PhD Synopsis

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Holographic Fluctuations at Finite Density

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Works that are included in the synopsis:

1. R. Loganayagam, Krishnendu Ray, Shivam K. Sharma, Akhil Sivakumar, *Holographic KMS relations at finite density*, JHEP 03 (2021) 233, [2011.08173].

2. Shivam K. Sharma, *Holographic Fluctuation-Dissipation Relations in Finite Density Systems*, [2501.17852].

Work that is not included in the synopsis but is related:

1. Godwin Martin, Shivam K. Sharma, *Open EFT for Interacting Fermions from Holography*, [2403.10604].

Contents

1 Introduction

Many fascinating physical systems, from charged *Quark-Gluon Plasma* (QGP) to non-Fermi liquids, exist at finite density. Understanding their transport properties and response to perturbations is difficult. This is primarily due to their strong coupling, which makes conventional perturbative techniques ineffective. Over the past two decades, the AdS/CFT correspondence has provided valuable insights into these strongly coupled systems [1].

The AdS/CFT is a concrete realization of holography that links a $(d+1)$-dimensional Anti-de Sitter (AdS) spacetime to a conformal field theory (CFT) on its boundary, see Fig. 1. It maps strongly coupled boundary dynamics to a weakly coupled bulk gravitational theory, offering a powerful tool for studying otherwise intractable systems. In particular, it allows us to study strongly correlated systems through classical computations in black hole geometries [1].

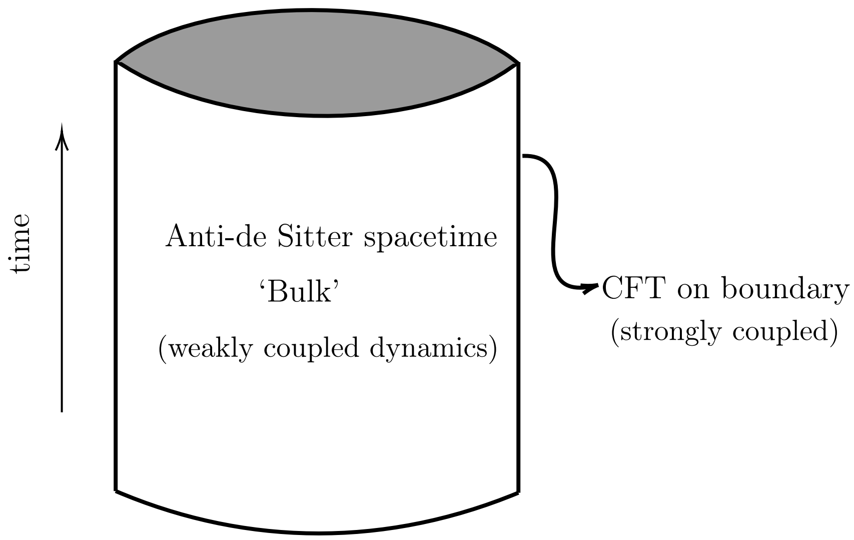


Figure 1: AdS/CFT Correspondence.

Black holes have recently emerged as a powerful tool for studying open quantum field theories (QFTs) [2]. The idea is to model *bath* as a holographic system and derive an effective field theory (EFT) for a *probe* coupled to this bath. Since holographic baths are strongly interacting and therefore *forgetful*, they naturally lead to a local

EFT for the probe. In contrast, weakly coupled baths retain *memory*, resulting in a non-local EFT. This distinction highlights the crucial role of holography in formulating local open EFTs. So, taking QGP as a bath, the resulting open EFT can capture the correlations within the strongly coupled plasma. Here, we note that a real-time formulation is essential for deriving this EFT [2].

Real-time holography has a rich history, beginning with the works of Son-Starinets and Skenderis-van Rees [3, 4]. However, these approaches did not incorporate fluctuations or Hawking effects in the bulk. To address this issue, [5] proposed a gravitational dual to the Schwinger-Keldysh (real-time) formalism of QFT, known as *grSK geometry*. The grSK construction involves stitching together two copies of the black hole exterior along their future horizons, as illustrated in Fig. 2. Yet, extending these results to finite-density systems remained an open problem for some time.

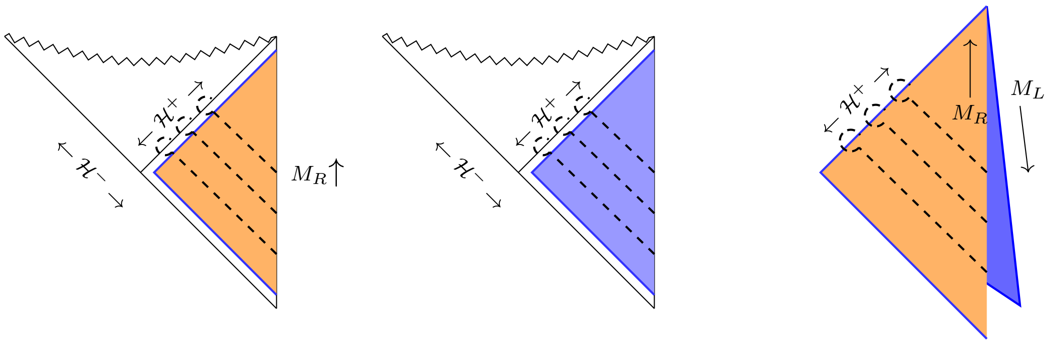


Figure 2: Schematic construction of grSK geometry with the black dashed lines denoting the radial direction.

Since charged black holes serve as natural bulk duals for finite-density systems (see [1]), a necessary step forward is to extend the grSK prescription to incorporate these backgrounds, leading to what we call the *RNSK geometry*[1] [6]. With this gravity dual to the Schwinger-Keldysh formalism at finite density, we can now ask about the role of fluctuations in such holographic systems. Specifically, can we construct real-time correlators from an *exterior field theory* living outside the black hole horizon that naturally incorporates these fluctuations? As we will see, the answer is yes, and the resulting structure closely mirrors the zero-density case [7].

[1]We will describe the details of RNSK geometry later.

Fluctuations in finite-density holographic systems raise an important question: How are the fluctuations from Hawking radiation connected to the dissipative effects of black hole quasi-normal modes? Since all thermal systems obey some form of the *fluctuation-dissipation theorem* (FDT),[2] establishing its validity in this setting could offer new insights into finite-density holography. Here, we emphasize the distinction between linear and non-linear FDTs.

Linear FDT has been extensively studied in systems ranging from weakly to strongly coupled regimes. A classic example is Brownian motion described by following Langevin's equation:

$$\ddot{q} + \gamma \dot{q} = \mathfrak{f}\,\eta(t)\;, \qquad \text{with linear FDT as:} \qquad \frac{2}{\beta}\gamma = \mathfrak{f}\;, \tag{1.1}$$

where γ is the damping coefficient, β denotes inverse temperature, η denotes noise and $\mathfrak{f}$ measures fluctuation strength. A common argument is that the random kicks from the environment not only hinder the particle's motion but also simultaneously generate noise. Then the FDT precisely asserts that, on average, the random kicks from the fluctuating force balance out the energy lost through dissipation. More generally, the underlying principle is straightforward: linear FDT follows directly from the two-point KMS condition [8].

However, calculating non-linear FDTs in strongly coupled field theories remains highly challenging and has only been carried out up to a certain order. Recent developments in grSK geometry have enabled the computation of non-linear FDTs at zero density to arbitrary order [2]. As of now, this is the only known method capable of deriving non-linear FDTs at all orders, with higher-order results remaining unverified by any other approach.

Regarding non-linear FDTs at finite density, to the best of our knowledge, they have not yet been explored in the context of strongly coupled field theories. This naturally raises two key questions: Do non-linear FDTs exist at finite density? And if they do, how do they differ from their zero-density counterparts? As we will

[2] In this note, we will use the terms FDT and FDR interchangeably.

demonstrate, the answer to this question is affirmative [9].

2 Our setup & its gravity dual

We start this section by considering a CFT at finite temperature and finite density within the real-time (SK) formalism and then propose its gravitational dual. Given that charged black holes are natural bulk descriptions of finite-density systems, we designate this dual as the *Reissner-Nordström Schwinger-Keldysh* (RNSK) geometry.

We define the RNSK geometry as one constructed by taking the RN–AdS black brane[3] and replacing the radial interval extending from the outer horizon to infinity by a doubled contour, parametrized by ζ in Fig. 3. We then obtain a geometry with two copies of RN–AdS exteriors smoothly stitched together by an 'outer-horizon cap':

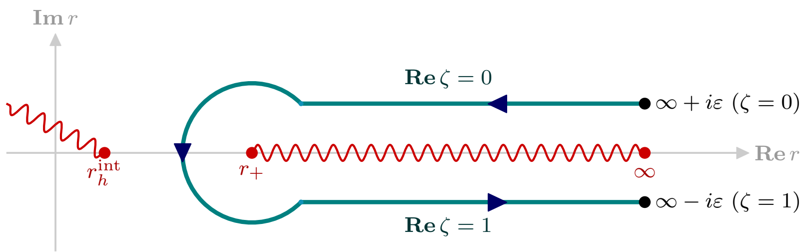


Figure 3: The radial contour parametrized by ζ, at fixed v.

The RNSK geometry in ingoing coordinates with the parameter ζ is:

$$ds^2 = -r^2 f\, dv^2 + i\,\beta\, r^2 f\, dv\, d\zeta + r^2\, d\mathbf{x}_{d-1}^2 \ , \qquad \mathcal{A}_M\, dx^M = -\mu \left(\frac{r_+}{r}\right)^{d-2} dv \ , \tag{2.1}$$

where f is the emblackening factor of the RN–AdS black brane,

$$f(r) \equiv 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d} \ , \qquad \text{where} \quad \{Q, M\} \in \{\text{Charge, Mass}\} \ , \tag{2.2}$$

with β is its inverse Hawking temperature, r_+ is its outer horizon radius, and μ is its chemical potential.

[3] The geometry is a solution for the Einstein-Maxwell bulk action.

Given that the RNSK geometry is elaborated, we next ask: If the ingoing modes are known, how do we construct the outgoing Hawking modes? A natural approach would be to apply time reversal, but the situation is more subtle. In reality, a full CPT symmetry is required to properly transform an ingoing solution into an outgoing one. This boundary CPT symmetry can be systematically implemented on bulk fields using standard AdS/CFT rules, as detailed in [6]. For a mathematically inclined reader, the boundary CPT transformation manifests as an automorphism of the RNSK principal bundle in the bulk.

Ultimately, we obtain a mapping that takes an ingoing solution to its corresponding outgoing solution. Here, we explicitly present this map for a scalar field of charge q:

$$\Phi^{\text{in}}_{\text{q}}(r,k) \longmapsto e^{-\beta k^0 \zeta} e^{\beta\mu_q \Theta}\, \Phi^{\text{in}}_{-\text{q}}(r,-k) \equiv \Phi^{\text{out}}_{\text{q}}(r,k) \ , \tag{2.3}$$

where we have defined

$$\mu_q \equiv \mu q \ , \qquad\qquad \Theta \equiv -\frac{1}{\mu}\int_0^{\zeta} \mathrm{d}\zeta' \mathcal{A}_v(\zeta') \ . \tag{2.4}$$

The prefactor in the exponent shows that this map takes ingoing solutions that are analytic in r to outgoing solutions which exhibit branch cuts.

The outgoing solution in a black hole geometry corresponds to the Hawking modes. To recover their statistical factors correctly, we must understand how these quantities behave across the horizon cap. Using the fact that ζ jumps by unity and the gauge field $\mathcal{A}_v$ has the value $-\mu$ at the horizon, we see that the scale factor in $e^{\beta\mu_q \Theta}$ is the fugacity of the boundary theory, as given below

$$\lim_{\zeta\to 0} e^{\beta\mu_q \Theta} = 1 \ , \qquad \lim_{\zeta\to 1} e^{\beta\mu_q \Theta} = e^{\beta\mu_q} \ . \tag{2.5}$$

For a field of spin s, one gets the statistical factor $(-1)^{2s}e^{-\beta(k^0-\mu_q)}$ which is identical to the factor gained in Euclidean path integrals as we traverse the thermal circle .

Consistency checks of RNSK prescription

After introducing the geometry, we examine its validity by probing it with fields and verifying the necessary consistency conditions the probe should obey. Let us consider a free Dirac field Ψ as a probe, with the action given by:

$$\mathbb{S}_\Psi = i \oint \mathrm{d}\zeta \int \mathrm{d}^d x \sqrt{-g}\, \overline{\Psi} \left(\Gamma^A \, \mathbb{D}_A - m\right) \Psi + i \int \mathrm{d}^d x \, \overline{\Psi}\, \mathcal{P}_- \, \Psi \, \Bigg|_{\zeta=0}^{\zeta=1} , \tag{2.6}$$

where the symbol $\mathbb{D}_A$ represents the covariant derivative, and $\mathbb{P}_\pm$ are the appropriate projection operators. Explicitly,

$$\mathbb{D}_A \equiv \partial_A - iq\mathcal{A}_A + \frac{1}{4}\omega_{abA}\Gamma^a\Gamma^b \ , \qquad \mathcal{P}_\pm \equiv \frac{1}{2}\left(\mathbb{1} \pm \Gamma^\zeta\right) \ , \tag{2.7}$$

where ω_{abA} are the spin connection coefficients and Γ^a are the bulk gamma matrices.[4] The second term is a variational boundary term, added in to ensure that this action admits a well-posed variational principle.

The Dirac equation $\left(\Gamma^M \, \mathbb{D}_M - m\right) \Psi = 0$, with the relevant AdS/CFT boundary conditions,

$$\lim_{\zeta\to 0} r^{\frac{d}{2}-m}\, \mathcal{P}_+ \, \Psi = \mathcal{P}_+ S_0 \, \psi_{\rm L} \ , \qquad \lim_{\zeta\to 1} r^{\frac{d}{2}-m}\, \mathcal{P}_+ \, \Psi = \mathcal{P}_+ S_0 \, \psi_{\rm R} \ . \tag{2.8}$$

can be solved using boundary-to-bulk Green's function. Here, $\psi_{\rm L}$ and $\psi_{\rm R}$ are the boundary sources in *left-right* basis.

Here S_0 is a constant boundary-to-bulk matrix, defined as the boundary limit of ingoing boundary-to-bulk propagator $S^{\rm in}$. These boundary conditions uniquely determine the solution in terms of ingoing and outgoing solutions:

$$\Psi(\zeta, k) = -S^{\rm in}(\zeta, k) \cdot \psi_{\bar{F}}(k) - e^{\beta\left(k^0-\mu_q\right)} S^{\rm out}(\zeta, k) \cdot \psi_{\bar{P}}(k) \ , \tag{2.9}$$

[4] Our notation for the indices is as follows: Lowercase Latin indices are for vielbein frame gamma matrices whereas uppercase Latin indices are for spacetime gamma matrices.

where we have defined the sources in the *Future-Past* (FP) basis as

$$\psi_{\bar{F}} \equiv n_k^{\rm FD}\left(\psi_{\rm R} - \psi_{\rm L}\right) - \psi_R\,, \qquad\qquad \psi_{\bar{P}} \equiv n_k^{\rm FD}\left(\psi_{\rm R} - \psi_{\rm L}\right)\ , \tag{2.10}$$

with $n_k^{\rm FD} \equiv \frac{1}{e^{\beta(k^0-\mu q)}+1}$ being the Fermi-Dirac factor at non-zero chemical potential.

Once we have the solution, we can evaluate the on-shell action or *SK generating functional* $\mathcal{Z}_{\rm SK}$ on the boundary. As explicitly seen in [6], the $\mathcal{Z}_{\rm SK}$ computed using the RNSK geometry satisfies both the SK *collapse rules* (microscopic unitarity), as well as the KMS conditions (thermality), i.e., the $\mathcal{Z}_{\rm SK}$ cannot contain terms with only $\psi_{\bar{F}}$ or terms with only $\psi_{\bar{P}}$.

It is crucial to note that the prescription has originally only been checked for free probes. However, if we want to trust this prescription, we need to go beyond free fields and understand interactions. Taking a cue from the Feynman diagrammatic understanding of interactions in grSK geometry [7, 10], we extended this analysis to finite-density systems (RNSK geometry) [9]. In bulk, this facilitates the study of scattering processes against black holes in the presence of Hawking radiation.

3 Interactions in the bulk geometry

As an illustrative example, we will only consider a massless complex scalar field interacting via a quartic term

$$S = \ -\oint d\zeta\, d^dx\sqrt{-g}\left[|D_M\Phi|^2 + \frac{\lambda}{2!\,2!}|\Phi|^4\right]\ , \qquad \text{with} \quad D_M \equiv \nabla_M - iq\mathcal{A}_M\ , \tag{3.1}$$

where q is the charge and λ is the interaction parameter of the field. Varying the above action, we obtain the field equation for Φ. We will solve this equation perturbatively in the coupling constant λ by expanding the $\Phi = \sum_{i=0}^{\infty}\lambda^i\Phi_{(i)}$ under the double-Dirichlet boundary conditions, given by

$$\lim_{\zeta\to 0,1}\Phi_{(0)} = J_{\rm L,R}\ , \qquad \lim_{\zeta\to 0,1}\Phi_{(i)} = 0 \quad \forall \quad i > 1\ , \tag{3.2}$$

where $J_{\text{L,R}}$ are the left and the right boundary sources respectively. The boundary conditions above are *doubled* versions of the standard conditions in AdS/CFT, modified specifically to fit the Schwinger-Keldysh formalism.

The solution can be expressed in terms of the ingoing (G^{in}) and outgoing (G^{out}) boundary-to-bulk Green's functions, along with the bi-normalizable ($\mathbb{G}$) bulk-to-bulk Green's function, as shown below

$$\begin{aligned}\Phi_{(0)}(\zeta,k) &= -G^{\text{in}}(\zeta,k)J_{\bar{F}}(k) + e^{\beta(k^0-\mu_q)}G^{\text{out}}(\zeta,k)J_{\bar{P}}(k)\ ,\\ \Phi_{(i)}(\zeta,k) &= \oint_{\zeta'}\mathbb{G}(\zeta|\zeta',k)\mathbb{J}_{(i)}(\zeta',k)\ ,\qquad \text{with}\quad \oint_{\zeta}\equiv\oint d\zeta\sqrt{-g}\ ,\end{aligned} \tag{3.3}$$

where $\mathbb{J}_{(i)}$ are the bulk sources for i-th order term in the solution and the boundary sources in the *Future-Past* (FP) basis, defined as

$$J_{\bar{F}} = -\left[1+n_k\right]J_{\text{R}} + n_k\,J_{\text{L}}\ ,\qquad\qquad J_{\bar{P}} = -n_k\left[J_{\text{R}} - J_{\text{L}}\right]\ , \tag{3.4}$$

where $n_k = \frac{1}{e^{\beta(k^0-\mu_q)}-1}$ is the Bose-Einstein statistical factor. Here, we also define the *average-difference* basis, which will be important in the subsequent discussion:

$$J_a \equiv \frac{J_{\text{R}}+J_{\text{L}}}{2}\ ,\qquad\qquad J_d \equiv J_{\text{R}} - J_{\text{L}}\ . \tag{3.5}$$

Inserting the perturbative solution into the bare action yields the *on-shell action* S_{os} . The on-shell action S_{os}, expanded in powers of λ, is expressed as:

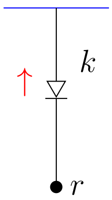
$$S_{\text{os}} = S_{(2)} + \lambda S_{(4)} + \lambda^2 S_{(6)} + \dots\ , \tag{3.6}$$

where the subscript indicates the number of boundary sources. In open EFT context, $S_{(n)}$ represents the n-point influence phase of the boundary theory [2].

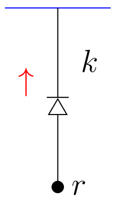
The explicit computation of the on-shell action in the bulk naturally gives rise to an *exterior field theory* [9]. This theory exists outside the outermost horizon of the RN-AdS black hole. Now, we will define Feynman diagrammatic rules for the exterior theory and then apply these rules to compute the on-shell action.

Feynman Rules for Exterior Diagrammatics

The *boundary-to-bulk propagators* for complex scalar are:

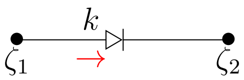


$$\equiv \frac{G^{\text{in}}(r,k)}{1+n_k}\; J_{\bar{F}}(k) \ ,$$

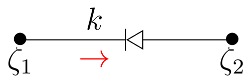


$$\equiv G^{\text{out}}(r,k)\; J_{\bar{P}}(k) \ ,$$

and the retarded and the advanced *bulk-to-bulk propagators* are:



$$\equiv -i\mathbb{G}_{\text{ret}}(r_2|r_1,k) \ ,$$

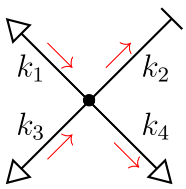


$$\equiv -i\mathbb{G}_{\text{adv}}(r_2|r_1,k) \ .$$

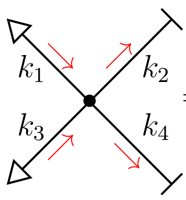
where the red-colored arrow indicates the direction of charge.[5] Before specifying the vertices in this exterior field theory, it is useful first to define

$$n_{k,\alpha} \equiv \frac{1}{e^{\beta(k^0-\alpha\mu_q)}-1} \ , \qquad \text{and} \quad k_{12\ldots m} \equiv k_1 + k_2 + \ldots k_m \ . \tag{3.7}$$

Now, let us come to the *vertices* in this exterior field theory with some of them given as:

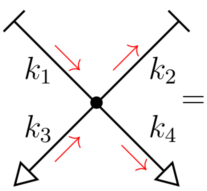


$$= -i\lambda \frac{n_{-k_2,1}}{n_{k_{134},-1}} \ ,$$

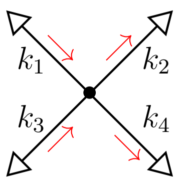


$$= -i\lambda \frac{n_{-k_2,1}\,n_{-k_4,1}}{n_{k_{13},-2}} \ ,$$

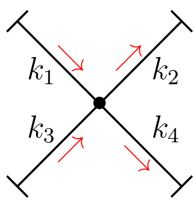
and



$$= -i\lambda \frac{n_{-k_1,-1}\,n_{-k_2,1}}{n_{k_{34},0}} \ ,$$



$$=$$



$$= 0 \ .$$

[5]Convention: momentum flows from the boundary to the bulk, while it flows from left to right in bulk-to-bulk propagators.

where one can easily see the charge conservation at the bulk vertex. For each of the vertices above, we have chosen the convention that all momenta flow into the vertex.

The fact that vertices with only 'triangle' legs or only 'bar' legs vanish follows from the collapse rule and the KMS condition, respectively. Notably, these vertices closely resemble those found in thermal field theory. One can interpret the legs with triangles as a single effective leg carrying the combined momentum and charge, which explains why some scattering processes seem to involve particles with zero or double the charge. This feature is quite general and holds for fermions as well, with the additional observation that two opposite fermions with opposite quantum numbers behave like a boson [10].

Along with the above Feynman rules for propagators and vertices, to obtain the *boundary Schwinger-Keldysh generating functional* $\mathcal{Z}_{\rm SK}$ (on-shell action),[6] we must further supply –

1. Multiply every diagram by i.
2. The vertices are integrated over the exterior of the black brane, with the following radial exterior integral

$$\int_{\rm ext} = \int_{r_+}^{r_c} dr\, r^{d-1} \;, \tag{3.8}$$

   where r_c is the radial cutoff required for holographic renormalisation [4].
3. Impose momentum conservation at each vertex and integrate over all momenta.
4. Divide by the symmetry factor of the diagram.

For instance, a diagram contributing to the $S_{(4)}$ is:

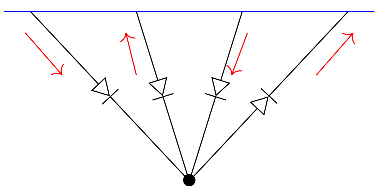


Figure 4: A Witten diagram that contributes to the four-point influence phase $S_{(4)}$.

[6]It is equal to the boundary influence phase in the large N limit.

and the term corresponding to this diagram is:

$$-\frac{\lambda}{2}\int_{k_{1,2,3,4}} \frac{\bar{J}_{\bar{F}}(k_1)J_{\bar{F}}(k_2)\bar{J}_{\bar{P}}(k_3)J_{\bar{P}}(k_4)}{n_{k_4,1}} \int_{\text{ext}} \bar{G}^{\text{in}}(r,k_1)G^{\text{in}}(r,k_2)\bar{G}^{\text{in}}(r,k_3)G^{\text{out}}(r,k_4)\ , \tag{3.9}$$

where we have used the following notation

$$\int_{k_1,\ldots,m} \equiv \int \frac{d^dk_1}{(2\pi)^d}\cdots\int \frac{d^dk_m}{(2\pi)^d}\ \delta^{(d)}\left(k_1+\ldots+k_m\right)\ . \tag{3.10}$$

The bulk interpretation of the above diagram is a scattering process, where three in-going modes (two with $-q$ charges and one with $+q$ charge) scatter into one outgoing mode with $-q$ charge. Similarly, other contact diagrams and higher-point exchange diagrams contributing to the $\mathcal{Z}_{\text{SK}}$ can be built using these Feynman rules.

4 Holographic Fluctuation-Dissipation Relations

Let us now come to the question we asked about the fluctuations in finite-density systems. The on-shell action at quadratic order in the *average-difference* basis is found to be:

$$S_{(2)} = \int \frac{d^dk}{(2\pi)^d}\left[K_{\text{ret}}\bar{J}_d(-k)J_a(k) + K_{\text{adv}}\bar{J}_a(-k)J_d(k) + K_{\text{kel}}\bar{J}_d(-k)J_d(k)\right]\ , \tag{4.1}$$

where K_{ret}, K_{adv} and K_{kel} denote the two-point retarded, advanced and Keldysh boundary correlators respectively. These correlators obey the two-point Kubo-Martin-Schwinger (KMS) condition, which is given by

$$K_{\text{kel}} = \frac{1}{2}\coth\left(\frac{\beta(k^0-\mu_q)}{2}\right)\left[K_{\text{ret}} - K_{\text{adv}}\right]\ . \tag{4.2}$$

The equation above explicitly demonstrates the linear *Fluctuation-Dissipation Relation* (FDR). To understand this, recall that $\text{Im}\left[K_{\text{ret}}\right]$ corresponds to the spectral function at finite temperature, while K_{kel} represents twice the anti-commutator. Commutators present in response functions naturally measure dissipation, whereas anti-commutators capture fluctuations. Consequently, Eq. (4.2) provides a correct description of the linear FDR in finite-density holographic systems.

We now extend this analysis to non-linear FDRs at finite density by considering the quartic term in the on-shell action $S_{(4)}$. Using contact diagrams (see Fig. 4) and considering terms up to linear order in derivatives,

$$S_{(4)} = \int d^d x \ \mathcal{L}_0 + \int d^d x \ \mathcal{L}_1 + \mathcal{O}\left((\beta\partial_t)^2\right) \ , \tag{4.3}$$

where the terms $\mathcal{L}_0$ and $\mathcal{L}_1$ represent the non-derivative and first-derivative contributions to $S_{(4)}$, respectively, given by:

$$\begin{aligned}
\mathcal{L}_0 &= \sum_{r,s=0}^{2} \mathcal{G}_{r,s} \ \left[\bar{J}_a\right]^r \left[J_a\right]^s \left[\bar{J}_d\right]^{2-r} \left[J_d\right]^{2-s} \ , \\
\mathcal{L}_1 &= \sum_{r=0}^{1}\sum_{s=0}^{2} \left\{ \mathcal{G}_{\dot{r},s}\left(\partial_t \bar{J}_a\right) + \mathcal{H}_{\dot{r},s}\left(\partial_t \bar{J}_d\right) \right\} \left[\bar{J}_a\right]^r \left[J_a\right]^s \left[\bar{J}_d\right]^{1-r} \left[J_d\right]^{2-s} \\
&\quad + \sum_{r=0}^{2}\sum_{s=0}^{1} \left\{ \mathcal{G}_{r,\dot{s}}\left(\partial_t J_a\right) + \mathcal{H}_{r,\dot{s}}\left(\partial_t J_d\right) \right\} \left[\bar{J}_a\right]^r \left[J_a\right]^s \left[\bar{J}_d\right]^{2-r} \left[J_d\right]^{1-s} \ .
\end{aligned} \tag{4.4}$$

These coefficients are not independent but related to each other [9]. Here, we only present two relations that hold at the level of $S_{(4)}$:

$$\begin{aligned}
\left(\mathcal{G}_{\dot{0},0} + \mathcal{G}_{\dot{0},0}\right) - \left(\mathcal{H}_{\dot{1},0} + \mathcal{H}_{0,\dot{1}}\right) - 2i\beta\mathcal{G}_{0,0} + \frac{i\beta}{12}\left(\mathcal{G}_{2,0} + \mathcal{G}_{1,1} + \mathcal{G}_{0,2}\right) = 0 \ , \\
-\frac{1}{3}\left(\mathcal{G}_{\dot{0},2} + \mathcal{G}_{\dot{1},1} + \mathcal{G}_{2,\dot{0}} + \mathcal{G}_{1,\dot{1}}\right) + \left(\mathcal{H}_{\dot{1},2} + \mathcal{H}_{2,\dot{1}}\right) + \frac{i\beta}{3}\left(\mathcal{G}_{2,0} + \mathcal{G}_{1,1} + \mathcal{G}_{0,2}\right) = 0 \ ,
\end{aligned} \tag{4.5}$$

where the terms containing time derivatives correspond to dissipation coefficients, while those without derivatives represent the fluctuation strength.[7] Therefore, these equations describe the non-linear FDRs at finite density.

[7] These terms correspond to various types of fluctuations, such as those arising from the Compton effect, pair production, etc in QED.

Zero-density limit: In the zero-density limit ($\mu_q \to 0$) along with the real scalar field limit, the corresponding expression of quartic on-shell action is-

$$S_{(4)} \propto \int d^d x \left\{ \sum_{k=1}^{4} \theta_k \frac{J_a^{4-k}}{(4-k)!} \frac{(iJ_d)^k}{k!} - \sum_{k=1}^{3} \bar{\theta}_k \frac{J_a^{4-k}}{(4-k)!} \frac{(iJ_d)^{k-1}}{(k-1)!} \partial_t (iJ_d) \right\} \tag{4.6}$$

with the coefficients related by the following equation

$$\frac{2}{\beta}\bar{\theta}_k + \theta_{k+1} + \frac{1}{4}\theta_{k-1} = 0 \ . \tag{4.7}$$

This is the same generalised FDRs for zero-density systems first obtained in [2], where stochastic field-theoretic description has also been developed. Thus, we have reproduced the correct effective action and the non-linear FDRs at zero density.

5 Outlook

In this work, we constructed open EFTs for a probe interacting with a holographic system at finite density, such as a charged plasma. This was achieved by extending the gravitational Schwinger-Keldysh framework to charged black holes [6], which serve as the holographic dual of a CFT evolving in real-time at finite temperature and chemical potential. In this geometry, we demonstrated how outgoing Hawking modes emerge from ingoing quasi-normal modes through a bulk CPT transformation. To validate our approach, we confirm that the resulting influence phase for the probe satisfies the expected KMS relations, incorporating the correct statistical factors.

In bulk, our approach constructs an *exterior field theory* to describe interactions in a charged black hole background, confined outside the outermost horizon [9]. Here, we extended the work of [7] by developing an exterior EFT for studying scattering against charged black holes in the presence of Hawking radiation. In particular, we developed a tree-level Witten diagrammatics to evaluate real-time correlators for a finite-density holographic CFT.

One of the major insights from our analysis is the identification of relationships among boundary correlators, which manifest as (non)-linear fluctuation-dissipation relations (FDRs) up to linear order in the small density. Here, we verified that our results correctly reproduce the expected zero-density ($\mu_q \to 0$) behaviour [9]. To our knowledge, this is the first derivation of FDRs for a holographic system at finite density. Although a complete understanding of holographic FDRs at arbitrary density remains an open question, our work represents significant progress in the small-density regime.

There are several ways to extend this work. A natural next step is to apply our analysis to fermionic systems, using the framework of open EFTs for interacting fermions [10]. Since fermions are present in nearly all physical systems, establishing the existence of FDRs in this context would provide deeper insights into strongly coupled physical systems.

Another important direction is to investigate the impact of loop corrections in the bulk exterior theory [11]. One notable effect of these corrections is the modification of the thermal mass of bulk fields. These corrections, corresponding to $\frac{1}{N}$ effects in the boundary theory, raise questions about how FDRs behave when such effects are included. Addressing this issue would also help to disentangle the statistical and quantum origins of fluctuations.

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