Random friend trees

Based on joint work with Louigi Addario-Berry, Simon Briend, Luc Devroye, Céline Kerriou and Gabor Lugosi

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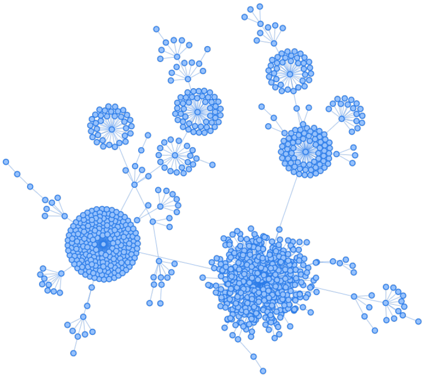
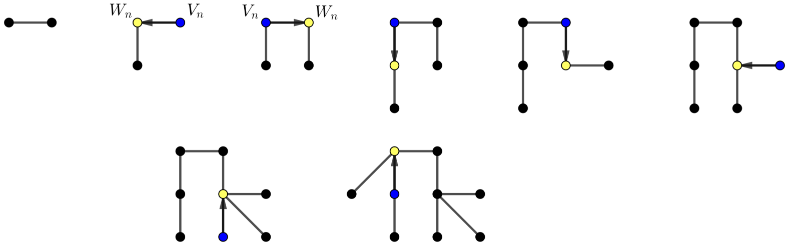
Introduction

- We study a random recursive tree model called *the random friend tree*
- Attachment via complete redirection
- 'Rich-getting-richer' dynamics
- Interesting emergent properties
- Local attachment rule



Definition of the model

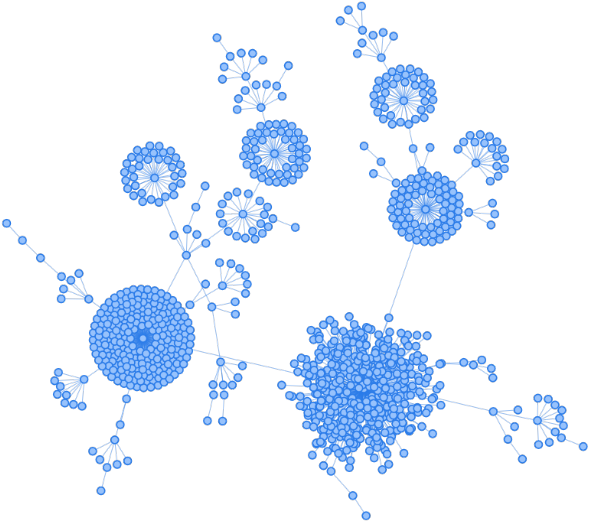
- Let T_2 be the tree consisting of vertices 1 and 2 connected via an edge.
- Given T_n, let V_n be a uniform vertex in T_n
- Let W_n be a uniform neighbour of V_n in T_n
- Include the vertex $n+1$ and the edge $\{n+1, W_n\}$ in T_n to obtain T_{n+1}



Earlier work

- 'Introduced' by Saramäki and Kaski (2004)
- Cannings and Jordan show that T_n contains $n - o(n)$ leaves almost surely (2014)
- More specific properties of the degree sequence were studied non-rigorously in the physics literature by Karpivsky and Redner (2017)

Intuition for dynamics



Results I - Hubs

For $u \in [n]$, let $N_n(u)$ be the number of neighbours of vertex u in T_n and let $L_n(u)$ be the number of leaf neighbours of vertex u in T_n.

Convergence of normalised degrees ⋆

There exists a random variable Z_u on $[0, \infty)$ such that

$$\left(\frac{L_n(u)}{n}, \frac{N_n(u)}{n} \right) \to (Z_u, Z_u) \text{ almost surely}$$

We call a vertex u a *hub* if $Z_u > 0$.

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All edges contain a hub $\star$

For any edge $\{m+1, W_m\}$, it holds that

$$Z_{m+1} + Z_{W_m} > 0 \text{ almost surely}$$

Results II - Frozen degrees

Vertices never acquire a new neighbour with positive probability

For each k, there is a p_k so that for any n, any degree k vertex present at time n has probability at least p_k to *never* acquire another neighbour.

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Most leaves stay leaves forever

Suppose v is a hub. Then, then the number of leaves adjacent to v at time n that *ever* acquire another neighbour is tight.

Results III - Distances

Diameter $\star$

Let D_n be the diameter of T_n. Then,

$$1 \leq \liminf \frac{D_n}{\log n} \leq \limsup \frac{D_n}{\log n} \leq 4e.$$

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Leaf depth

Let M_n be the maximal distance of any vertex to its nearest leaf in T_n. Then,

$$M_n = \Theta\left(\frac{\log n}{\log\log n}\right) \text{ in probability.}$$

Results IV - Degree sequence

Let $X_n^{\geq k}$ be the number of vertices in T_n with degree at least k.

Large-ish degrees

For any sequence $(m_n)_{n\geq 1}$ with $m_n = o(n)$,

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Small degrees

There exist constants $0.1 < \alpha < \beta < 0.9$ such that for any $k \geq 2$, as $n \to \infty$,

$$\frac{X_n^{\geq k}}{n^\alpha} \to \infty \text{ and}$$
$$\frac{X_n^{\geq k}}{n^\beta} \to 0.$$

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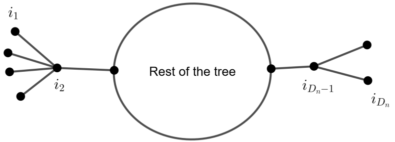
- In preferential attachment trees and uniform attachment trees, the development of the sequence of vertex degrees, or even of a single vertex degree, can be studied without keeping track of the geometry of the tree.
- In the random friend tree, this is not possible.
- To know $\mathbb{E}[\Delta N_n(v) \mid T_n]$, you need to know the degrees of the neighbours of v. But to sample their development, you need to know the degrees of *their* neighbours, etc.
- More generally, in the random friend tree, the global structure affects local properties.

Proof lower bound diameter $\liminf D_n/n \geq 1$



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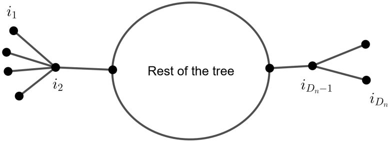

$$\mathbb{P}(\Delta D_n = 1 \mid T_n) \geq \mathbb{P}(V_n = i_2, W_n \neq i_3) + \mathbb{P}(V_n = i_{D_n-1}, W_n \neq i_{D_n-2})$$
$$\geq \frac{1}{n} \times \frac{1}{2} + \frac{1}{n} \times \frac{1}{2} = \frac{1}{n}.$$

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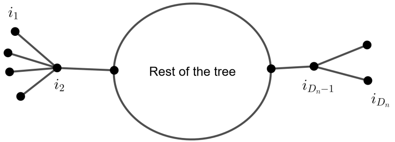

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So D_n stochastically dominates $B_1 + \cdots + B_n$, for B_i a Bernoulli random variable with $\mathbb{P}(B_i = 1) = \frac{1}{i}$. Kolmogorov's strong law of large numbers implies that

$$\liminf \frac{D_n}{\log n} \geq 1 \text{ almost surely.}$$

Proof almost sure limit $L_n(v)/n, N_n(v)/n$.

$$\begin{aligned}\mathbb{E}[L_{n+1}(v) \mid T_n] &= L_n(v) - \mathbb{P}(V_n = v, W_n \text{ a leaf}) + \mathbb{P}(W_n = v) \\ &\geq L_n(v) - \mathbb{P}(V_n = v) + \mathbb{P}(W_n = v, V_n \text{ a leaf}) \\ &= L_n(v) - \frac{1}{n} + \frac{L_n(v)}{n}.\end{aligned}$$

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Rearrangement yields that

$$\mathbb{E}\left[\frac{L_{n+1}(v) - 1}{n+1} \Big| T_n\right] \geq \frac{L_n(v) - 1}{n}.$$

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$$\frac{L_n(v)}{n} \leq \frac{N_n(v)}{n} \leq \frac{L_n(v) + X_n^{\geq 2}}{n}.$$

Joint convergence follows from $X_n^{\geq 2} = o(n^{0.9})$ almost surely.

Proof sketch $Z_u + Z_v > 0$ for $\{u, v\}$ an edge

Set $(u, v) = (m + 1, W_m)$ and for $n \geq m + 1$, write $N_n = N_n(u) + N_n(v)$ and $L_n = L_n(u) + L_n(v)$. We show that $\liminf L_n/n > 0$ almost surely.

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Further directions I - Extend results

- Does the degree of an older vertex stochastically dominate the degree of a younger vertex in T_n?

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- What does the graph restricted to the small degree vertices look like?

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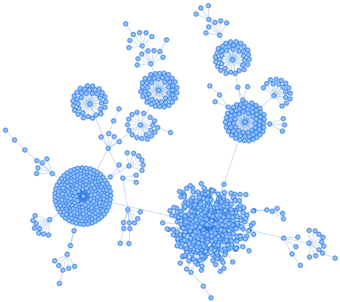
Can we show that the average proportion of leaf neighbours across non-leaf vertices converges almost surely?

Krapivsky and Redner conjecture that $\gamma = \mu \approx 0.566$

Further directions III - A potential master key

Intuitively, the different parts of the tree should 'decouple' at the hubs. The hubs cut the tree in (almost) independent parts that, up to a time change, evolve like rooted random friend trees.



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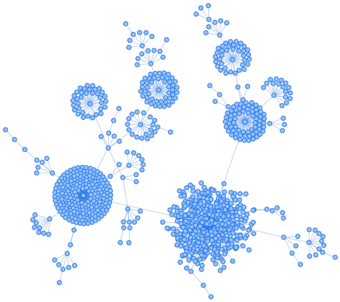

If we can make this formal, averaging across the tree allows for use of the law of large numbers, bringing many more complicated questions within reach.

Further directions IV - Generalisations

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- We may consider partial redirection, where $n+1$ connects to V_n instead of W_n with probability $p > 0$.
- We can let new vertices connect to multiple vertices.