Classical Gravitational Scattering

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Refs:

- [1]: ArXiv:1910.14392 S. Duttachowdhury, A. Gadde, I. Halder, T. Gopalka, L. Janagal and S. M.
- [2]: ArXiv 2001.07117, S. Chakraborty , S. Duttachowdhury, T. Gopalkam S. Kundu, A. Mishra and S.M.
- [3] 2103.02122, D. Chandorkar, S. Dutta Chowdhury, S. Kundu and, S.M.

Contents

- Universal tree level gravitational scattering in string compactifications.
- Conjectures I, II and III
- Causality and the CEMZ Result.
- Regge Boundedness and the from the chaos bound.

Introduction: Type II Scattering

- Consider type II string compactifications of the form $R^p \times M$ where M is an internal manifold. E.g. $p = 4$ and $M =$ a CY manifold.
- Such a compactification defines a quantum theory of gravity on R^p. Spectrum includes p dimensional gravitons.
- The scattering amplitudes of these gravitons are given by the following schematic formula

$$\mathcal{A} = \sum_g \int d\tau_i dz_i < V_1(z_1) \ldots V_n(z_n) >$$

where V_n are the graviton vertex operators, z_i are their insertion locations, and τ_i are the moduli of the genus g Reimann surface. Expectations values are taken in the sigma model on $R^p \times M$.

Introduction: Type II Scattering loop amplitudes

- As graviton vertex operators all lie in the R^p part of the CFT, the formula above can be simplified to

$$\mathcal{A} = \sum_g \int d\tau_i \, Z_M(\tau_i) \, C_{R^p}(\tau_i)$$

$$C_{R^p}(\tau_i) = \int dz_i < V_1(z_1) \ldots V_n(z_n) > |_{R^p}$$

  where $Z_M(\tau_i)$ is partition function of the sigma model on M on the Reimann surface. The vertex operator expectation values are taken purely in the R^p part of the CFT.
- Even though $C_{R^p}(\tau_i)$ are universal - independent of M - $Z_M(\tau_i)$ - and hence the integral over τ_i above - clearly depends on M. It follows that graviton scattering amplitudes at generic values of g depend on details of the compactification manifold M (e.g. are intricate functions the CY moduli).

Introduction: Type II Scattering Tree Amplitudes

- The story above holds for generic g. Let us now, however, focus on the special case $g = 0$. As the Reimann sphere has no moduli, the integral over τ_i is absent at $g = 0$. It follows that
$$\mathcal{A}_{g=0} = Z_M^{S^2}\ C_{R^p}^{S^2}$$
Here $Z_M^{S^2}$ is the partition function of the M CFT on the 2 sphere. $Z_M^{S^2}$ is a multiplicative factor for all scattering amplitudes. It is an overall number that sets the value of the effective p dimensional Newton constant.
- $C_{R^p}^{S^2}$, the nontrivial part of the scattering amplitude is universal (i.e. independent of M). It is not hard to convince oneself that $C_{R^p}^{S^2}$ is the same for type IIA, IIB and Type I theory.

Introduction: Aside on Consistent Truncations

- Recall that a spacetime theory with two groups of degrees of freedom - $\{a_i\}$ and $\{b_j\}$ is said to admit a consistent truncation to a if its action takes the form $S = S(a) + S_{int}(a, b)$ where every term in S_{int} is of quadratic or higher order in b_i. In this situation the theory admits solutions with $b_i = 0$. These simple solutions obey the equations of motion derived from the action $S(a)$.
- Simple situation in which we find consistent truncations: System has symmetry under some group G. a_i are G singlets while b_i transform in a nontrivial representation of G.

- Spacetime implication of worldsheet discussion above: Classical type II theory on $R^p \times M$ admits a consistent truncation to a universal (i.e. M independent) theory which describes the interaction of gravitons and an infinite number of additional fields.
- The a_i type fields here are the gravitons as well as an infinite number of other fields. These are the fields whose vertex operators lie completely in the R^p sector (and have $(-1)^{F_L} = (-1)^{F_R} = 1$. These are the fields that appear as poles in graviton S matrices. The b type fields are those associated with vertex operators on M + ...
- The consistent truncation to this universal sector is a remarkable fact as there is no known spacetime symmetry explanation.

- Heterotic (and Bosonic) compactifications also admit consistent truncations to their own universal sectors.
- Finally, there is another, more elementary example of a 'classical' S matrix (defined as having only poles and no cuts). This is the classical Einstein S matrix.
- As far as I am aware, these examples exhaust the set of classical S matrices that emerge in any parameteric limit of string theory. The parametric limits relevant to the enumerated examples is $g_s \to 0$ (with no restriction on energy) for the string S matrices and $E/m_p \to 0$ with no restrictions on g_s for the Einstein S matrix.

Intro: A set of Bold (Crazy?) Conjectures

- The observations of the previous transparency motivate the following bold conjecture.
- Conjecture I: The classical Einstein S matrix, the tree level type II S matrix, and the tree level Heteroitic S matrix constitute the exhaustive list of 'consistent' tree level S matrices of gravity. Restated, every 'consistent' classical gravitational theory admits a consistent truncation to one of these three universal sectors.
- 'Tree level': no singularities apart from poles corresponding to the exchange of a massive or massless particle transforming in some representation of the Little group.
- 'Consistent': means emerges as the parametric limit of a consistent quantum theory. Implies, in particular, that it as all good properties expected of classical theories including causality and boundedness of energy, ...(see below).

Intro: Attribution

- The conjectures above were cleanly formulated (and motivated) in [1].
- That something like Conjecture I should hold has been suggested on several occasions by Nima Arkani Hamed (though perhaps not in print).
- The validity - or otherwise - of conjecture was also one of my "Two Questions About Gravity" in the talk by that title that I gave in the 50th Anniversary of String Theory session at Strings 2018.

Intro: Implications

- The provocative Conjecture I implies the following two successively weaker subconjectures.
- Conjecture II: The only consistent tree level gravitational S matrix with poles of bounded spin is the Einstein S matrix
- Conjecture III: The only consistent tree level gravitational S matrix with only gravitational poles is the Einstein S matrix.
- This is a heirarchy of Russian dolls of conjectures. $I \implies II \implies III$ but the reverse implications do not hold.
- In these lectures I will attempt to say something more concrete about conjecture II. III is a special case of II. I will have nothing further to say about the strongest - and most interesting conjecture, namely conjecture I, which has been included just for motivation.

Intro: No conjecture II for photons

- Could something like conjecture II be true for Maxwel Electrodynamics (or Yang Mills theory)? Obviously not. Simple counterexample: coupling with a scalar field via

$$\int \phi F_{\mu\nu} F^{\mu\nu}.$$

- This is an allowed coupling in a perfectly ordinary 2 derivative theory. Should be an allowed coupling. Leads to a tree level scalar exchange contribution to 4 photon scattering. Could have many such couplings with different masses. Also could have couplings to non scalar fields. Thus there clearly are infinitely many 'consistent' tree level 4 photon S matrices.

Intro: Conjecture II for gravitons

- Might initially think it would be easy to construct similar counter examples to Conjecture II for gravity in the space of manifestly good 2 derivative theories. However this is not the case.
- As one attempt to construct a two derivative (so manifestly good) counter example to Conjecture II consider the coupling $\int \phi R$
- Might at first think that this coupling leads yields a new pole contribution to 4 graviton scattering. Not the case. Coupling modifies propagators as well as 3 pt function. Need field redefinition to diagonalize propagators. Same field redefinition removes ϕhh coupling. 'Going to Einstein Frame'
- In lecture 2 we will show this example generalizes. No 2 derivative counter example to Conjecture II for 4 graviton scattering.

Causality and Scattering

- To make the conjectures above useful we need to ennumerate a set of mathematically precise consequences of 'consistency'. Theories that do not meet these criteria will then be ruled out as inconsistent.
- In these lectures I will list two such criteria. The first of these is simply causality.
- In a relativistic quantum field theory, causality implies commutator of any two spatially separated gauge invariant operators must vanish.
- In a gravitational theory, local operators do not exist in gravity. E.g consider

$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi e^{-\int \sqrt{g}R - \sqrt{g}J(x)\phi(x)}$$

$T_{\mu\nu} \propto g_{\mu\nu}\phi(x)J(x)$. Not conserved for generic $J(x)$ (the way to make this conserved is to make J dynamical - in which case it is no longer a source). Rel to diff inv.

Low Energy Constraints: Causality

- Given this, one might wonder what it even means to assert that a classical theory of gravity respects causality.? Can't demand that'the response to a source lies within its lightcone' (because there are sources).
- One criterion for cauasality goes as follows. Consider a spacetime S generated by the initial data D. Then consider a second spacetime generated by new data D', which agrees precisely with D outside a compact region R. Our theory is causal provided S and S' agree with each other outside the lightcone of R. GR is causal by this criterion.
- In these lectures we employ another criterion - better suited for the study of quantum theories (also for effective field theories).

- Consider an asymptotically flat spacetime. Send some gravitational pulses into this spacetime from $\mathcal{I}^-$ and watch them emerge at $\mathcal{I}^+$. Compare the time of emergence of the outgoing pulse with the lightcones emanating out of the ingoing pulses of an auxilliary pure flat space spacetime obtained by filling in the flat shells of spacetime around infinity (such an auxilliary flat spacetime can be uniquely constructed is true only in $D \geq 5$) If the outgoing pulse emerges outside its lightcone then the theory is acausal.
- In other words, our theory is acausal if interactions speed particles up rather than slowing them down, i.e. if interactions cause a time advance rather than a time delay.

Causality and 3 graviton scattering

- The conjectures of the previous subsection apply to the scattering of n gravitons, for all $n = 3, 4, 5 \ldots$. The case $n = 3$ is especially simple.
- This simplicity has its root in the fact that 3 graviton S matrices are highly kinematically constrained. The most general 3 graviton S matrix - classical or quanutm - is necessrily a linear combination of three structures.

$$T_1 = (\epsilon_1.\epsilon_2\epsilon_3.p_1 + \text{perm})^2 \qquad \text{2 der : Einstein}$$

$$T_2 = (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge p_1 \wedge p_2)^2 \qquad \text{4 der : GaussBonnet}$$

$$T_3 = \text{tr}(f_1 f_2)\text{tr}(f_2 f_3)\text{tr}(f_1 f_3) \qquad \text{6 der : Reimann}^3$$

$$f_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu .$$

The formulae above are actually valid only for $D \geq 5$. In $D = 4$, T_2 vanishes but a new parity odd structure appears.)

- Note in particular that all 3 graviton scattering amplitudes - classical or quantum - are always analytic in momenta.

CEMZ result on 3 graviton scattering from causality

- The most general 3 graviton S matrix takes the form
$$aT_1 + bT_2 + cT_3$$
where a, b and c are numbers (they have mass dimensions but are independent of momenta).
- CEMZ demonstrated that any theory in which either b or c is nonzero is necessarily acausal unless it couples to higher spin particles of arbitrarily high spin.
- I very briefly describe outline the nature of the argument that gives this result. Consider the contribution to four graviton scattering from a graviton pole exchange. The non analytic (pole) contribution to this amplitude is completely determined by the 3 graviton scattering amplitude.
- Using the explicit form of the exchange amplitude, CEMZ were able to show that when b and c are both nonzero, there always exist choices of polarizations for the intial graviton for which scattering at high enough energies leads to a time advance.

CEMZ from causality

- This acausality cannot be cured by adding contact terms to the Lagrangian. This is because contact terms only affect the scattering amplitude at zero impact parameter, while pole exchanges affect the amplitude even at nonzero impact parameter.
- CEMZ showed that the acuasaility is of a nature that also cannot be cured by the additional exchanges of a finite number of higher spin particles.
- On the other hand exchanges with an infinite number of higher spins can cure this problem. This happend in classical string theory. Loop diagrams can also cure the problem. This happens in, e.g. M theory.
- In other words classical gravitational theories with bounded spin are only causal if their 3 graviton scattering amplitude is that of Einstein gravity

CEMZ and Conjecture 2

- It follows that the CEMZ argument has already established conjecture 2 for the special case of three graviton scattering.
- This is very encouraging. However note that 3 graviton scattering is special as it is parameterized by finite data (there were only 3 allowed structures).
- Scattering with 4 or more gravitons has qualitatively greater complexity; scattering amplitudes are parameterized my infinite amount of data. Consequently, conjecture 2 is a much more dramatic statement for 4 or higher gravitational scattering amplitudes. Topic of rest of these lectures.

Boundedness of scattering amplitudes

- Two ingredients went into the CEMZ proof of conjecture 2 for the case $n = 3$. The first, and most important element was the physical principle (causality in that case). The second element was the complete kinematic classification of possibilities. Together these two ingredients gave us our result.
- In order to argue for Conjecture 2 for four graviton scattering, we will need anlogues of both these elements.
- We will first argue for a physical principle that can be used to constrain scattering amplitudes. We will then turn to the second element - namely find a complete kinematical classification of classical 4 gravitation amplitudes. Right at the end of these lectures we will put these two elements together to obtain our final results.

Regge Boundedness

- The physical principle we will use to constrain 4 graviton scattering is the following Classical Regge Growth or CRG conjecture
- The CRG conjecture states that “A classical theory whose scattering grows faster than s^2 at fixed t is physically unacceptable.
- There are many intuitive reasons to believe that the CRG conjecture is true. The simplest (but also weakest) argument for this conjecture comes from the fact that it is true in all classical theories that we know for sure to be consistent (i.e. to emerge as parametrically exact descriptions in a controlled limit of a good quantum theory).

CRG: Argument from examples

- Most classical theories that we know for sure to be consistent are two derivative theories.
- It follows immediately from dimensional analysis that contact diagrams in such theories cannot grow faster than s at fixed t, and that exchange diagrams cannot grow faster than $\frac{s^2}{t}$. Thus two derivative theories, almost trivially, obey the CRG conjecture. Note Einstein gravity saturates CRG.
- The only non two derivative classical theory that I feel I know for sure to be consistent is classical string theory. In string theory the S matrix in the Regge limit scales like s^{2+at} where a is a positive number. As t is negative in physical scattering, classical string theory also obeys CRG.

- In Appendix D of their now classic paper, CEMZ used a 'signal model' to intuitively argue for the CRG conjecture. As both Juan has reviewed it, and Simone will also touch on it, my review will be brief.
- First suppose an effect is related to a cause via the linear relation
$$E(t) = \int dt' G(t-t')C(t')$$
Define $E(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{E}(\omega)$, etc, so that
$$\tilde{E}(\omega) = \tilde{G}(\omega)C(\omega)$$
Now causality is the requirement that $G(t) = 0$ for $t < 0$. It implies that
$$\tilde{G}(\omega) = \int dt \; e^{i\omega t} G(t)$$
is analytic in the upper half ω plane.

CRG: CEMZ argument II

- Note that the effect at time t only depends on the cause at times $t' < t$. The upper half of the ω complex plane is singled out becuase this is where the cause

$$C(t) = \int d\omega \tilde{C}(\omega) e^{-i\omega t'}$$

is bounded for all $t' < t$ (the same is not true in the lower half of the complex plane).

- Now let us supppose that we know, for some reason, that $|G(\omega)|^2 \leq 1$, for all real ω. This tells us that

$$\int dt |E(t)|^2 = \int \frac{d\omega}{2\pi} |E(\omega)|^2 \leq \int \frac{d\omega}{2\pi} |C(\omega)|^2 = \int dt |C(t)|^2$$

In other words the integrated output 'flux' is smaller than the integrated input 'flux'.

- Let us now specialize to the study of the response, $E(t)$, to a cause that is zero for $t < 0$. It follows that $E(t)$ also vanishes for $t < 0$. For any such response ω in the upper half plane

$$|\tilde{E}(\omega)|^2 = \left| \int_0^\infty dt e^{i\omega t} E(t) \right|^2 \leq \int_0^\infty \left| e^{i\omega t} \right|^2 \int_0^\infty |E(t)|^2 \quad (1)$$

where we have used the Cauchy Shwarz inequality for the inner product on the space of square integrable functions $\langle g|f\rangle = \int g^* f$.

- Now

$$\left| \int_0^\infty e^{i\omega t} \right|^2 = \frac{1}{2\mathrm{Im}(\omega)} \implies |\tilde{E}(\omega)|^2 \leq \frac{1}{2\mathrm{Im}(\omega)} \left| \int_0^\infty |E(t)|^2 \right|^2$$

- The arguments above apply for the response to any source, whatsoever, provided it vanishes for $t<0$. In order to find an upper bound for the function $|\tilde{G}(\omega)|^2 = \frac{|\tilde{E}(\omega)|^2}{|\tilde{C}(\omega)|^2}$ we need an upper bound on $|\tilde{C}(\omega)|^2$ in one cleverly chosen example.
- CEMZ realized we could make the following clever choice. For the purpose of bounding $\tilde{G}(\omega)$ at $\omega=\omega_0+i\gamma$ ($\gamma>0$), it is useful to use the source function

$$C(t) = e^{-\gamma t}e^{-i\omega_0 t}\theta(t).$$

For this choice of function

$$\begin{aligned}\tilde{C}(\omega) &= \int_0^\infty e^{i\omega_0 t-\gamma t}e^{-\gamma t}e^{-i\omega_0 t} = \frac{1}{2\gamma} = \frac{1}{2\mathrm{Im}(\omega)} \\ \int dt|C(t)|^2 &= \frac{1}{2\mathrm{Im}(\omega)}\end{aligned} \tag{2}$$

CRG: The CEMZ argument V

- It follows that

$$
\begin{aligned}
|\tilde{G}(\omega)|^2 &= \left|\frac{\tilde{E}(\omega)}{\tilde{C}(\omega)}\right|^2 \\
&\leq 2\mathrm{Im}(\omega)\int_0^\infty |E(t)|^2 \leq 2\mathrm{Im}(\omega)\int_0^\infty |C(t)|^2 = 1
\end{aligned} \tag{3}
$$

- It follows that causality upgrades the (assumed) 'unitarity relation into one that applies everywhere in the entire upper half plane.

CRG: The CEMZ argument VI

- What does any of this have to do with scattering? Consider a two graviton partial wave scattering amplitude. Atleast in channels with unit degeneracy (see below for what this means) the unitarity tells us that $|\tilde{S}(\omega))|^2 \leq 1$ for all real $\omega > 0$.
- Moreover, atleast intuitively, the S matrix is the response (final state) to a cause (initial state) and so plausibly obeys the constraints of causality (however it is confusing here that, in the physical situation, both cause and response have $\omega > 0$, preventing pulses from being completely localized)
- If we ignore subtleties, its a small step from here to show that a perturbative partial wave (or fixed impact parameter) S matrix is cannot grow faster than s at large s (because $|1 + igAs^b|$, when expanded at order g, cannot be less than unity for any choice of sign of A when $b > 1$. This observation translates to the CRG conjectue.

CRG: Argument from the chaos bound

- The third stream of arguments in support of the CRG conjecture is the suspicion that, in the context of the AdS/CFT correspondence, the bulk CRG conjecture follows from the boundary chaos bound.
- This suggestion has been made by many people, including the authors of the original chaos bound paper. Juan alluded to this connection in his lectures at this school.
- While its nice to have intuitive arguments, it is safer (and more satisfying) to have precise results - results you are sure are true. In other words it would be nice to turn one of these sets of intuitive arguments into a proof. It would be very interesting to turn the intuitive signal model argument into a clear proof of the CRG conjecture. However I dont (atleast yet) know how to do this.
- It has, however, proved possible to make the chaos bound argument, atleast in a particular context. I will explain how that has been done (reviewing the paper [3]).

Kinematics I: Insertion points

- Consider four operators inserted at the boundary of global AdS_{D+1} space at the following two parameter set of locations.

$$\begin{aligned}
P_1 &= (\cos\tau, \sin\tau, 1, 0, \vec{0}) \\
P_3 &= (\cos\tau, \sin\tau, -1, 0, \vec{0}) \\
P_2 &= (-1, 0, -\cos\theta, -\sin\theta, \vec{0}) \\
P_4 &= (-1, 0, \cos\theta, \sin\theta, \vec{0})
\end{aligned} \tag{4}$$

- We study these insertions assuming that

$$0 \leq \tau \leq \pi, \quad 0 \leq \theta \leq \frac{\pi}{2} \tag{5}$$

Kinematics II: Insertion diagram

These insertion points may be visualized as follows:

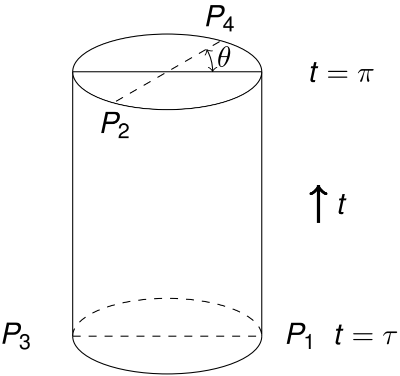


Figure: Insertion points in global AdS

Kinematics III: Three causal configurations

- Our insertion points are causally related in the following way:
- $$\begin{array}{lll} \tau > \pi - \theta & \text{Causally Euclidean,} & \\ \pi - \theta > \tau > \theta & \text{Causally Regge} & (P_4 > P_1, \text{ and } P_2 > P_3) \\ \tau < \theta & \text{Causally Scattering} & (P_4, P_2) > (P_1, P_3) \end{array} \tag{6}$$
- These three different causal configurations lie on three different sheets in conformal cross ratio space (more below)

Kinematics IV: $i\epsilon$ corrected insertions and cross ratios

- The $i\epsilon$ corrected insertions

$$\begin{aligned} P_1 &= (\cos(\tau - i\epsilon\tau), \sin(\tau - i\epsilon\tau), 1, 0) \\ P_3 &= (\cos(\tau - i\epsilon\tau), \sin(\tau - i\epsilon\tau), -1, 0) \\ P_2 &= (\cos(\pi - i\pi\epsilon), \sin(\pi - i\pi\epsilon), -\cos\theta, -\sin\theta) \\ P_4 &= (\cos(\pi - i\pi\epsilon), \sin(\pi - i\pi\epsilon), \cos\theta, \sin\theta) \end{aligned} \tag{7}$$

- Yield the following conformal cross ratios

$$\begin{aligned} z &= \frac{1}{2}(1 - \cos(\theta - \tau - i(\pi - \tau)\epsilon)), \\ \bar{z} &= \frac{1}{2}(1 - \cos(\theta + \tau + i(\pi - \tau)\epsilon)), \end{aligned} \tag{8}$$

- For some purposes useful to define new cross ratios

$$z = \sigma e^{\rho}, \quad \bar{z} = \sigma e^{-\rho} \tag{9}$$

σ and ρ easily determined on for our configurtion.

Kinematics V: Passage in cross ratio space

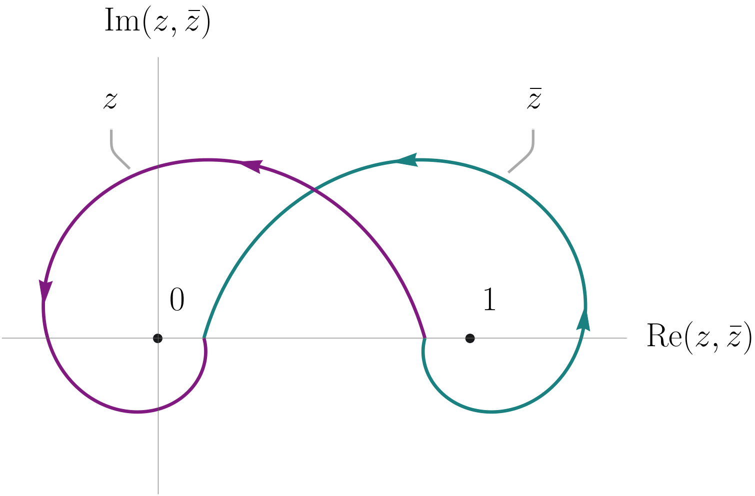


Figure: The path traversed in the complex plane by the variables z (purple) and $\bar{z}$ (green) as we lower τ from π to 0 at fixed θ. The vertical scale in these graphs is greatly exaggerated.

Two limits will play an important role in our analysis. The first of these is the limit $\tau \to 0$ with σ held fixed. This limit lies on the Causally Scattering sheet.

- $$\begin{aligned} z &= \sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - \tau\cos\frac{\theta}{2}\right) + \mathcal{O}(\tau^2) \\ \bar{z} &= \sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \tau\cos\frac{\theta}{2}\right) + \mathcal{O}(\tau^2) \\ \sigma &= \sin^2\frac{\theta}{2} + \mathcal{O}(\tau^2) \\ \rho &= -\tau\cot\frac{\theta}{2} + \mathcal{O}(\tau^3) \end{aligned} \tag{10}$$

- Note that $\rho \to 0$ in this limit

Kinematics VII: $\tau \to 0$ as bulk point

- $\tau = 0$ is special for the following reason. The four boundary points P_a ($a = 1 \ldots 4$) are generically linearly independent vectors in embedding space $SO(D, 2)$. The space they span is generically either an $R^{3,1}$ or an $R^{2,2}$.
- A one parameter tuning of cross ratios, however, can sometimes make the 4 P_a linearly dependent. In the current context this happens when $\tau = 0$ at which point

$$P_1 + P_2 + P_3 + P_4 = 0$$

- Whenever this happens it can be shown on general grounds that ρ always vanishes. If, in additon, some other causal constraints are met, it can be shown that the correlator develops a bulk point singularity - i.e. diverges like an inverse power of ρ.
- Our correlator develops a bulk point singularity in the $\tau \to 0$ limit (more below).

Kinematics VIII: Regge limit

- The second limit of interest to us is the (generalized) Regge limit defined by

$$\tau \to 0, \quad \theta \to 0, \quad \frac{\tau}{\theta} = a = \text{fixed} \tag{11}$$

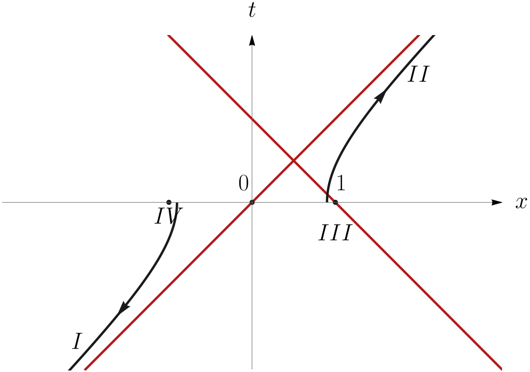
- In this limit

$$\begin{aligned} z &= \frac{(\theta - \tau - i\epsilon)^2}{4} = \frac{\theta^2}{4}(1 - a - i\epsilon)^2 + \mathcal{O}(\theta^4) \\ \bar{z} &= \frac{(\theta + \tau + i\epsilon)^2}{4} = \frac{\theta^2}{4}(1 + a + i\epsilon)^2 + \mathcal{O}(\theta^4) \\ \sigma^2 &= \frac{\theta^4(1 - a^2)}{16} + \mathcal{O}(\theta^6) \\ e^{2\rho} &= \left(\frac{1 - a - i\epsilon}{1 + a + i\epsilon}\right)^2 + \mathcal{O}(\theta^2) \end{aligned} \tag{12}$$

- Note that in this limit $z \to 0$, $\bar{z} \to 0$, $\sigma \to 0$ but $e^{2\rho}$ is nontrivial.

Kinematics IX: Regge limit significance

- When $a < 1$ the Regge limit lies on the Causally Scattering sheet (same as the $\tau \to 0$ limit). However when $a > 1$, it lies on the Causally Regge sheet. At $a = 1$ $z = 0$.
- The part of the Regge limit that lies on Causally Regge sheet has recently been extensively studied. This is most usually done by studying the following path integral in $R^{1,1}$



Aside: Chaos Bound

- We choose to normalize the path integral of the previous slide it by a suitable product of two point functions. As is usual, the operator interpretation of the path integral -in usual quantization - is the time ordered correlator

$$\frac{\langle O_2 O_3 O_4 O_1 \rangle}{\langle O_2 O_1 \rangle \langle O_4 O_3 \rangle} \tag{13}$$

- However there is another operator interpretation of the same path integral which makes it clear that this normalized path integral is constrained by the chaos bound, as we now explain. This operator interpretation works in so called angular quntization - a slicing of the Euclidean path integral in which the angular coordinate is regarded as time.

- Within angular quantization, the path integral above computes an OTOC

$$\frac{\langle O_1 O_4 O_2 O_3 \rangle}{\langle O_2 O_1 \rangle \langle O_4 O_3 \rangle} \tag{14}$$

Note that the boost in Minkowski space is simply time translation in (analytically continued) angular time.

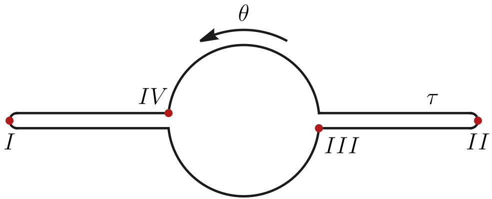


Figure:

Specializing to a large N field theory, the chaos bound applies to this new correlator. It tells us that the normalized correlator cannot grow faster than $\frac{1}{\sigma}$ as $\sigma \to 0$ at every fixed ρ.

Kinematics X: Overlap of Regge and $\tau \to 0$.

- It will be of importance to this paper that the $\tau \to 0$ limit and the Regge limit overlap.
- This statement can be quantified as follows. If we take the cross ratios computed in the $\tau \to 0$ limit and expand them to leading order in θ, we get the same cross ratios that we do upon first taking the Regge limit and then working to leading order in a.
- In other words the $\tau \to 0$ and Regge limits overlap. We can access one corner of the the $\tau \to 0$ limit by working within the Regge limit. The importance of this fact will become clear below.

- In the rest of this talk we focus attention on boundary theories with a local bulk dual. We further focus on the computation of boundary four point functions obtained by the rules of AdS/CFT via a local bulk contact term. We discuss possible generalizations at the end of this talk.
- Such correlators are all of given by experessions of the form

$$\int d^{D+1}X \frac{N}{(-P_1.X+i\epsilon)^{\tilde{a}_1}(-P_2.X+i\epsilon)^{\tilde{a}_2}(-P_3.X+i\epsilon)^{\tilde{a}_3}(-P_4.X+i\epsilon)^{\tilde{a}_4}} \tag{15}$$

  where $N = N(Z_i, X)$ is a polynomial function of Z_i and X and $\tilde{a}_i$ are positive numbers (not necessarily integers).
- We now follow the classic analysis of HPPS to study such correlators of the form (15) in the Regge limit.

Regge Limit II: Small quantities

- In the strict Regge limit
$$P_1 + P_2 = P_3 + P_4 = 0$$
The collection of points P_a span an $R^{1,1}$ in $R^{D,2}$. The subspace orthognal to this $R^{1,1}$ is an $R^{D-1,1}$.
- It is thus useful to parameterize points on AdS_{D+1} as
$$X = (\frac{u+v}{2}, y_0, \frac{v-u}{2}, y_1, y_i) \qquad (16)$$
with
$$-(y^\mu)^2 + uv = 1 \qquad (17)$$
When $u = v = 0$ gives the intersection of the orthogonal $R^{D-1,1}$ with the AdS_{D+1} hyperboloid. In other words $P_a.X = 0$ for all a when $u = v = 0$.
- It is thus intuitive (and may be shown) that the dominant contribution to the integral in the Regge limit comes from the neighbourhood of $u = v = 0$; infact from u and v that are of the same order as τ and θ.

Regge Limit III: Taylor Expansion

- In fact the correlator can be systematically expanded in a power series in θ by expanding the integrand in the four small parameters, θ, τ, u and v, and then performing the integral, first over u and v and then over y_i and y_1.
- In our paper we have studied this expansion carefully and established the following.
  - The normalized correlator admits an expansion of the form

$$\frac{1}{\theta^{2A'-2}a^{\Delta+r-3}}\left(\sum_{n=0}^{\infty}\theta^n h_n(a)\right)$$

  - The functions $h_n(a)$ are analytic in the upper half complex a plane, and all admit a power series expansion in a around $a=0$. In particular each of the functions $h_n(a)$ are well behaved (either constant or zero) in the $a \to 0$ limit.

Regge Limit IV: analyticity in a

- At leading order in the Regge limit, , in particular, the normalized four point correlators take the form

$$\frac{1}{\theta^{2A'-2} a^{\Delta+r-3}} h_0(a) \tag{18}$$

  where $h_0(a)$ is an analytic function in the upper half a plane.

- Recall $a < 1$ lies on the Causally Scattering sheet, $a > 1$ lies on the Causally Regge sheet. On the real axis $h_0(a)$ has a branch cut lightcone singularity at $a = 1$.

Regge Limit V: Analytic continuation in a

- When we take careful account of the $i\epsilon$ in denominators we find that, for physical purposes, $h_0(a)$ is evaluated at a with a infinitesimal positive imaginary piece. In other words physics instructs us to evaluate $h_0(a)$ on the real axis but to continue past any singularities via the upper half of the complex a plane

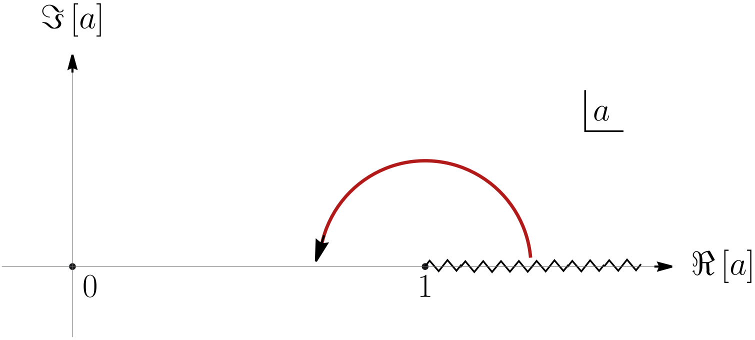
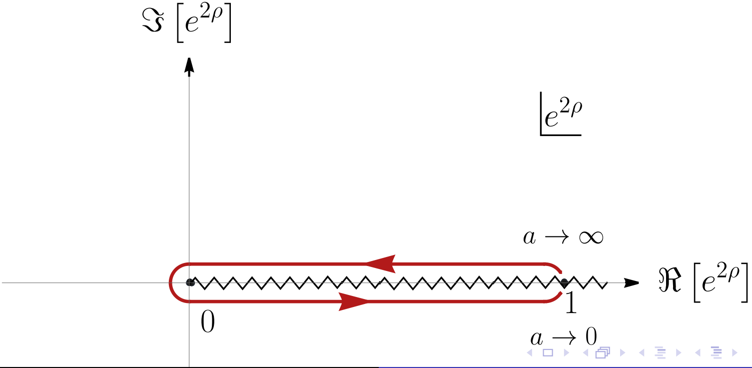


Figure:

- Recall

$$e^{2\rho} = \left(\frac{1-a-i\epsilon}{1+a+i\epsilon}\right)^2 \tag{19}$$

It follows that as a decreases from ∞ to 0 $e^{2\rho}$ (and hence z) traverses the path, in agreement with the 'branch' discussion early in this talk.



Regge Limit: Summary

- We have reached three important conclusions. First that the leading small θ scaling of correlators in the Regge limit is very simple; it takes the form $\frac{1}{\theta^{2A'-2}}$ where A' is a constant rather than a function of a
- Second that the the coefficient of this scaling, $h_0(a)$, while not actually analytic in a at $a = 1$, is the boundary value of an analytic function in the upper half plane, allowing us to analytically continue from $a < 1$ to $a > 1$. This fact makes it impossible for $h_0(a)$ to be proprtional to, e.g. $\theta(1 - a)$. In other words the constant A is the same in the Causally Regge and Causally scattering sheets.
- Third that all $h_n(a)$ are nonsingular in the $a \to 0$ limit.

Bulk Point Limit : I

- In a remarkable paper already in 2009, Gary, Giddings and Penedones demonstrated that CFTs dual to local bulk theories have unusual singularities in ρ in four point functions. For the special case of external scalar operators they also wrote down an explicit expression for the coefficient of this 'bulk point singularity' in terms of the flat space S matrix of bulk scalar fields.
- As we are principally interested in the scattering of gravitons, we needed to generalize the analysis of GGP to the study of correlators of the stress tensor (and also conserved currents). The generalization involves a few extra element (having to do with polarizations) that had no counterpart in the GGP study. To perform the generalizations we employed the method of the 'bulk point' paper of Maldacena, David Simons Duffin and Sasha Zhibeodov. The details are a bit involved and are presented in our paper in detail.

Bulk Point Limit II: Formula

- With our kinematics in the $\tau \to 0$ limit we find that the leading bulk point singularity is given by the expression

$$G_{\text{sing}} = i\left(2\pi^3\left(\tilde{\mathcal{C}}_{\Delta,J}\right)^4\right)\Gamma(\Delta+r-3)e^{-\frac{i(\Delta+r)}{2}} \frac{\sqrt{1-\sigma}^{(\Delta+r-4)}}{\sigma^{\frac{\Delta+r-2}{2}}\rho^{\Delta+r-3}} \times$$
$$\int_{H_{D-2}} d\Omega_{D-3} d\zeta \frac{\sinh^{D-3}\zeta}{\cosh\zeta^{\Delta+r-3}}\left(\frac{\mathcal{S}_X(\omega)}{\omega^r}\right) \tag{20}$$

- The H_{D-2} is the intersection of the AdS_{D+1} hyperboloid and the $R^{D-2,1}$ orthogonal to the $R^{2,1}$ spanned by the insertion points.
- $\mathcal{S}_{\mathcal{X}}(\omega)$ is the flat space S matrix generated by the bulk local interaction term (assumed to be of r order in derivatives), where the scattering waves for the S matrix are given by

Bulk Point Limit III: Formula

- $$\begin{aligned} \phi &= e^{ik.x} && \textit{Scalar} \\ A_M &= Z^{\perp}_{M}\, e^{ik.x} && \textit{Vector} \\ h_{MN} &= \left(Z^{\perp}_{M_1} Z^{\perp}_{M_2} - \frac{(X.Z)^2 \left(\eta_{M_1 M_2} + X_{M_1} X_{M_2}\right)}{D-1} \right) e^{ik.x} && \textit{Graviton} \end{aligned} \tag{21}$$

- And where the scattering momenta are given by

$$\begin{aligned} k_1 &= \omega(1, 0, 1, 0, \vec{0}) \\ k_3 &= \omega(1, 0, -1, 0, \vec{0}) \\ k_2 &= -\omega(1, 0, \cos\theta, \sin\theta, \vec{0}) \\ k_4 &= -\omega(1, 0, -\cos\theta, -\sin\theta, \vec{0}) \end{aligned} \tag{22}$$

The Bulk Point limit IV: Regge scaling

- The formulae above capture the leading small ρ behaviour of the correlator. The coefficient of this ρ singularity is a function of σ. Using the explicit expression given above, it is not difficult to demonstrate that (after normalizing the correlator) the coefficient of this singularity scales with σ like

$$G^{\rm norm}_{\rm sing} \propto \frac{1}{\sigma^{A-1}} \tag{23}$$

- whenever the flat space S matrix generated by the same contact term scales like

$$s^A$$

in the Regge limit.

CRG conjecture and the chaos bound

- With all these results in place, it is now easy to argue that - in the context studied in this paper (i.e. S matrices for scalars, photons or gravitons generated by local bulk contact terms) - the CRG conjecture follows from the chaos bound.
- Recall that we found that in the $\sigma \to 0$ Regge limit at generic values of $e^{2\rho}$, the scaling of the correlator with σ is proportional to $\frac{1}{\sigma^{A'-1}}$.
- When we first took the $\rho \to 0$ bulk point limit, on the other hand, we found that the correlator scaled with σ like $\frac{1}{\sigma^{A-1}}$ where A is the CRG scaling coefficient of the flat space S matrix.
- We will now examine the relationship betwen the variables A and A'.

Relationship between A and A'

- Recall that our correlator takes the form

$$\frac{1}{\theta^{2A'-2}a^{\Delta+r-3}}\left(\sum_{n=0}^{\infty}\theta^n h_n(a)\right)$$

- When we take the $\theta \to 0$ limit first, we find the scaling $\frac{1}{\theta^{2A'-2}} \sim \frac{1}{\sigma^{A'-1}}$.
- On the other hand if we take the $a \to 0$ limit first and then take θ to zero we find the scaling $\frac{1}{\sigma^{A-1}} = \frac{1}{\sigma^{A'-1-\frac{n}{2}}}$ where n is the smallest value such that $h_n(a) \neq 0$. Generically we expect $n = 0$, in which case $A' = A$. However even if this generic expectation is not met, we find $A' = A + \frac{n}{2}$ and so it is always true that $A' \geq A$.

CRG conjecture from the chaos bound

- Our central result now follows immediately. It follows from the chaos bound that
$$A' \leq 2$$

- On the other hand we have demonstrated that
$$A \leq A'$$

- It follows that
$$A \leq 2$$
In other words the CRG conjecture follows from the chaos bound.

Lecture 2

Contents

- The road forward
- Kinematics of 4 graviton scattering. Momenta and Polarizations.
- Classification of Pole contributions. Classification of ggP couplings
- Relation to partial waves for gravitons
- Mathematical statement of the problem of classification of polynomial 4 graviton S matrices.
- Module Structure of Polynomial S matrices.
- The mathematical structure of modules
- $Z_2 \times Z_2$, S_3 and quasi invariant modules.
- S_3 representation theory, and Verma Modules of primaries in different S_3 representations.

The road forward

- At the end of the previous lecture we decided what our criterion we would impose to rule out classical S matrices as unphysical. Any classical S matrix that scales faster than s^2 at fixed t is unphysical.
- We will now proceed as follows. In imitation of the strategy adopted by CEMZ, we will first proceed to kinematically ennumerate all possible structures in the four graviton scattering S matrix. Unlike for 3 graviton scattering, four graviton scattering is parameterized by infinite data. However we will find a way to usefully organize this data
- Once we have the most general gravitational S matrix parameterized, we proceed to constrain the data in this S matrix by imposing the CRG cut.
- Remarkably enough we will find that this criterion is hugely constraining, leaving us with only a finite number of possibilities. Indeed, in $D \leq 6$, the only possibility that remains is the Einstein S matrix.

Scattering Momenta

- Consider scattering of 4 massless particles in D-dimensional Minkowski space. Let p_i^μ be momentum of the i^{th} particle. The masslessness of the scattering particles and momentum conservation means

$$p_i^2 = 0, \qquad \sum_{i=1}^{4} p_i^\mu = 0. \tag{24}$$

- Mandelstam variables:

$$\begin{aligned}
s &:= -(p_1+p_2)^2 = -(p_3+p_4)^2 = -2p_1.p_2 = -2p_3.p_4 \\
t &:= -(p_1+p_3)^2 = -(p_2+p_4)^2 = -2p_1.p_3 = -2p_2.p_4 \\
u &:= -(p_1+p_4)^2 = -(p_2+p_3)^2 = -2p_1.p_4 = -2p_2.p_3. \\
& s+t+u = 0
\end{aligned} \tag{25}$$

Polarizations: Photons

- $$A_\mu = \epsilon^i_\mu e^{-ik^i.x}, \qquad i = \text{particle index}$$

- Adopt Lorentz gauge (to maintain manifest Lorentz invariance)
$$k^i.\epsilon^i = 0$$

- Residual gauge invariance
$$\epsilon^\mu_i \to \epsilon^\mu_i + \zeta(p_i)p^\mu_i \tag{26}$$

- As p_i different, amplitude invariant under
$$\epsilon^\mu_i \to \epsilon^\mu_i + \zeta_i p^\mu_i \tag{27}$$
separately for each i

Polarizations: Gravitons

- $$h_{\mu\nu}(x) = h^i_{\mu\nu} e^{-ik^i.x}$$

- Lorentz gauge and Einstein equation

$$h_i^{\mu\nu} p_i^\nu = 0, \quad (h_i)^\mu_\mu = 0 \tag{28}$$

- Residual gauge invariance

$$h_i^{\mu\nu} \to h_i^{\mu\nu} + \zeta_i^\mu p_i^\nu + \zeta_i^\nu p_i^\mu, \qquad \text{where} \quad \zeta_i \cdot p_i = 0. \tag{29}$$

- Choice of polarization (no loss of generality):

$$h^i_{\mu\nu} = \epsilon^i_\mu \epsilon^i_\nu \qquad \text{where} \quad k_i \cdot \epsilon_i = 0, \;\; \epsilon_i \cdot \epsilon_i = 0 \tag{30}$$

Polarizations: Gravitons

- Polarization form preserved by gauge transformation of the form $\zeta_i^\mu = \zeta_i \epsilon^i_\mu$.
- Induced effective transformation

$$\epsilon_i^\mu \to \epsilon_i^\mu + \zeta_i p_i^\mu. \tag{31}$$

- Same as for gauge field.

Poles

- In these lectures we will focus on local classical theories. By definition these are theories with a finite number of fields, and where interactions have no more than finitely many derivatives. 4 graviton S matrices in such theories consist of a finite number of poles
- can probably be extended to S matrices with an infinite number of particles provided these are all bounded in spin plus a polynomial of finite degree.
- The pole contribution to the S matrix is given by sewing two copies of the *ggR* three particle S matrix through an *R* propagator (*R* is some other particle). If we can enumerate all kinematically allowed ggR couplings, we will have effectively classified all pole contributions to gggg scattering. This classification is easy to do.

Detail: Three particle scattering, Counting 1

- Consider the scattering of 2 gravitons and a massive particle in a representation R of the little group $SO(D-1)$. Spacetime can be divided up into the 'scattering 2 plane' - spanned by k_1, k_2 - together with its orthogonal compliment.
- The polarization ϵ_1 of the graviton with momentum k_1 obeys $k_1.\epsilon_1 = 0$. Implies $\epsilon_1 = \epsilon_1^\perp + a_1 k_1$ where $\epsilon_1^\perp$ lies in the orthogonal compliment. As far as gauge invariant amplitudes go, $\epsilon_1 = \epsilon_1^\perp$. Sim for ϵ_2. It follows that graviton polarization states are labelled by traceless symmetric tensors of $SO(D-2)$ that stabilizes the scattering two plane.
- On the other hand the representation R of $SO(D-1)$ (which stabilizes $k_1 + k_2$) descends, via $SO(D-1)$ branching rules, to a finite set of representations of the $SO(D-2)$ that also stabilizes $k_1 - k_2$, and so the full scattering two plane

Detail: Three particle scattering: Counting 2

In order to enumerate all possible ggP three point functions we

- Enumerate all $SO(D-2)$ singlets in the product of two $SO(D-2)$ tranceless symmetric tensors and one copy of any of the $SO(D-2)$ reps that descend from R.
- Retain only those singlets that respect the Bose symmetry of gravitons.

This exercise is not difficult to undertake. Once we have enumerated all structures it is also easy to explicitly construct them all, and also to list the Lagrangians from which they follow.

Detail: 3 particle scattering, Listing 1

- It is also not difficult to explicitly construct all relevant 3 point functions. For instance for $D \geq 8$. Yellow boxes denote indices effectively contracted with $k_1 - k_2$ in order to facilitate comparison with counting outlined above.

$\begin{ytableau} *(yellow) a & *(yellow) b \end{ytableau} : \nabla_a \nabla_b R_{cdef} R_{cdef} S_{ab}$ $\qquad$ $\begin{ytableau} a & b \end{ytableau} : R_{efga} R_{efgb} S_{ab}$

$\begin{ytableau} a & b & \alpha & \beta \end{ytableau} : R_{cadb} R_{c\alpha d\beta} S_{ab\alpha\beta}$

$\begin{ytableau} a & *(yellow) b \\ c \end{ytableau} : \nabla_d R_{acef} R_{bdef} S_{[ac]b}$ $\qquad$ $\begin{ytableau} a & *(yellow) b & *(yellow) c & *(yellow) d \\ e \end{ytableau} : \nabla_h R_{aedi} R_{hbci} S_{[ae]bcd}$

$\begin{ytableau} a & b & *(yellow) c \\ *(yellow) d \end{ytableau} : R_{efch} \nabla_b \nabla_h R_{efad} S_{[ad]bc}$ $\qquad$ $\begin{ytableau} a & b & c & d & *(yellow) e \\ *(yellow) f \end{ytableau} : \nabla_\beta R_{\alpha bch} \nabla_h \nabla_e \nabla_\alpha R_{\beta daf} S_{[af]bcde}$

$\begin{ytableau} a \\ b \\ *(yellow) c \end{ytableau} : \nabla_f R_{abde} R_{cfde} S_{[abc]}$ $\qquad$ $\begin{ytableau} a & d & e \\ b \\ *(yellow) c \end{ytableau} : \nabla_h R_{abdf} R_{chef} S_{[abc]de}$

Detail: 3 point structures, Listing 2

$\begin{array}{|c|c|}\hline r & t \\ \hline s & u \\ \hline\end{array}$ **and** $\begin{array}{|c|c|}\hline r & t \\ \hline s & u \\ \hline\end{array}$: $R_{pqrs}R_{pqtu}S_{[rs][tu]}$ **and** $R_{prqt}R_{psqu}S_{[rs][tu]}$

$\begin{array}{|c|c|c|c|}\hline a & c & e & f \\ \hline b & d \\ \cline{1-2}\end{array}$: $R_{abeh}R_{cdfh}S_{[ab][cd]ef}$

$\begin{array}{|c|c|c|}\hline c & i & d \\ \hline a & j \\ \cline{1-2}\end{array}$: $R_{abdk}\nabla_k R_{bcij}S_{[ca][ij]d}$

$\begin{array}{|c|c|c|c|}\hline a & d & f & i \\ \hline b & e \\ \cline{1-2} c \\ \cline{1-1}\end{array}$: $\nabla_i R_{abfj}R_{cjde}S_{[abc][de]fi}$

$\begin{array}{|c|c|c|}\hline a & d & f \\ \hline b & e \\ \cline{1-2} c \\ \cline{1-1}\end{array}$: $R_{abfi}R_{cide}S_{[abc][de]f}$

$\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & f \\ \hline\end{array}$: $R_{abdh}R_{chef}S_{[abc][def]}$

Detail: 3 point structures, Listing 3

$$\begin{array}{|c|c|c|c|}\hline c & i & k & e \\ \hline a & j & d \\ \cline{1-3}\end{array} : R_{abkd}\nabla_e R_{bcij} S_{[ca][ij][kd]e}$$

$$\begin{array}{|c|c|c|c|c|}\hline a & b & c & d & e \\ \hline f & i & j \\ \cline{1-3}\end{array} : \nabla_k R_{afcj}\nabla_e R_{bidk} S_{[af][bi][cj]de}$$

$$\begin{array}{|c|c|c|}\hline a & d & f \\ \hline b & e & i \\ \hline c \\ \cline{1-1}\end{array} : \nabla_j R_{abde} R_{cjfi} S_{[abc][de][fi]}$$

$$\begin{array}{|c|c|c|c|}\hline a & c & e & i \\ \hline b & d & f & j \\ \hline\end{array} : R_{abcd} R_{efij} S_{[ab][cd][ef][ij]}$$

Three Point Couplings: Important take away

- Observation: Every coupling on our list is of fourth or higher order in derivatives. To start with this result gives us a proof of a fact we had suspected above - namely that graviton scattering amplitudes in 2 derivative theories of gravity interacting with additional fields never have massive exchange pole contributions.
- The fact that all couplings are of 4th or higher order in derivatives has a simple physical interpretation. Every *genuine ggS* interaction term is constructed out of a product of (derivatives of) two Reimann tensors with the field S.
- Recall mechanism for removal of two derivative hhP interactions is field redefinitions that are forced on us by the need to diagonalize the quadratic part of the action. Consistent truncation to Einstein gravity at 2 derivative level.

Aside: Partial Waves I

- With all *ggP* S matrices in hand, all pole exchange *S* matrices (one might think of these as scattering blocks) generated by these three particle scattering amplitudes are, in principle, completely determined
- Unfortunately, no one has ever, as yet, explicitly listed all relevant structures (atleast in higher than 4 dimensions - there is a sense in which the spinor helicity formalism makes gravitons as easy as scalars in 4 dimensions). Luckily, for the purposes of my lectures, we will be able to get away without this explicit knowledge. For other applications (like those Simone is telling you about) we will not be so lucky.
- These structures are basic kinematical data for 4 graviton scattering. With two students (Shoaib and Suman) I am in the process of explicitly constructing them.

- As we do not really need the details these structures for these lectures (and as their construction is not yet complete) I describe them very briefly.
- Let us continue to work in the rest frame of the massive particle (as above). Then 'final state' of the ggP scattering process, above, has a definite angular momentum, namely the spin of the massive particle. As all interactions conserve angular momentum, it is clear that only that part of the two graviton state that has the same angular momentum as our massive particle can interact with it.
- The three point functions we have constructed may be thought of as the projection of our plane wave two graviton initial states onto states of definite centre of mass energy and angular momentum.

Aside: Partial Waves III

- We have studied the scattering of two particle states to a massive spin state with a definite polarization. For the purposes of the scattering block, however, we must sum over all intermediate polaraizations.
- Once we perform that summation, the structure we are left over with is the overlap between the part of the initial state that transforms in a particular angular momentum, and the part of the final state that transforms in the same angualar momentum
- These projected overlaps are not difficult to compute explicitly using group theory - for any given three point couplings - give rise to definite angular dependences in scattering. These angular dependences are captured by functions that appear in the study of (vector, tensor, ...) spherical harmonics in higher dimensions.

Aside: Partial Waves IV

- Once we view the ‘scattering blocks’ in these terms, we realize that these scattering blocks will be useful in contexts that have nothing to do with the exchange of massive particles.
- Indeed any two particle S matrix can be expanded in a basis of these blocks. These blocks have a name in scattering theory. They are called partial waves.
- The general untility of partial waves is the following: in the partial wave basis the $2 \to 2$ S matrix becomes diagonal (in the case of scalar scattering) or a finite S matrix (in the case of phton or graviton scattering). For this reason the constraints of unitarity are very easy to impose in this basis. I’m sure you will here much more about this in Simone’s lectures. I end my aside here. We now turn from the study of exchange S matrices to contact S matrices.

Polynomial 4 particle S matrices

The most general polynomial S matrix is simply the most general polynomial built out of polarizations and momenta that is

- 1: Lorentz Invariant
- 2: Separately quadratic in all polarzations
- 3: Gauge Invariant
- 4: Bose symmetric, i.e. invariant under S_4 permutations.

Moreover the polynomials above are evaluated only onshell. Two polynomials that agree onshell but differ otherwise are counted as the same S matrix. It is possible to obtain a completely explicit listing of all such S matrices. Next several slides, explain in detail..

Contact S matrices

The most general polynomial S matrix is simply the most general polynomial built out p^i_μ and ϵ^i_μ that is

- 1: Lorentz Invariant.
- 2: Separately quadratic in each ϵ_i^μ where $p_i.\epsilon^i = 0$ and $\epsilon_i.\epsilon_i = 0$.
- 3: Invariant under $\epsilon_i^\mu \to \epsilon_i^\mu + \zeta_i p_i^\mu$. separately for each ϵ_i.
- 4: Bose symmetric, i.e. invariant under S_4 permutations.

Moreover the polynomials above are evaluated only onshell, i.e. when all momenta obey

$$p_i^2 = 0, \quad p_i.\epsilon_i = 0, \quad \epsilon_i.\epsilon_i = 0.$$

Two polynomials that agree when these conditions are imposed but difer otherwise are counted as the same polynomial S matrix.

Module Structure

- To begin with let us put aside the requirement of S_4 invariance. Let us call the most general polynomial of momenta and polarizations that obeys conditions 1-3 (but not necessarily condition 4) of the previous page as the space of ‘unsymmetrized polynomial S matrices’.
- Let M be any unsymmetrized polynomial S matrix. It is then obvious that $P(s,t)M$ is also such an S matrix (here P is any polynomial of the Mandlestam variables).
- In mathematical language, the space of unsymmetrized polynomial S matrices is a ‘module over the the ring of polynomials of Mandlestam variables’.

Modules

- Recall that vector spaces are defined over a 'field' F. A field is an abstract space within which one can perform several operations. Two field elements can be added, subtracted, multiplied and divided to yield another field element. E.g. real or complex numbers
- Modules are defined over a 'Ring' R. A ring is an abstract space within which one can perform all the operations one can perform within a field except division. The space of polynomials of s, t, u is an example of a ring.
- Consider two module elements a and b and a nontrivial ring element r. If $b = ra$ we say b is a descendent of a. A module element that is not the descendent of any other module element is said to be a generator of the module (think Virasoro generator).

More about modules I

- The ‘generators’ (think Virasoro primaries) of a module play the same role that basis elements play for a vector space.
- A subset $G = \{g_i\}$ of the module M is said to generate M if the smallest submodule which contains G is M itself. In other words, the union of spans of all descendants of g_i is M itself.
- A module M is said to be finitely generated if it has a finite generator set (i.e. a generator set with a finite number of elements).

More about modules II

- A generator set G is said to generate M freely if the following condition holds,

$$\sum_i r_i \cdot g_i = 0 \qquad \text{iff} \qquad \text{all } \ r_i = 0. \tag{32}$$

  In other words if every element of M has a unique decomposition in terms of descendents of generators.

- A module M is a free module if there exists a G set that generates it freely. In this case the generator set G is called the basis of M.

- A free module is a very simple structure. Understanding its structure is equivalent to understanding its basis elements. In this case the full module is the (nonoverlapping) linear sum of the 'Verma Modules' of its generators.

More about modules III

- When the module is not free, by definition, it has a collection of nontrivial ‘relations’ (think null states).
- Let
$$\sum_i r_i \cdot g_i = 0$$
Clearly it is also true that
$$r\left(\sum_i r_i \cdot g_i\right) = 0.$$
Thus the set of all relations themselves form a module.
- A general module can be thought of as the linear sum of the ‘Verma Modules’ built on top of each of its generators, minus the relation module.
- The relation module in term is the sum of ‘Verma modules’ of *its* generators minus *its* relation module. The complete specification of the relations, and relations for relations ... is called the free resolution of a module.

'Quasi Invariant' S matrices

- Now it is not difficult to verify that the $Z_2 \times Z_2$ subgroup of S_4, consisting of I, $P_{12}P_{34}$, $P_{13}P_{24}$ and $P_{14}P_{23}$ leaves the Mandlestam variables s, t and u invariant.
- E.g. under $P_{12}P_{34}$

$$
\begin{aligned}
&s = -(p_1 + p_2)^2 \to s, \quad t = -(p_1 + p_3)^2 \to -(p_2 + p_4)^2 = t, \\
&\quad u = -(p_1 + p_4)^2 \to -(p_2 + p_3)^2 = u.
\end{aligned}
\tag{33}
$$

- Let us call the collections of polynomials of polarizations and momenta that obey conditions 1-3 above - but are also $Z_2 \times Z_2$ invariant - the space of 'Qasi Invariant' polynomial S matrices.
- The $Z_2 \times Z_2$ invariance of Mandlestam variables immeditely tells us that the space of Quasi Invariant Polynomial S matrices is also a module over the ring of polynomials of s, t and u.

Action of S_3

- The space of Quasi Invariant S matrices can be decomposed into irreps of $S_4/(Z_2 \times Z_2) = S_3$.
- Can check that this S_3 coset group acts on s, t and u by permutations. S_3 has 3 irreps; the one dimensional completely symmetric irrep **S**, the one dimensional completely antisymmetric irrep **A**. And the two dimensional mixed representation **M**. The **3** dimensional defining representation of S_3 is not irreducible. It decomposes into on **S** and one **M**
- Our strategy to enumerate all S matrices is the following. First we construct module of quasi invariant S matrices. The space of polynomial S matrices is simply the projection of the quasi invairant module onto the space of S_3 singlets. Over the next several slides we will try to implement this strategy. First some technical details.

Counting polynomials of s, t and u

- Like module elements, generators of a module also appear in representations of S_3.
- The 'Verma Module' built over a singlet generator is simply the collection of all polynomials built out of s, t and u.
- We will now count all such polynomials graded by scaling dimension ($\Delta[s] = \Delta[t] = \Delta[u] = 2$) as well as S_3 reps.
- Let the number of S_3 *representations* of type α at degree n (so dimension $2n$) be denoted as $n_\alpha(n)$. Can show

$$\text{Let} \quad Z_\alpha(x) = \sum_n n_\alpha(n) x^{2n},$$

$$Z_{\mathbf{S}}(x) = \mathcal{D}, \qquad Z_{\mathbf{A}}(x) = x^6 \mathcal{D}, \qquad Z_{\mathbf{M}}(x) = (x^2 + x^4)\mathcal{D},$$

$$\mathcal{D} = \frac{1}{(1-x^4)(1-x^6)}$$

Interpretation: $s^2 + t^2 + u^2 \to x^4$, $stu \to x^6$.

Verma Module of a **A** or **M** generator

- The Verma Module of a generator in the **A** or **M** representations is given by the tensor product of the generator states and the set of polynomials made out of s, t and u.
- This tensor product is easily decomposed into S^3 representations using the fusion rules

$$\mathbf{S}\times\mathbf{R}=\mathbf{R},\quad \mathbf{A}\times\mathbf{A}=S,\quad \mathbf{A}\times\mathbf{S}=\mathbf{A},\quad \mathbf{A}\times\mathbf{M}=\mathbf{M},$$
$$\mathbf{M}\times\mathbf{M}=\mathbf{S}+\mathbf{A}+\mathbf{M} \tag{34}$$

- (34) together with the equations on the previous slide allow us to evaluate the partition function of the Verma modules of a generator in either the **A** or **M**.
- In particular the partition function over singlets in a Verma Module whose representation transforms in the representation **R** is simply given by the functions $Z_{\mathbf{R}}(x)$ evaluated above. Because S appears on the RHS of the fusion rules only of identical reprs.

Unconstrained Polarizations

- With Verma Modules under complete control, all that remains is to characterize the generators of the Local S matrix module, and also the generators of its relations, relations for relations, etc. To do this it is useful to view the Local module as a submodule of a simpler 'Bare Module'.
- The main difficulty in classifying and ennumerating polynomial S matrices is in implementing the gauge invariance condition, which, in turn, reflects the fact that the D component vector ϵ_μ is a redundant specification of the $D-2$ components of polarization data.
- For this reason it is useful, for some purposes, to use 'independent data' to characterize polarizations. We find it useful to do this as follows.
- Consider the 3 dimensional timelike subspace of Minkoski space that is spanned by the vectors p_i^μ. We refer to this subspace as the 'scattering 3 plane'.

Unconstrained Polarizations

- Every polarization vector can be decomposed as

$$\epsilon_i = \epsilon_i^{\perp} + \epsilon_i^{||} \tag{35}$$

where $\epsilon_i^{\perp}$ is orthogonal to the scattering 3 plane, and $\epsilon_i^{||}$ lies within this 3 plane.

- $\epsilon_i^{\perp}$ is completely free data. The same is not true of $\epsilon_i^{||}$. $\epsilon_i.p_i = \epsilon_i^{||}.p_i = 0$ forces $\epsilon_i^{||}$ to lie in a two dimensional subspace of the scattering plane. Moreover the constraint that S-matrices are invariant under the shifts $\epsilon_i^{||} \to \epsilon_i^{||} + p_i$

Contents

- Unconstrained Polarizations. Bare Modules. Physical module as a submodule of the bare module.
- Counting and constructing the generators of bare modules.
- Condition for the submodule of physical S matrices to be freely generated.
- Two way map between Lagrangins and S matrices. Two way map between module generators and Lagrangians.
- Generators and complete characterizatin of 4 photon scattering modules

Unconstrained Polarizations

- $$\begin{aligned}
\epsilon_1^{\parallel} &= \alpha_1 \sqrt{\frac{st}{u}} \left(\frac{p_2}{s} - \frac{p_3}{t} \right) + a_1 p_1 \\
\epsilon_2^{\parallel} &= \alpha_2 \sqrt{\frac{st}{u}} \left(\frac{p_1}{s} - \frac{p_4}{t} \right) + a_2 p_2 \\
\epsilon_3^{\parallel} &= \alpha_3 \sqrt{\frac{st}{u}} \left(\frac{p_4}{s} - \frac{p_1}{t} \right) + a_3 p_3 \\
\epsilon_4^{\parallel} &= \alpha_4 \sqrt{\frac{st}{u}} \left(\frac{p_3}{s} - \frac{p_2}{t} \right) + a_4 p_4 .
\end{aligned} \tag{36}$$

- a_i pure gauge. α_i physical.

$$\epsilon_i^{\parallel} . (\epsilon_i^{\parallel})^* = |\alpha_i|^2 \tag{37}$$

Unconstrained Polarizations

- For photons, pair $(\epsilon_i^{\perp}, \alpha_i)$ are unconstrained data. For gravitons single constraint

$$\epsilon_i^{\perp}.\epsilon_i^{\perp} + \alpha_i^2 = 0 \tag{38}$$

- In enumerating contraction structures we simply omit all terms containing factors of $\epsilon_i^{\perp}.\epsilon_i^{\perp}$. For counting purposes, therefore, $\epsilon_i^{\perp}$ can effectively be treated as null.
- S matrix function of $(\epsilon_i^{\perp}, \alpha_i)$ and (s, t). Separately quadratic in each $(\epsilon_i^{\perp}, \alpha_i)$. Note explicit momentum dependence only through (s, t) (follows from $\epsilon_i^{\perp}.p_j = 0$). Also any such expression is a good S matrix using

Unconstrained to $\rightarrow$ Lorentz Invariant

$$
\alpha_1 = 2\frac{p_2.F^1.p_3}{\sqrt{stu}}, \qquad \alpha_2 = 2\frac{p_1.F^2.p_4}{\sqrt{stu}},
$$
$$
\alpha_3 = 2\frac{p_4.F^3.p_1}{\sqrt{stu}}, \qquad \alpha_4 = 2\frac{p_3.F^4.p_2}{\sqrt{stu}}. \tag{39}
$$

$$
\begin{aligned}
\epsilon_1^{\perp} &= -2\frac{p_2.F^1}{s} + 2\frac{p_2.F^1.p_3}{tu}p_3 - 2\frac{p_2.F^1.p_3}{ts}p_1 - 2\frac{p_2.F^1.p_3}{su}p_2 \\
\epsilon_2^{\perp} &= -2\frac{p_1.F^2}{s} + 2\frac{p_1.F^2.p_3}{tu}p_3 - 2\frac{p_1 F^2.p_3}{us}p_2 - 2\frac{p_1.F^2.p_3}{ts}p_1 \\
\epsilon_3^{\perp} &= -2\frac{p_2.F^3}{u} + 2\frac{p_2.F^3.p_1}{ts}p_1 - 2\frac{p_2.F^3.p_1}{tu}p_3 - 2\frac{p_2.F^3.p_1}{su}p_2 \\
\epsilon_4^{\perp} &= -2\frac{p_2.F^4}{t} + 2\frac{p_2.F^4.p_3}{su}p_3 - 2\frac{p_2.F^4.p_3}{ts}p_4 - 2\frac{p_2.F^4.p_3}{tu}p_2. \\
F^i_{\mu\nu} &= p^i_\mu \epsilon^i_\nu - p^i_\nu \epsilon^i_\mu
\end{aligned} \tag{40}
$$

Action of permutations on α_i

- How do $(p_i, \epsilon_i^{\perp}, \alpha_i)$ transform under S_4? Let $B = (p_1, p_2, p_3, p_4)$, $C = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $D = (\epsilon_1^{\perp}, \epsilon_2^{\perp}, \epsilon_3^{\perp}, \epsilon_4^{\perp})$. Let P be any permutation in S_4. Let $M(P)$ denote the representation of S_4 in its 'defining' (reducible) 4 dimensional representation.
- Clearly

$$B \to M(P)B, \qquad D \to M(P)D. \tag{41}$$

- α_i, however, characterizes not just the vector ϵ_i but its projection onto the scattering 3 plane. Explicit computation using (39) gives

$$C \to (-1)^{\mathrm{sgn(P)}} M(P)C. \tag{42}$$

Parity even S matrices are even functions of α_i. 'Anomaly' above important only for parity odd.

Bare Module

- We now come to a crucial point. By performing a long set of comptuations we have shown that any polynomial parity and gauge invariant S matrix is a polynomial in momenta, α_i and $\epsilon_i^{\perp}$. There is probably an elegant proof of this fact but we have not found it.
- Define the 'Bare' module as the collection of all $Z_2 \times Z_2$ invariant polynomials of $(\epsilon_i^{\perp}, \alpha_i)$ and s, t that are separately quadratic in each $(\epsilon_i^{\perp}, \alpha_i)$.
- This Bare Module over the ring of polynomials of s and t is very simple. Its generators are all polynomials of $(\epsilon_i^{\perp}, \alpha_i)$ - but not of s, t - of appropriate homogeniety. The module is freely generated.
- We have just argued that the physical module of Local S matrices is a submodule of this Bare Module.

Embedding

- Our aim is to characterize the physical 'Local' module. We can now break this up into two steps. First completely understand the bare module (i.e. enumerate its generators)
- Second, completely characterize the embedding of the physical Local module into the Bare module. That is find the expressions

$$E_J(p_i, \epsilon_i) = \sum_{e_I \in B} p_{IJ}(s,t) e_I(\alpha_i, \epsilon_i^{\perp}). \tag{43}$$

for the generators $E_J(p_i, \epsilon_i)$ of the Local module in terms of the generators $e_I(\alpha_i, \epsilon_i^{\perp})$ of the Bare Module. We turn to these tasks one by one.

Counting Generators of the Bare Module I

- The first task- characterizing the Bare module - is relatively simple. The generators of the bare module are simply $Z_2 \times Z_2$ and $SO(D-3)$ invariant polynomials in $(\epsilon_i^\perp, \alpha_i)$ that are separately linear in each group (for photons) or separately quadratic in each group and also obey (38) (for gravitons)
- Working with $SO(D-3)$ representations, the number of generators of the Bare module equals

$$(\mathtt{s}\oplus\mathtt{v})^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} \text{ for photons,} \qquad (\mathtt{s}\oplus\mathtt{v}\oplus\mathtt{t})^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} \text{ for gravitons} \tag{44}$$

  where the notation $|_G$ stands for projection onto G invariants (we also only count $SO(D-3)$ invariants).
- Ennumerating $SO(D-3)$ invariants is a straightforward exercise in Clebsh Gordans. The only tricky thing is that we wish to keep only $Z_2 \times Z_2$ invariants. Not too tough.

Counting Generators of the Bare Module II

- Consider any ‘single particle Hilbert space’ with a basis single particle eigenstates $|i\rangle$ with definite values of the commuting charges J_m and a single particle partition function

$$\mathrm{Tr}_\rho \left(\prod_m y_m^{J_m} \right) = \sum_i \langle i| \prod_m y_m^{J_m} |i\rangle = z(y_m) \tag{45}$$

- Next consider the Hilbert space of two identical bosons/fermions, each of whose single particle Hilbert space is ρ. Let the corresponding Hilbert spaces be denoted by $S^2\rho$ and $\wedge^2\rho$ respectively.

$$\mathrm{Tr}_{S^2\rho}\left(\prod_m y_m^{J_m}\right) = \sum_{i_1,i_2,}\langle i_i i_2| \left(\prod_m y_m^{J_m}\right)\left(\frac{1+P_{(12)}}{2}\right)|i_1 i_2\rangle = \frac{z^2(y_m)+z(y_m^2)}{2}$$

$$\mathrm{Tr}_{\Lambda^2\rho}\left(\prod_m y_m^{J_m}\right) = \sum_{i_1,i_2,}\langle i_i i_2| \left(\prod_m y_m^{J_m}\right)\left(\frac{1-P_{(12)}}{2}\right)|i_1 i_2\rangle = \frac{z^2(y_m)-z(y_m^2)}{2} \tag{46}$$

We have used

$$\langle i_1 i_2|\left(\prod_m y_m^{J_m}\right) P_{(12)}|i_1 i_2\rangle = \langle i_1 i_2|\left(\prod_m y_m^{J_m}\right)|i_2 i_1\rangle = \delta_{i_1,i_2}\langle i_1|\left(\prod_m (y_m^2)^{J_m}\right)|i_1\rangle \tag{47}$$

Counting Generators of the Bare Module IV

- Next consider the Hilbert space ρ of four distinguishable particles, each of whose single particle state space is spanned by $|i\rangle$. The partition function over this Hilbert space is, of course, given by
- $$\mathrm{Tr}_{\rho^{\otimes 4}}\left(\prod_m y_m^{J_m}\right) = \sum_{i_1,i_2,i_3,i_4} \langle i_i i_2 i_3 i_4| \prod_m y_m^{J_m} |i_1 i_2 i_3 i_4\rangle = z^4(y_m) \tag{48}$$

Counting Generators of the Bare Module V

- $$\begin{aligned}
&\mathrm{Tr}_{\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}}\left(\prod_m y_m^{J_m}\right)\\
&= \sum_{i_1,i_2,i_3,i_4} \langle i_i i_2 i_3 i_4| \left(\prod_m y_m^{J_m}\right)\left(\frac{1+P_{(2143)}+P_{(3412)}+P_{(4321)}}{4}\right)|i_1 i_2 i_3 i_4\rangle\\
&=\frac{1}{4}\sum_{i_1,i_2,i_3,i_4}\Big(\langle i_1 i_2 i_3 i_4|\prod_m y_m^{J_m}|i_1 i_2 i_3 i_4\rangle + \langle i_1 i_2 i_3 i_4|\prod_m y_m^{J_m}|i_2 i_1 i_4 i_3\rangle\\
&\qquad + \langle i_1 i_2 i_3 i_4|\prod_m y_m^{J_m}|i_3 i_4 i_1 i_2\rangle + \langle i_1 i_2 i_3 i_4|\prod_m y_m^{J_m}|i_4 i_3 i_2 i_1\rangle\Big)\\
&=\frac{z^4(y_m)+3z^2(y_m^2)}{4}\\
&=z^4(y_m)-3\left(\frac{z^2(y_m)+z(y_m^2)}{2}\right)\times\left(\frac{z^2(y_m)-z(y_m^2)}{2}\right)
\end{aligned} \tag{49}$$

Counting Generators of the Bare Module VI

- $$\mathrm{Tr}_{\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2}}\left(\prod_m y_m^{J_m}\right) = \mathrm{Tr}_{\rho^{\otimes 4}}\left(\prod_m y_m^{J_m}\right) - 3\mathrm{Tr}_{S^2\rho}\left(\prod_m y_m^{J_m}\right)\mathrm{Tr}_{\Lambda^2\rho}\left(\prod_m y_m^{J_m}\right) \quad (50)$$

- Schematically

$$\rho^{\otimes 4}|_{\mathbb{Z}_2\times\mathbb{Z}_2} = \rho^{\otimes 4} - 3S^2\rho\otimes\Lambda^2\rho \quad (51)$$

- It is now a simple matter to project onto $SO(D-3)$ singlets.

Counting Generators of the Bare Module: Results

Results:

photons	even	odd
$D \geq 8$	7	0
$D = 7$	7	1
$D = 6$	7	1
$D = 5$	7	0
$D = 4$	5	2
$D = 3$	1	1

gravitons	even	odd
$D \geq 8$	29	0
$D = 7$	29	7
$D = 6$	28	9
$D = 5$	22	3
$D = 4$	5	2
$D = 3$	-	-

Table: Number of parity even and parity odd index structures for 4-photon and 4-graviton S-matrix as various dimensions.

We have also explicitly constructed all these basis elements and have thereby grouped them into S_3 multiplets.

Counting Generators of the Bare Module: Results 2

photons	even			odd		
	n_{1_S}	n_{2_M}	n_{1_A}	n_{1_S}	n_{2_M}	n_{1_A}
$D \geq 8$	3	2	0	0	0	0
$D = 7$	3	2	0	1	0	0
$D = 6$	3	2	0	0	0	1
$D = 5$	3	2	0	0	0	0
$D = 4$	3	1	0	2	0	0
$D = 3$	1	0	0	0	0	1

gravitons	even			odd		
	n_{1_S}	n_{2_M}	n_{1_A}	n_{1_S}	n_{2_M}	n_{1_A}
$D \geq 8$	10	9	1	0	0	0
$D = 7$	10	9	1	3	2	0
$D = 6$	9	9	1	0	3	3
$D = 5$	7	7	1	0	1	1
$D = 4$	3	1	0	2	0	0
$D = 3$	-	-	-	-	-	-

When is the embedding Freely Generated? - I

- In $D \geq 5$ it turns out that there are exactly as many generators of in the Local module as in the Bare module.
- In such a situation the Local module has relations if and only if the equation

$$\sum_{E_J \in L} r^J(s,t) E_J(p_i, \epsilon_i) = 0, \tag{52}$$

has non-trivial solutions for polynomials $r^J(s,t)$.

When is the embedding is Freely Generated? -II

- Plugging (43) into (52) and equating coefficients of e_I, it follows there exist module relations if and only if

$$\sum_J p_{IJ}(s,t) r^J(s,t) = 0. \tag{53}$$

has nontrivial solutions, i.e. iff

$$\mathrm{Det}\left[p_{IJ}(s,t)\right] = 0. \tag{54}$$

- This is an extremely stringent condition, and we find it is never met. Consequently the Local modules in $D \geq 5$ are all freely generated.
- On the other hand when the number of E_I exceeds the number of e_I (this turns out to be the case in $D \leq 4$), the Local module always has relations. We will completely uncover and understand these. For this we now discuss the relationship between Local Modules and Lagrangians.

- Local S matrices are in one to one correspondence with Local Lagrangians modulo total derivatives and field redefinition. I will explain this relationship in the simplest case of a scalar theory.
- Consider a Z_2 invariant scalar field theory. If this theory describes a single massless at mass m, as we require, then its Lagrangian at quadratic order takes the form

$$S_2 = -\frac{1}{2}\int d^D x\, \partial_\mu \phi \partial^\mu \phi. \tag{55}$$

- At quartic order

$$S_4 = \int d^D x L_4, \quad L_4 = \sum a_{m_1,m_2,m_3,m_4} \partial^{m_1}\phi\, \partial^{m_2}\phi\, \partial^{m_3}\phi\, \partial^{m_4}\phi \tag{56}$$

schematic summation runs over number of derivatives as well as ways of contracting the derivative indices.

Local S Matrices and Local Lagrangians II

- Consider an field redefinition of the schematic form

$$\phi \to \phi + \delta\phi$$
$$\delta\phi = \left(\sum b_{m_1,m_2,m_3} \partial^{m_1}\phi\, \partial^{m_2}\phi\, \partial^{m_3}\phi\right) \tag{57}$$

- Up to terms of sextic and higher terms, (57) shifts L_4 by

$$\delta L_4 = \partial^2\phi \left(\sum b_{m_1,m_2,m_3} \partial^{m_1}\phi\, \partial^{m_2}\phi\, \partial^{m_3}\phi\right) \tag{58}$$

- Also the addition of total derivatives to the Lagrangian shift L_4 schematically by

$$\delta L_4 = \partial \left(\sum a_{m_1,m_2,m_3,m_4} \partial^{m_1}\phi\, \partial^{m_2}\phi\, \partial^{m_3}\phi\, \partial^{m_4}\phi\right)$$

Local S Matrices and Local Lagrangians III

- In momentum space

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik.x}\ \tilde{\phi}(k)$$

$$L_4 = \int \prod_i \frac{d^d k}{(2\pi)^d} e^{i(\sum_j k_j x_j)}\ \tilde{L}_4(k_1, k_2, k_3, k_4)\ \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \quad (59)$$

- Clearly the equivalence classes of Lagrangians modulo field redefinitions and total derivatives are labelled by $\tilde{L}_4(k_1, k_2, k_3, k_4)$ at values s.t. $k_i^2 = 0$ and $\sum_i k_i = 0$, i.e. the S matrix.
- Moreover this map is invertible (given an S matrix, an arbitrary offshell extension of the same gives a local Lagrangian). This establishes the one to one map.

- Recall that any set of not necessarily S_3 symmetric generators M_a of the local module is naturally associated with an infinite class of genuine (S_3 invariant) S-matrices $S(M_a)$as follows.
- $S(M_a)$ is defined as the restriction of the span of M_a to S_3 singlets. In other words $S(M_a)$ are all the S_3 invariant descendants of the generators.
- Similarly any Lagrangian L can be associated with an infinite class of Lagrangians $C(L)$ defined as follows. $C(L)$ is defined as the set of Lagrangians obtained by taking derivatives the fields that appear in the Lagrangian and contracting the indices of these derivatives in pairs.

- We say that a Lagrangian L is associated with the generators M_a if the set of S-matrices obtained from the Lagrangians $C(L)$ coincide with $S(M_a)$.
- This association allows us to use Lagrangians to label generators (and more generally elements) of the local module. As an example consider the photon Lagrangian $\mathrm{Tr}(F^2)\mathrm{Tr}(F^2)$. The corresponding generators of the local Module are $\mathrm{Tr}(F_1F_2)\mathrm{Tr}(F_3F_4)$, $\mathrm{Tr}(F_1F_3)\mathrm{Tr}(F_2F_4)$ and $\mathrm{Tr}(F_1F_4)\mathrm{Tr}(F_3F_2)$; this set of generators transforms in the **3** of S_3.
- Note that as a Lagrangian $\mathrm{Tr}(F^2)\mathrm{Tr}(F^2)$ transforms in the **S** because the S matrix corresponding to this Lagrangian - like every other S matrix- is S_3 invariant. But the generators it labels transforms in the **3**.

The embedding of Local in Bare I

- In determining the embedding of the Local module in the Bare module we made crucial use of the Lagrangian picture.
- Let us first start with the example of parity invariant electromagnetism. From the Lagrangian viewpoint it is obvious that the dimension 8 Lagrangians are $\mathrm{Tr}(F^2)\mathrm{Tr}(F^2)$ and $\mathrm{Tr}(F^4)$ are generators of the Local module (they both transform in the **3**).
- It is also easy to show that all terms of dimension 14 or higher (i.e. with 6 derivatives on 4 Fs) are necessarily descendents.

The embedding of Local in Bare II

- For example consider

$$\partial_a F_{\mu\nu} \partial_\mu F_{ab} \partial_b \partial_\nu \partial^p F_{mn} \partial^m F_{pn} \qquad (60)$$

- Using the Bianchi identity

$$\partial_a F_{\mu\nu} = -\partial_\mu F_{\nu a} - \partial_\nu F_{a\mu}$$

we re express the first field field strength in (60) as a sum of two other terms. Both of these new terms have a pair of derivatives with contracted indices, and so are descendents.

The embedding of Local in Bare III

- Working a bit harder in the same way one can also show that all dimension 12 terms (4 derivatives on 4 field strengths) are descendents, but that there is exactly one Lagrangian at dimension 10 (2 derivatives on 4 field strengths) that is not a descendent.
- In summary the generators of the Local module are dual to the Lagrangians

$$\mathrm{Tr}(F^2)\mathrm{Tr}(F^2), \quad \mathrm{Tr}(F^4), \quad -F^{ab}\partial_a F^{\mu\nu}\partial_b F^{\nu\rho}F^{\rho\mu} \tag{61}$$

The embedding of Local in Bare III

Explicitly the generators are

$$\begin{aligned}
E_{\mathbf{3},1}^{(1)} &= 8\mathrm{Tr}(F_1F_2)\mathrm{Tr}(F_3F_4), \quad E_{\mathbf{3},1}^{(2)} = 8\mathrm{Tr}(F_1F_3)\mathrm{Tr}(F_2F_4), \\
E_{\mathbf{3},1}^{(3)} &= 8\mathrm{Tr}(F_1F_4)\mathrm{Tr}(F_3F_2), \\
E_{\mathbf{3},2}^{(1)} &= 8\mathrm{Tr}(F_1F_3F_2F_4), \qquad E_{\mathbf{3},2}^{(2)} = 8\mathrm{Tr}(F_1F_2F_3F_4), \\
E_{\mathbf{3},2}^{(3)} &= 8\mathrm{Tr}(F_1F_3F_4F_2), \\
E_{\mathbf{S}} &\simeq -6F_1^{ab}\partial_a F_2^{\mu\nu}\partial_b F_3^{\nu\rho}F_4^{\rho\mu}|_{\mathbb{Z}_2\times\mathbb{Z}_2}
\end{aligned} \tag{62}$$

The embedding of Local in Bare IV

For $D \geq 5$ the generators of the Bare module are

$$
\begin{aligned}
\mathbf{e}^{(1)}_{\mathbf{3},1}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp} \cdot \epsilon_2^{\perp})(\epsilon_3^{\perp} \cdot \epsilon_4^{\perp}), \\
\mathbf{e}^{(2)}_{\mathbf{3},1}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp} \cdot \epsilon_3^{\perp})(\epsilon_2^{\perp} \cdot \epsilon_4^{\perp}), \\
\mathbf{e}^{(3)}_{\mathbf{3},1}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp} \cdot \epsilon_4^{\perp})(\epsilon_3^{\perp} \cdot \epsilon_2^{\perp}) \\
\\
\mathbf{e}^{(1)}_{\mathbf{3},2}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp}.\epsilon_2^{\perp}\, \alpha_3\alpha_4 + \epsilon_3^{\perp}.\epsilon_4^{\perp}\, \alpha_1\alpha_2), \\
\mathbf{e}^{(2)}_{\mathbf{3},2}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp}.\epsilon_3^{\perp}\, \alpha_2\alpha_4 + \epsilon_2^{\perp}.\epsilon_4^{\perp}\, \alpha_1\alpha_3), \\
\mathbf{e}^{(3)}_{\mathbf{3},2}(\alpha_i, \epsilon_i^{\perp}) &= (\epsilon_1^{\perp}.\epsilon_4^{\perp}\, \alpha_3\alpha_2 + \epsilon_3^{\perp}.\epsilon_2^{\perp}\, \alpha_1\alpha_4) \\
\\
\mathbf{e}_{\mathbf{S}}(\alpha_i, \epsilon_i^{\perp}) &= \alpha_1\alpha_2\alpha_3\alpha_4.
\end{aligned} \tag{63}
$$

The embedding of Local in Bare V

We can explicitly express the 7 generators of Local as descendents of the 7 generators of Bare.

$$
\begin{aligned}
&E^{(1)}_{\mathbf{3},1} - 8s^2 e^{(1)}_{\mathbf{3},1} + 8s^2 e^{(1)}_{\mathbf{3},2} - 8s^2 e_{\mathbf{S}}, \\
&E^{(2)}_{\mathbf{3},1} = -8t^2 e^{(2)}_{\mathbf{3},1} + 8t^2 e^{(2)}_{\mathbf{3},2} - 8t^2 e_{\mathbf{S}}, \\
&E^{(3)}_{\mathbf{3},1} = -8u^2 e^{(3)}_{\mathbf{3},1} + 8u^2 e^{(3)}_{\mathbf{3},2} - 8u^2 e_{\mathbf{S}},
\end{aligned}
$$

$$
\begin{aligned}
&E^{(1)}_{\mathbf{3},2} = -2(u^2 e^{(2)}_{\mathbf{3},1} + t^2 e^{(3)}_{\mathbf{3},1}) + 2(u(s-t) e^{(2)}_{\mathbf{3},2} + t(s-u) e^{(3)}_{\mathbf{3},2}) - 2(t^2+u^2) e_{\mathbf{S}}, \\
&E^{(2)}_{\mathbf{3},2} = -2(s^2 e^{(3)}_{\mathbf{3},1} + u^2 e^{(1)}_{\mathbf{3},1}) + 2(s(t-u) e^{(3)}_{\mathbf{3},2} + u(t-s) e^{(1)}_{\mathbf{3},2}) - 2(u^2+s^2) e_{\mathbf{S}}, \\
&E^{(3)}_{\mathbf{3},2} = -2(t^2 e^{(1)}_{\mathbf{3},1} + s^2 e^{(2)}_{\mathbf{3},1}) + 2(t(u-s) e^{(1)}_{\mathbf{3},2} + s(u-t) e^{(2)}_{\mathbf{3},2}) - 2(s^2+t^2) e_{\mathbf{S}},
\end{aligned}
$$

$$
E_{\mathbf{S}} = 3\,stu\,(e^{(1)}_{\mathbf{3},2} + e^{(2)}_{\mathbf{3},2} + e^{(3)}_{\mathbf{3},2} - 2e_{\mathbf{S}}).
$$

Det $[p_{IJ}(s,t)] = 393216 s^5 t^5 u^5$. Nonzero so Freely Generated!

The embedding of Local in Bare VI

- The generators of the Local module are the same in every dimension. Howevever the explicit counting of the Bare module presented earlier shows that it has only 5 generators in $D = 4$ (similar story in $D = 3$).
- We can also see this explicitly. The three generators $e^{(i)}_{\mathbf{3},1}$, which are all distinct in $D \geq 5$, are all the same in $D = 4$ (in this dimension $\epsilon_i^{\perp}$ are numbers rather than vectors)
- It follows that the Bare module is not freely generated. The generators of the relation module can be found. In $D = 4$, for instance, they turn out to be

$$\begin{aligned} &s\tilde{E}^{(1)} + t\tilde{E}^{(2)} + u\tilde{E}^{(3)} = 0 \\ &(s^2 + 2ut)\tilde{E}^{(1)} + (t^2 + 2us)\tilde{E}^{(2)} + (u^2 + 2st)\tilde{E}^{(3)} = 0 \end{aligned} \tag{64}$$

The relation module turns out to be freely generated. This completes our characterization of the Local module

Contents

- Explicit listing of all parity invariant 4 photon polynomial S matrice in $D \geq 5$.
- Gravity Lagrangians with 1, 2, 3 and 4 factors of Reimann.
- Generators of the parity even gravitational scattering module in $D \geq 7$. Explicit listing of the corresponding S matrices.
- Tabulation of S matrix partition functions in various dimensions.
- Counting from Plethystics.
- CRG allowed contact S matrices.
- CRG constraints on exchange S matrices.
- Conjecture II for 4 graviton scattering.
- Discussion and Conclusions.

Review: Parity even photon S matrices in $D \geq 5$.

- For $D \geq 5$ the most general local parity invariant S matrix for 4 photons is freely generated.
- The most general S matrix parameterized by by three polynomials in the **S** representation and two in the **M** representations. Equivalently - and more conveniently for some purposes, these set of polynomials may be shown to be characterized by 2 Z_2 invariant functions (i.e. functions that are symmetric under u goes to t interchange) $A^{0,1}(t,u)$ and a single S_3 invariant function $A^{2,1}(s,t,u)$.
- $A^{0,1}$ and $A^{0,2}$ parameterize descendents of the four derivative structures $(TrF^2)^2$ and $Tr(F^4)$ respectively while $A^{1,2}$ parameterizes descendents of the six derivative term

$$F_{ab}\mathrm{Tr}(\partial_a F \partial_b F F)$$

Explicit parameterization of 4 photon S matrices 1

- Explicitly the most general parity even 4 photon S matrix in $D \geq 5$ is given by the sum of
-
$$
\begin{aligned}
&A^{0,1}(t,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\\
+&A^{0,1}(s,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\\
+&A^{0,1}(t,s)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)
\end{aligned}
\tag{65}
$$

- and

$$
\begin{aligned}
&A^{0,2}(t,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)\,(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta)\\
+&A^{0,2}(s,u)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha)\,(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta)\\
+&A^{0,2}(t,s)\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha)\,(p^2_\beta\epsilon^2_\mu - p^2_\mu\epsilon^2_\beta)
\end{aligned}
\tag{66}
$$

Explicit parameterization of 4 photon S matrices 2

- $$\begin{aligned}
&\left(A^{2,1}(s,t)+A^{2,1}(t,u)+A^{2,1}(u,s)\right)\times\\
&\Big[\left(p^1_a\epsilon^1_b-p^1_b\epsilon^1_a\right)p^2_a\left(p^2_\mu\epsilon^2_\nu-p^2_\nu\epsilon^2_\mu\right)p^3_b\left(p^3_\nu\epsilon^3_\alpha-p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\mu-p^4_\mu\epsilon^4_\alpha\right)\\
&+\left(p^2_a\epsilon^2_b-p^2_b\epsilon^2_a\right)p^1_a\left(p^1_\mu\epsilon^1_\nu-p^1_\nu\epsilon^1_\mu\right)p^4_b\left(p^4_\nu\epsilon^4_\alpha-p^4_\alpha\epsilon^4_\nu\right)\left(p^3_\alpha\epsilon^3_\mu-p^3_\mu\epsilon^3_\alpha\right)\\
&+\left(p^3_a\epsilon^3_b-p^3_b\epsilon^3_a\right)p^4_a\left(p^4_\mu\epsilon^4_\nu-p^4_\nu\epsilon^4_\mu\right)p^1_b\left(p^1_\nu\epsilon^1_\alpha-p^1_\alpha\epsilon^1_\nu\right)\left(p^2_\alpha\epsilon^2_\mu-p^2_\mu\epsilon^2_\alpha\right)\\
&+\left(p^4_a\epsilon^4_b-p^4_b\epsilon^4_a\right)p^3_a\left(p^3_\mu\epsilon^3_\nu-p^3_\nu\epsilon^3_\mu\right)p^2_b\left(p^2_\nu\epsilon^2_\alpha-p^2_\alpha\epsilon^2_\nu\right)\left(p^1_\alpha\epsilon^1_\mu-p^1_\mu\epsilon^1_\alpha\right)\Big]
\end{aligned} \tag{67}$$

- The most general local S matrices are given by the form listed above with $A^{0,1}$, $A^{0,2}$ and $A^{1,2}$ polynomials of s, t and u. We have counted the data in such S matrices above- our photon S matrix has 7 degrees of freedom. The most general S matrices - not necessarily local - are also given by the forms above allowing for more general (not necessarily polynomial) dependences of the unknown functions.

Gravity Lagrangians I

- A key feature of the electromagnetic Lagrangian is that all S matrices were generated by Lagrangians with (derivatives on) atleast 4 field strengths.
- The situation is a bit more involved for gravity, as we now describe.
- It is convenient to categorize gravity Lagrangians by the number of factors of (symmetrized derivatives of) the Reimann tensor they contain.
- The unique diffeomorphism invariant action that is linear in Riemann tensors is, of course, the Einstein action

$$S_E = \int \sqrt{-g}R. \tag{68}$$

Gravity Lagrangians II

- Now consider Lagrangians quadratic in Reimann tensors. It is possible to demonstrate that the field redefinition

$$\delta g_{\mu\nu} = H^{(1)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}] \tag{69}$$

may be used to cast the most general Lagrangian, quadratic in Riemann tensors, into the form

$$S = S_E + S_{GB} + \int \mathcal{O}(R_{\alpha\beta\gamma\delta})^3, \tag{70}$$

- where,

$$\begin{aligned} S_{GB} &= \int \sqrt{-g}\ \delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d]}\ R_{ab}{}^{gh} R_{cd}{}^{ij} \\ &\propto \int \sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right). \end{aligned} \tag{71}$$

- In other words Einstein-Gauss-Bonnet is the most general action quadratic in the Riemann tensor up to total derivatives or terms terms that involve explicit factors of $R_{\mu\nu}$ and the Ricci scalar R
- When evaluated in a spacetime of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{72}$$

it turns out that the Gauss-Bonnet term in (70) starts out at order h^3 (up to total derivatives). Thus the Gauss-Bonnet term does not modify the Einstein propagator but does contribute to three point scattering of gravitons. This contribution is proportional to

$$\mathcal{A}^{R^2} = (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge p_1 \wedge p_2)^2 \tag{73}$$

Gravity Lagrangians IV

- Continuing, it is possible to show that field redefinitions of the form

$$\delta g_{\mu\nu} = H^{(2)}_{\mu\nu}[R_{\alpha\beta\gamma\delta}] \tag{74}$$

can be used to cast the most general cubic correction to the Einstein-Gauss-Bonnet action into the form

$$S = S_E + S_{GB} + aS^{(1)}_{R^3} + b\chi_6 + \int \sqrt{-g}\left(\mathcal{O}(R_{\alpha\beta\gamma\delta})^4\right) \tag{75}$$

-

$$\begin{aligned} S^{(1)}_{R^3} &= \int \sqrt{-g}\left(R^{pqrs}R_{pq}{}^{tu}R_{rstu} + 2R^{pqrs}R_p{}^t{}_r{}^u R_{qtsu}\right) \\ \chi_6 &= \int \sqrt{-g}\left(\frac{1}{8}\,\delta^g_{[a}\delta^h_b\delta^i_c\delta^j_d\delta^k_e\delta^l_{f]}\; R_{ab}{}^{gh}R_{cd}{}^{ij}R_{ef}{}^{kl}\right) \end{aligned} \tag{76}$$

Gravity Lagrangians V

- When evaluated on the metric (72), the term χ_6 starts out at order $h_{\mu\nu}^4$ (up to total derivatives). It follows in particular that this term does not contribute to three graviton scattering.
- While the GB and R^6 correct both pole as well as contact corrections to the Einstein's S matrix. However χ_6 is yields a purely contact correction to the Einstein S matrix.
- It follows, in summary, that the most general contact correction to the Einstein S matrix is given by χ_6 plus (derivatives of) 4 Reimann terms (can show that all descenents of χ_6 can themselves be written as descendents of 4 Reimann terms). Note χ_6 vanishes identically in $D \leq 5$ and is a total derivative in $D = 6$.
- With this framework in place it is possible to perform an analysis for gravity similar to that for electromagnetism described above. We now describe the results for parity invariant S matrices in $D \geq 7$.

S matrices for 4 identical gravitons

- As another example we present the most general parity even gravity S matrix in $D \geq 7$.
- This S matrix turns out to be parameterized by 7 Z_2 invariant, one function that enjoys no permutation symmetry and two functions that are completely permutation symmetric. or a total of 29 degrees of freedom.
- In more detail we have one completely symmetric generator at 6 derivatives (Riemann^3) term

$$\chi_6 = \int \sqrt{-g} \left(\frac{1}{8} \, \delta^g_{[a} \delta^h_b \delta^i_c \delta^j_d \delta^k_e \delta^l_{f]} \; R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \right) \tag{77}$$

Second Lovelock term. One d.o.f.

S matrices for 4 identical Gravitions: parity even $D \geq 7$.

- At 8 derivative order we have 5 generators in the **3** and one generator in the **6** rep of S_3. Total 21 dofs.
- At 10 derivative order there are 2 generators in the **3** rep. 6 degrees of freedom.
- Finally at 12 derivative order there is a single generator in the **S** rep. One d.o.f.
- Note: If we set $g_{\mu\nu}(k) = \eta_{\mu\nu} + \epsilon_\mu(k)\epsilon_\nu(k)e^{ik.x}$ with $k^2 = 0$ then it turns out that R_{abmn} evaluated to linearized order is proportional to $F_{ab}(k)F_{mn}(k)$ where $F_{mn} = k_m\epsilon_n - k_n\epsilon_n$. In our Lagrangian terms below we will sometimes replace R_{abmn} with $F_{ab}F_{mn}$.

- Explicitly, the most general 4 gravition S matrix is given by the sum of

$$S_1 = 3B^{0,0}(s,t,u)\left(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2 \quad (78)$$

  with $B^{0,0}(s,t,u)$ completely symmetric (this is from descendents of the Reimann^3 structure) and

-

$$\begin{aligned} &B^{0,1}(s,t)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right. \\ &\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right] \\ &+ B^{0,1}(s,u)\,[3 \leftrightarrow 4] + B^{0,1}(t,s)\,[2 \leftrightarrow 3] + B^{0,1}(t,u)\,[2 \leftrightarrow 3 \ \text{ then } \ 2 \leftrightarrow 4] \\ &+ B^{0,1}(u,t)\,[2 \leftrightarrow 4] + B^{0,1}(u,s)\,[2 \leftrightarrow 4 \ \text{ then } \ 2 \leftrightarrow 3] \end{aligned} \quad (79)$$

  where $B^{0,1}$ has no special symmetry property; this term is from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$

Explicit parameterization of the gravity S matrix:2

- $$B^{0,2}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^3_b\epsilon^3_c - p^3_c\epsilon^3_b\right)\left(p^2_c\epsilon^2_d - p^2_d\epsilon^2_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right]$$
$$+ B^{0,2}(s,u)\,[3\leftrightarrow 2] + B^{0,2}(s,t)\,[2\leftrightarrow 4] \tag{80}$$
where
$$B^{0,2}(t,u) = B^{0,2}(u,t) \tag{81}$$
From descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$.

- $$B^{0,3}(s,u)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
$$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_q\epsilon^2_r - p^2_r\epsilon^2_q\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
$$+ B^{0,3}(t,u)\,[3\leftrightarrow 2] + B^{0,3}(s,t)\,[3\leftrightarrow 4] \tag{82}$$
$$B^{0,3}(s,u) = B^{0,3}(u,s) \tag{83}$$
(from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$)

- $$B^{0,4}(s,t)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
$$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q\right)\left(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
$$+ B^{0,4}(s,u)\,[3 \leftrightarrow 4] + B^{0,4}(u,t)\,[2 \leftrightarrow 4] \tag{84}$$
$$B^{0,4}(s,t) = B^{0,4}(t,s) \tag{85}$$
from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$

- $$B^{0,5}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c\right)\right]$$
$$+ B^{0,5}(s,u)\,[3 \leftrightarrow 2] + B^{0,5}(s,t)\,[2 \leftrightarrow 4] \tag{86}$$
$$B^{0,5}(t,u) = B^{0,5}(u,t) \tag{87}$$
from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)$

- $$B^{0,6}(s,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p\right)\left(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c\right)\right]$$
$$+ B^{0,6}(t,u)\,[3 \leftrightarrow 2] + B^{0,6}(s,t)\,[3 \leftrightarrow 4] \tag{88}$$

$$B^{0,6}(s,u) = B^{0,6}(u,s) \tag{89}$$

  from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^4)\mathrm{Tr}(F^2F^3)$
- This completes the listing of the S matrices of denscendents of 6 and 8 derivative terms. We now turn to the listing of S matrices that follow from descendents of the two 10 derivative and one 12 derivative terms.

Explicit prameterization of the general 4 graviton S matrix: 5

- $$\begin{aligned}&+\left(B^{2,1}(s,u)\left(p_p^1\epsilon_q^1-p_q^1\epsilon_p^1\right)\left(p_q^2\epsilon_r^2-p_r^2\epsilon_q^2\right)\left(p_r^3\epsilon_s^3-p_s^3\epsilon_r^3\right)\left(p_s^4\epsilon_p^4-p_p^4\epsilon_s^4\right)\right.\\&\quad B^{2,1}(t,u)\left(p_p^1\epsilon_q^1-p_q^1\epsilon_p^1\right)\left(p_q^3\epsilon_r^3-p_r^3\epsilon_q^3\right)\left(p_r^2\epsilon_s^2-p_s^2\epsilon_r^2\right)\left(p_s^4\epsilon_p^4-p_p^4\epsilon_s^4\right)\\&+\left.B^{2,1}(t,s)\left(p_p^1\epsilon_q^1-p_q^1\epsilon_p^1\right)\left(p_q^3\epsilon_r^3-p_r^3\epsilon_q^3\right)\left(p_r^4\epsilon_s^4-p_s^4\epsilon_r^4\right)\left(p_s^2\epsilon_p^2-p_p^2\epsilon_s^2\right)\right)\\&\quad\left(\left(p_a^1\epsilon_b^1-p_b^1\epsilon_a^1\right)p_a^2\left(p_\mu^2\epsilon_\nu^2-p_\nu^2\epsilon_\mu^2\right)p_b^3\left(p_\nu^3\epsilon_\alpha^3-p_\alpha^3\epsilon_\nu^3\right)\left(p_\alpha^4\epsilon_\mu^4-p_\mu^4\epsilon_\alpha^4\right)\right.\\&\quad+\left(p_a^2\epsilon_b^2-p_b^2\epsilon_a^2\right)p_a^1\left(p_\mu^1\epsilon_\nu^1-p_\nu^1\epsilon_\mu^1\right)p_b^4\left(p_\nu^4\epsilon_\alpha^4-p_\alpha^4\epsilon_\nu^4\right)\left(p_\alpha^3\epsilon_\mu^3-p_\mu^3\epsilon_\alpha^3\right)\\&\quad+\left(p_a^3\epsilon_b^3-p_b^3\epsilon_a^3\right)p_a^4\left(p_\mu^4\epsilon_\nu^4-p_\nu^4\epsilon_\mu^4\right)p_b^1\left(p_\nu^1\epsilon_\alpha^1-p_\alpha^1\epsilon_\nu^1\right)\left(p_\alpha^2\epsilon_\mu^2-p_\mu^2\epsilon_\alpha^2\right)\\&\quad+\left.\left(p_a^4\epsilon_b^4-p_b^4\epsilon_a^4\right)p_a^3\left(p_\mu^3\epsilon_\nu^3-p_\nu^3\epsilon_\mu^3\right)p_b^2\left(p_\nu^2\epsilon_\alpha^2-p_\alpha^2\epsilon_\nu^2\right)\left(p_\alpha^1\epsilon_\mu^1-p_\mu^1\epsilon_\alpha^1\right)\right)\end{aligned} \tag{90}$$

  $$B^{2,1}(s,u)=B^{2,1}(u,s) \tag{91}$$

  from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)F^1_{ab}\mathrm{Tr}(p_a^2F^2p_b^3F^3F^4)$.

Explicit parameterization of the general 4 graviton S matrix: 6

- $$\begin{aligned}&\Big(B^{2,2}(t,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\\
&+ B^{2,2}(s,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_p\epsilon^3_q - p^3_q\epsilon^3_p)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\\
&+ B^{2,2}(t,s)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\Big)\\
&\Big((p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,p^2_a\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,p^3_b\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha)\\
&+ (p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,p^1_a\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,p^4_b\,(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu)\,(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha)\\
&+ (p^3_a\epsilon^3_b - p^3_b\epsilon^3_a)\,p^4_a\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,p^1_b\,(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu)\,(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha)\\
&+ (p^4_a\epsilon^4_b - p^4_b\epsilon^4_a)\,p^3_a\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,p^2_b\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha)\Big)\end{aligned} \tag{92}$$

$$B^{2,2}(t,u) = B^{2,2}(u,t) \tag{93}$$

from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

- $$\begin{aligned} &\left(B^{4,1}(s,t)+B^{4,1}(t,u)+B^{4,1}(u,s)\right)\times \\ &\Big[\left(p^1_a\epsilon^1_b-p^1_b\epsilon^1_a\right)p^2_a\left(p^2_\mu\epsilon^2_\nu-p^2_\nu\epsilon^2_\mu\right)p^3_b\left(p^3_\nu\epsilon^3_\alpha-p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\mu-p^4_\mu\epsilon^4_\alpha\right) \\ &\left(p^1_p\epsilon^1_q-p^1_q\epsilon^1_p\right)p^2_p\left(p^2_\beta\epsilon^2_\gamma-p^2_\gamma\epsilon^2_\beta\right)p^3_q\left(p^3_\gamma\epsilon^3_\delta-p^3_\delta\epsilon^3_\gamma\right)\left(p^4_\delta\epsilon^4_\beta-p^4_\beta\epsilon^4_\delta\right) \\ &+(1\leftrightarrow 2)+(1\leftrightarrow 3)+(1\leftrightarrow 4)\Big] \end{aligned} \tag{94}$$

  $$B^{4,1}(s,t)=B^{4,1}(u,t)=B^{4,1}(t,s)=B^{4,1}(u,s)=B^{4,1}(s,u)=B^{4,1}(t,u) \tag{95}$$

  from descendents of
  $F^1_{pq}\mathrm{Tr}(p^2_pF^2p^3_qF^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

Results: Counting

dimension	Even partition function	Odd partition function
$D \geq 10$	$x^8(x^{-2}+6+9x^2+10x^4+3x^6)\mathbb{D}$	0
$D=9$	$x^8(x^{-2}+6+9x^2+10x^4+3x^6)\mathbb{D}$	0
$D=8$	$x^8(x^{-2}+6+9x^2+10x^4+3x^6)\mathbb{D}$	0
$D=7$	$x^8(x^{-2}+6+9x^2+10x^4+3x^6)\mathbb{D}$	$x^8(2x^{-1}+3x+2x^3)\mathbb{D}$
$D=6$	$x^8(6+9x^2+10x^4+3x^6)\mathbb{D}$	$3x^{10}(x^2+x^4+x^6)\mathbb{D}$
$D=5$	$x^8(4+7x^2+8x^4+3x^6)\mathbb{D}$	$x^{11}(x^2+x^4+x^6)\mathbb{D}$
$D=4$	$x^8(2+2x^2+3x^4-x^6-x^8)\mathbb{D}$	$x^8(1+x^2+2x^4-x^6-x^8)\mathbb{D}$

Table: Partition function over 4 graviton S-matrices. $\mathbb{D}=\frac{1}{(1-x^4)(1-x^6)}$. The coefficient of x^m in these expressions gives the number of independent polynomial S matrices at m derivative order.

See paper for analogeous results for electromagnetism.

Tests- Plethystics

- We believe we have achieved a complete classification of polynomial 4 graviton S matrices. However the analysis was long and complicated (each dimension had its own detailed subtleties). To be sure there are no mistakes, useful to have an independent check.
- In order to check our results we reproduced the table of the previous slide by a completely independent compuatation. Our second independent computation proceeds by explicitly ennumerating local Lagrangians upto field refinitions and total derivatives rather than S matrices.
- We enumerate by evaluating an $SO(D)$ matrix integral that projects the 'four graviton letter' partition function onto the space of $SO(D)$ singlets after removing total derivatives. The computation is not completely trivial, but we managed to carry it through. Both methods give exactly the same final results, giving a highly nontrivial test of our module constructions. Next few slides: some details.

- Explain the procedure for scalars. Define the single letter partition function i.e. partition function over all the operators that involve a single field, modulo the free equation of motion. Space spanned by

$$\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_l}\phi, \qquad \text{subject to} \quad \partial_\mu\partial^\mu\phi = 0. \tag{96}$$

- Not difficult to check

$$\begin{aligned} i_{\rm s}(x,y) &= \mathrm{Tr}\; x^\Delta y_i^{H_i} = (1-x^2)\mathrm{D}(x,y). \\ \mathrm{D}(x,y) &= \left(\prod_{i=1}^{D/2}(1-xy_i)(1-xy_i^{-1})\right)^{-1} && \text{for D even} \\ &= \left((1-x)\prod_{i=1}^{\lfloor D/2\rfloor}(1-xy_i)(1-xy_i^{-1})\right)^{-1} && \text{for D odd.} \end{aligned}$$

- H_i are Cartan elements of $SO(D)$. Have kept track of the Cartans because will eventually project to $SO(D)$ singlets.

- The partition function of polynomials of the expressions (96) - the so called multi-letter partition function is given by the formula of Bose statistics

$$\sum_{k=1}^{\infty} t^k i_s^{(k)}(x,y) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} i_s(x^n, y^n)\right). \qquad (97)$$

- $i_s^{(k)}$ to be the partition function over k-letter partition function, $i_s^{(1)} = i_s$.
- The four-letter partition function - relevant for counting quartic Lagrangians - is easily read off from equation (97):

$$i_s^{(4)}(x,y) = \frac{1}{24}\Big(i_s^4(x,y) + 6 i_s^2(x,y) i_s(x^2,y^2) + 3 i_s^2(x^2,y^2) + 8 i_s(x,y) i_s(x^3,y^3) + 6 i_s(x^4,y^4)\Big). \qquad (98)$$

- This partition function over four particle states includes operators that are total derivatives which we wish to remove.
- Atleast naively: The partition function over polynomials of (96), modulo total derivatives is given by

$$i_{\mathrm{S}}^{(4)}(x,y)/\mathrm{D}(x,y). \tag{99}$$

  (96) is exact for scalars, but turns out to have some subtleties for gauge fields and gravitions in low dimensions, which have to be dealt with in a case by case manner. Ignore this subtlety here.
- The partition function over scalar operators is now obtained simply by projecting onto $SO(D)$ invariant states. This is achieved by integrating $i_{\mathrm{S}}^{(4)}(x,y)/\mathrm{D}(x,y)$ over the Haar measure of the group.

Plethystics III

- Analysis easy to repeat for gauge fields and gravitons. For photons have the same final formula as above but with $i_{\mathrm{s}} \to i_{\mathrm{v}}$ with
-

$$i_{\mathrm{v}}(x, y) = x\,\chi_{\yng(1,1)} + x^2\,\chi_{\yng(2,1)} + x^3\,\chi_{\yng(3,1)} + x^4\,\chi_{\yng(4,1)} + x^5\,\chi_{\yng(5,1)} + x^6\,\chi_{\yng(6,1)} \ldots . \quad (100)$$

- Summing we obtain

$$i_{\mathrm{v}}(x, y) = (((x - x^3)\chi_{\square} - (1 - x^4))\mathrm{D}(x, y) + 1)/x. \quad (101)$$

Plethystics IV

- Similarly for gravitons we have the same formula with $i_{\mathrm{s}} \to i_{\mathrm{t}}$
- $$i_{\mathrm{t}}(x,y) = x^2\,\chi_{\ydiagram{2,2}} + x^3\,\chi_{\ydiagram{3,2}} + x^4\,\chi_{\ydiagram{4,2}} + x^5\,\chi_{\ydiagram{5,2}} + x^6\,\chi_{\ydiagram{6,2}} + x^7\,\chi_{\ydiagram{7,2}} \ldots . \tag{102}$$
- Summing we obtain

$$i_{\mathrm{t}}(x,y) = x^2\,\chi_{\ydiagram{2,2}} + x^3\,\chi_{\ydiagram{3,2}} + x^4\,\chi_{\ydiagram{4,2}} + x^5\,\chi_{\ydiagram{5,2}} + x^6\,\chi_{\ydiagram{6,2}} + x^7\,\chi_{\ydiagram{7,2}} \ldots . \tag{103}$$

Plethystics V

- What remains is to evaluate the integral over $SO(D)$. Easy to do analytically at large D using saddle points. Recover precisely the tabulated results above for $D \geq 10$
- For lower dimensions there are two complications. First the reasoning above not completely precise. E.g. misses out Chern Simons terms. Also sometimes dividing out by the derivative denomonator is an overkill. However both these subtleties only occur for a small finite number of operators that can be explicitly ennumerated in every dimension. Results of integral have to be corrected in dimension dependent way.
- Other complication. Hard to evaluate Haar integrals exactly. However have used mathematica to evaluate to high order in Taylor expansion in x. Use results to guess answer and verify by going to even higher order. End result - perfect agreement with table above.

Constraining Polynomial S matrices

- Recall that the CEMZ programme for constraining 3 graviton scattering had 2 steps. The first step was to use symmetry considerations to minimally parameterize the S matrix. We are now done with the analogous step for the 4 graviton S matrix.
- As you can see the result here is much more complicated; as opposed to 3 numbers it is given in high enough dimensions in terms of terms of 10 unknown functions of s and t.
- We now turn to the second step of the programme, namely to use a physical principle to constrain the parameters that appear in the S matrix.

Implications of CRG: contact graviton interactions

- We can now use our painstakingly constructed explicit parameterization of polynomial graviton S matrices to list the most general S matrid of this form that obeys CRG scaling. We find that there is only one such S matrix namely

$$a\left(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2$$

- This 6 derivative S matrix - which, (roughly speaking) scales like *stu* and so is CRG allowed is generate by the Lagrangian

$$\chi_6 = \int \sqrt{-g}\left(\frac{1}{8}\;\delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d}\delta^{k}_{e}\delta^{l}_{f]}\;R_{ab}{}^{gh}R_{cd}{}^{ij}R_{ef}{}^{kl}\right)$$

Minimum Regge growth of contact interactions I

- I give a brief sketch of the argument for the result described in the previous slide. I present the argument for parity even S matrices though in our paper we also consider the parity even case.
- We use only the obvious fact that every gravity S matrix is *some* descendent of *some* generator of the Bare module.
- If the generator $|e_{\mathbf{S}}\rangle$ in question is in the **S** representation then its most general S_3 invariant descendent takes the form

$$\left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)|e_{\mathbf{S}}\rangle \tag{104}$$

- Let us define

$$P_S = \left(\sum_{k,m} a_{k,m}(stu)^k(s^2+t^2+u^2)^m\right)$$

Minimum Regge growth of contact interactions II

- If the generator $|e_{\mathbf{A}}\rangle$ in question is in the **A** representation, its most general S_3 invariant descendent is given by

$$P_S\left(s^2u - u^2s + t^2s - t^2u - s^2t + u^2t\right)|e_{\mathbf{A}}\rangle \qquad (105)$$

- Finally if the generators $|e^{(3)}_{\mathbf{M}}\rangle$ transform in the **M** then its most general S_3 invariant descendent is given by

$$P_S\left(s|e^{(1)}_{\mathbf{M}}\rangle + t|e^{(2)}_{\mathbf{M}}\rangle + u|e^{(3)}_{\mathbf{M}}\rangle\right) \qquad (106)$$

or

$$P_S\Big(\left(t^2+u^2-2s^2\right)|e^{(1)}_{\mathbf{M}}\rangle + \left(u^2+s^2-2t^2\right)|e^{(2)}_{\mathbf{M}}\rangle + \left(s^2+t^2-2u^2\right)|e^{(3)}_{\mathbf{M}}\rangle\Big) \qquad (107)$$

Minimum Regge growth of contact interactions III

- A simple inspection of (104), (105), (106) and (107) immediately reveals the following facts.
- No S_3 invariant descendent of the bare module at dimension 8 or higher has Regge growth slower than s^3.
- There is only one kind of S_3 invariant descendent of the bare module at dimension 6 that has Regge growth slower than s^3. This is a descendent of the form $|e_{\mathbf{S}}\rangle$.
- It is not difficult to verify that the generator corresponding to the Lagrangian χ_6 is of the form $|e_{\mathbf{S}}\rangle$.
- It follows that χ_6 is the unique gravitational contact interaction that leads to CRG allowed growth. No such interaction exists in dimensions in which χ_6 vanishes or is a total derivative.

- In summary, that the most general purely gravitational CRG action (upto terms that cannot affect 4 graviton scattering) is

$$a(Einstein) + b(GB) + c(Reimann^3) + d\chi_6$$

- Recall again

$$\chi_6 = \int \sqrt{-g}\left(\frac{1}{8}\,\delta^g_{[a}\delta^h_b\delta^i_c\delta^j_d\delta^k_e\delta^l_{f]}\;R_{ab}{}^{gh}R_{cd}{}^{ij}R_{ef}{}^{kl}\right)$$
$$= \int \sqrt{-g}\Big(4R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} - 8R_a{}^c{}_b{}^dR_c{}^e{}_d{}^fR_e{}^a{}_f{}^b - 24R_{abcd}R^{ab}$$
$$+ 24R_{abcd}R^{ac}R^{bd} + 16R_a{}^bR_b{}^cR_c{}^a - 12R_a{}^bR_b{}^aR + R^3\Big) \qquad (108)$$

$D \leq 6$

- $$\chi_6 = \int \sqrt{-g}\left(\frac{1}{8}\ \delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d}\delta^{k}_{e}\delta^{l}_{f]}\ R_{ab}{}^{gh}R_{cd}{}^{ij}R_{ef}{}^{kl}\right)$$
- It is obvious that χ_6 vanishes identially for $D \leq 5$. In $D = 6$ this term is a total derivative. The term is classically nontrivial only for $D \geq 7$. This fact is already apparent from the form of its S matrix.
- In $D \leq 6$ it thus follows that the most general CRG allowed purely gravitational action (upto terms that cannot impact the four graviton scattering) in $D \leq 6$ is

$$a(Einstein) + b(GB) + c(Reimann^3).$$

Exchange Contributions: Graviton Exchange

- So far we have considered the implications of the CRG conjecture on the polynomial contributions to 4 graviton scattering. As we have already discussed above the most general contribution also has exchange contributions.
- The exchange contributions relevant to Conjecture III are graviton poles, so lets study those first.
- These amplitudes can be thought of as a quadratic form in the coefficients a, b and c (of the allowed 3 point structures of 3 graviton scattering).
- We have explicitly constructed the most general exchange contribution of this nature (and also decomposed it in terms of our 'generator' index structures above). The final answer is a bit complicated. Main important result, however, is that this pole contribution to the amplitude grows faster than s^2 in the Regge limit unless $b = c = 0$. It follows that the most general CRG allowed purely gravitational action in $D \leq 6$ is Einstein. In particular we CRG implies CEMZ + more.

s^J?

- In the last slide I claimed that exchange contributions from, e.g., two GB vertices grows faster than s^2. On the other hand we often hear the following claim: the contribution to the S matrix from the exchange of paricles of spin J scales like s^J. Given that GB exchange contributions capture only gravity exchange (i.e. $J = 2$) don't we have a contradiction?
- The resolution is the following. The contribution of spin J particles to the S matrix scales like s^J only in the t channel. The contribution from the s and u channels is not universal. They depend on the details of the three point couplings. These contributions grow faster than s^2 in for GB-GB exchange.
- Note that t channel contributions are special. They are non polynomial in t even in the Regge limit. These are thus the only contributions that contribute to scattering at nonzero impact parameter in the Regge limit. In other words GB 4 graviton scattering violates the CRG conjecture - but not in a way that can be seen at nonzero impact parameter.

- On the other hand s and u channel contributions are typically analytic in t in the Regge limit, and (like contact terms) contribute to scattering only at zero impact parameter.
- In order to conclude that a GB coupling is unphysical, it is thus not sufficient to check that the exchange contribution from two GB vertices grows faster than s^2. We must also check that this growth is of the form that that cannot be cancelled by addition of a local counterterm. We have indeed checked this.

Regge Growth for Massive Exchange: Argument I

- A sketch of the argument for the result described in the previous transparency goes as follows.
- A general massive exchange contribution to 4 graviton takes the form

$$\mathcal{S} = \frac{|\alpha_1\rangle}{s-m^2} + \frac{|\alpha_2\rangle}{t-m^2} + \frac{|\alpha_3\rangle}{u-m^2}. \tag{109}$$

  where $|\alpha_1\rangle$ are local S matrices of dimension 8 or higher (we use the fact that *ggP* 3 point functions are all of dimension 4 or higher)
- Define

$$A \equiv (s-m^2)(t-m^2)(u-m^2)\mathcal{S} \tag{110}$$

  A is local. The highest dimension part of A is of dimension 12 or greater. By inspecting (104), (105), (106) and (107) we see that the minimum Regge growth of A is s^4, and this is achieved when $A = (stu)^2|e_{\mathbf{S}}\rangle$.

Regge Growth for Massive Exchange: Argument II

- In this case at high energies

$$S \sim stu|e_{\mathbf{S}}\rangle.$$

- This CRG allowed behaviour only occurs if

$$|\alpha_1\rangle = s(stu)|g_{\mathbf{S}}\rangle, \quad |\alpha_2\rangle = t(stu)|g_{\mathbf{S}}\rangle, \quad |\alpha_3\rangle = u(stu)|g_{\mathbf{S}}\rangle. \tag{111}$$

- However $|\alpha_i\rangle$ are elements of the Local module of dimension 8. We have a complete classification of such elements, and the only ones of the form presented in (111) are the descendents of χ_6. In dimensions in which χ_6 vanishes or is trivial there are no dimension 8 Local module elements of the form (111). In such dimensions S must grow faster than s^2.

Discussions and Conclusions

- In this talk we first presented a complete classification of 4 graviton (and four photon) classical S matrices in the theory whose Lagrangian has a finite number of derivatives and has a finite number of fields.
- We then presented a conjecture about the allowed growth of S matrices in classical theories. We then used this conjecture to completely classify allowed classical theories of gravity, upto Lagrangian terms of order Riemann^5 or higher that do not impact 4 graviton scattering.
- It would be very nice to understand our s^2 conjecture better - and if possible to replace it with a clear physical argument directly in flat space. We have some ideas that we are working on.
- It would also be interesting to understand the status of the ambiguity of the action in $D \geq 7$. Is this a genuine ambiguity, or does another physical argument set a to zero?

Discussions and Conclusions

- Using AdS/CFT one can turn our results into a constraint on stress tensor four point functions in the large N limit. Our results suggest that the only Chaos bound allowed large N $TTTT$ four point function that receives contributions from a finite number of single trace exchange blocks (in addition to double stress tensor exchanges) is the result generated by the pure Einstein action in the bulk.
- It would be very interesting to generalize the results of this talk to the scattering of more than 4 gravitions, and complete the process of characterizing the most general classical local theory of gravity consistent with general principles.
- Finally, if all this works out we could get more ambitious and generalize the study of this talk beyond local S matrices, with the hope of establishing Conjecture I: i.e the uniqueness of string scattering.

Rough Work