There exists a continuous decreasing function $r : [s_0, \infty) \to \mathbb{R}_{>0}$ uniquely determined by $\psi(t) = \frac{1-a(t)}{t}$ such that

$$A \notin DI(\psi) \iff g_s\Lambda_A \in K_{r(s)} \text{ for an unbounded set of } s > 0,$$

where $g_s := diag(e^{s/m} I_m, e^{-s/n} I_n)$ and $K_r := \left\{\Lambda \in X : \Lambda \cap (-e^{-r}, e^{-r})^d = \{\mathbf{0}\}\right\}$. Moreover,

$$\sum_k k^{-1} a(k)^\alpha \log^\beta\left(\frac{1}{a(k)}\right) < \infty \iff \sum_k r(k)^\alpha \log^\beta\left(\frac{1}{r(k)}\right) < \infty.$$

In other words,

$$\mathrm{DI}(\psi)^c = \limsup_{s\to\infty}(g_{-s}K_{r(s)} \cap Y).$$

Discretize it: define $B_k := \bigcup_{0\le s<1} g_{-s}K_{r(k+s)}$, then

$$\mathrm{DI}(\psi)^c = \limsup_{k\to\infty}(g_{-k}B_k \cap Y).$$

Measure of limsup sets: Borel-Cantelli lemma

Lemma (Convergence case)

Let (X, ν) *be a probability space. Given* $\{A_k\}_{k\in\mathbb{N}}$ *a sequence of measurable sets. If* $\sum_k \nu(A_k) < \infty$, *then* $\nu(\limsup_{k\to\infty} A_k) = 0$.

Lemma (Divergence case)

If $\sum_k \nu(A_k) = \infty$ *and* $\{A_k\}_{k\in\mathbb{N}}$ *further satisfies the following quasi-independence condition that*

$$\liminf_{k_2\to\infty} \frac{\left|\sum_{k_1\le i\ne j\le k_2} \nu(A_i\cap A_j) - \nu(A_i)\nu(A_j)\right|}{\left(\sum_{i=k_1}^{k_2} \nu(A_i)\right)^2} = 0 \quad \text{for some } k_1 \in \mathbb{N}, \tag{QI}$$

then $\nu(\limsup_{k\to\infty} A_k) = 1$.

The convergence case

Recall $\mathrm{DI}(\psi)^c = \limsup_{k\to\infty}(g_{-k}B_k \cap Y)$. Borel-Cantelli lemma tells us

$$\begin{cases} \sum_k \mathrm{Leb}(g_{-k}B_k \cap Y) < \infty & \Longrightarrow \mathrm{Leb}(\mathrm{DI}(\psi)^c) = 0, \\ \sum_k \mathrm{Leb}(g_{-k}B_k \cap Y) = \infty \ \&\ (\mathrm{QI}) & \Longrightarrow \mathrm{Leb}(\mathrm{DI}(\psi)^c) = 1. \end{cases}$$

Need to restate the convergence or divergence of the series $\sum_k \mathrm{Leb}(g_{-k}B_k \cap Y)$ in terms of $\psi = \frac{1-a(t)}{t}$.

Theorem (Kleinbock-Strömbergsson-Y. 2021)

$$\sum_k k^{-1} a(k)^\alpha \log^\beta\left(\frac{1}{a(k)}\right) < \infty \iff \sum_k Leb(g_{-k}B_k \cap Y) < \infty.$$

- This theorem settles the convergence case.
- Proof consists of two steps:
  Step 1. Estimate $\mu(B_k)$.
  Step 2. Relate $\mathrm{Leb}(g_{-k}B_k \cap Y)$ with $\mu(B_k)$.

Step 1: A quantitative Hajós' theorem

Recall $B_k = \bigcup_{0\le s<1} g_{-s}K_{r(k+s)}$ and

$$K_r = \left\{\Lambda \in X : \Lambda \cap (-e^{-r}, e^{-r})^d = \{\mathbf{0}\}\right\}.$$

$\{K_r\}_{r>0}$ are compact neighborhoods of the critical locus for the supremum norm in $\mathbb{R}^d$:

$$K_0 := \left\{\Lambda \in X : \Lambda \cap (-1, 1)^d = \{\mathbf{0}\}\right\}.$$

Theorem (Hajós, 1941)

Let U be the subgroup of upper triangular unipotent matrices in $\mathrm{SL}_d(\mathbb{R})$, and let W be the subgroup of permutations. Then

$$K_0 = \bigcup_{w\in W} (wUw^{-1})\mathbb{Z}^d.$$

- $\mu(K_r) \to \mu(K_0) = 0$ as $r \to 0^+$. Need a more precise asymptotic formula.

Theorem (Kleinbock-Strömbergsson-Y. 2021)

Let $\alpha = \frac{d^2+d-4}{2}$ *and* $\beta = \frac{d^2-d}{2}$.

$$\mu(K_r) \asymp_d r^{\alpha+1} \log^{\beta}\left(\tfrac{1}{r}\right), \quad \text{as } r \to 0^+.$$

Step 1: A quantitative Hajós' theorem

Up to permutations, $g\mathbb{Z}^d \in K_r$ roughly means each diagonal entry satisfies $g_{ii} = 1 + O(r)$, contributing

$$\int_{1-cr}^{1+cr} dx \asymp r$$

with $d-1$ copies, and each pair of symmetric off-diagonal entries (g_{ij}, g_{ji}) satisfies $|g_{ij}g_{ji}| \ll r$ and $\max\{|g_{ij}|, |g_{ji}|\} \ll 1$, contributing

$$\int_{\{(x,y):\max\{|x|,|y|\}\ll 1, |xy|\ll r\}} dxdy \asymp r\log(\frac{1}{r})$$

with $\frac{d(d-1)}{2}$ copies. In total

$$r^{d-1}(r\log(\tfrac{1}{r}))^{\frac{d(d-1)}{2}} = r^{\alpha+1}\log^{\beta}(\tfrac{1}{r}).$$

Corollary

$$\mu(\bigcup_{0\le s<1} g_{-s}K_r) \asymp r^{\alpha}\log^{\beta}\left(\tfrac{1}{r}\right), \quad as\ r \to 0^+.$$

In particular,

$$\sum_k k^{-1}a(k)^{\alpha}\log^{\beta}\left(\tfrac{1}{a(k)}\right) < \infty \Leftrightarrow \sum_k r(k)^{\alpha}\log^{\beta}\left(\tfrac{1}{r(k)}\right) < \infty \quad \Leftrightarrow \quad \sum_k \mu(B_k) < \infty.$$

Step 2: Effective equidistribution of expanding horoshperes

To relate $\mathrm{Leb}(g_{-k}B_k \cap Y)$ with $\mu(B_k)$ we use

Proposition (Kleinbock-Margulis, 1996)

There exists $\delta > 0$ such that for any $f \in C_c^\infty(X)$ and for any $k \in \mathbb{N}$,

$$\int_Y f(g_k\Lambda_A)\, dA = \mu(f) + O(e^{-\delta k}\mathcal{S}(f)),$$

where $\mathcal{S}(\cdot)$ is some Sobolev norm.

Note

$$\mathrm{Leb}(g_{-k}B_k \cap Y) = \int_Y \chi_{B_k}(g_k\Lambda_A)\, dA.$$

- Take $f_k \approx \chi_{B_k}$ a smooth function approximating χ_{B_k}.
- $\mathcal{S}(f_k) \asymp r(k)^{-L}$ for some $L > 0$.

Step 2: Effective equidistribution of expanding horoshperes

To summarize we show

$$\mathrm{Leb}(g_{-k}B_k \cap Y) = \int_Y \chi_{B_k}(g_k\Lambda_A)\, dA \approx \int_Y f_k(g_k\Lambda_A)\, dA \approx \mu(f_k) \approx \mu(B_k).$$

Making all these "$\approx$" precise one shows that

$$\sum_k \mathrm{Leb}(g_{-k}B_k \cap Y) < \infty \quad \Longleftrightarrow \quad \sum_k \mu(B_k) < \infty.$$

We have shown from step 1 that

$$\sum_k k^{-1} a(k)^{\alpha} \log^{\beta}\left(\frac{1}{a(k)}\right) < \infty \quad \Longleftrightarrow \quad \sum_k \mu(B_k) < \infty$$

Combining these two we get

$$\sum_k k^{-1} a(k)^{\alpha} \log^{\beta}\left(\frac{1}{a(k)}\right) < \infty \quad \Longleftrightarrow \quad \sum_k \mathrm{Leb}(g_{-k}B_k \cap Y) < \infty.$$

Divergence case: Effective doubly mixing of expanding horospheres

To verify quasi-independence condition, need to show

$$\mathrm{Leb}(g_{-i}B_i \cap g_{-j}B_j \cap Y) \approx \mathrm{Leb}(g_{-i}B_i \cap Y)\mathrm{Leb}(g_{-j}B_j \cap Y)$$

on average. For this we use

Proposition (Kleinbock-Shi-Weiss, 2017; Björklund-Gorodnik, 2019)

For any $f_1, f_2 \in C_c^\infty(X)$ and for any $i \neq j$,

$$\int_Y f_1(g_i\Lambda_A)f_2(g_j\Lambda_A)\,dA = \mu(f_1)\mu(f_2) + O\left(e^{-\delta \min\{i,j,|i-j|\}}\mathcal{S}(f_1)\mathcal{S}(f_2)\right).$$

Note

$$\mathrm{Leb}(g_{-i}B_i \cap g_{-j}B_j \cap Y) = \int_Y \chi_{B_i}(g_i\Lambda_A)\chi_{B_j}(g_j\Lambda_A)\,dA.$$

- Not so useful when $|i-j|$ is small.

Divergence case: Effective doubly mixing of expanding horospheres

- When $|i-j|$ is large, taking $f_i \approx \chi_{B_i}$ and $f_j \approx \chi_{B_j}$ and applying this effective doubly mixing we get

$$\begin{aligned}\mathrm{Leb}(g_{-i}B_i \cap g_{-j}B_j \cap Y) &\approx \int_Y f_i(g_i\Lambda_A)f_j(g_j\Lambda_A)\,dA \approx \mu(f_i)\mu(f_j)\\ &\approx \mu(B_i)\mu(B_j) \approx \mathrm{Leb}(g_{-i}B_i \cap Y)\mathrm{Leb}(g_{-j}B_j \cap Y).\end{aligned}$$

- For $i \le j$ with $j-i$ small we use the trivial bound

$$\begin{aligned}\mathrm{Leb}(g_{-i}B_i \cap g_{-j}B_j \cap Y) &= \int_Y \chi_{B_i}(g_i\Lambda_A)\chi_{B_j}(g_j\Lambda_A)\,dA\\ &\le \int_Y \chi_{B_j}(g_j\Lambda_A)\,dA = \mathrm{Leb}(g_{-j}B_j \cap Y).\end{aligned}$$

- The extra condition ($\star$) is needed to ensure (QI).

Thank you for your attention!