Secondary Deformation Classes

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Geometric Structures and Stability

ICTS-TIFR, Bengaluru

February 18th, 2026

European Research Council

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Introduction

Given a fibration $f : X \to S$, suppose we have a subspace $S' \subset S$ such that the fibration $f' : X' := X \times_S S' \to S'$ has the property that $\pi_1(S')$ acts trivially on the cohomological or homotopical invariants of the fiber.

The topological structure of X'/S' is determined by higher cohomology classes.

Let X'_k denote the quotient of the relative Postnikov tower, using a basic hypothesis that f' is relatively 1-connected. Then $X'_{k+1} \to X'_k$ is a fibration with fiber $K(A_{k+1}, k+1)$, where A_{k+1} is a constant local system over S.

Introduction

The cohomological invariants determining these fibrations may be viewed as secondary or higher monodromy classes of X'/S'.

We will be interested in the case where S' has the structure of a torus. On the one hand, its fundamental group is abelian, so not very complicated. But on the other hand, a torus is a space having a large but understandable cohomology, so the secondary invariants may be nontrivial yet computable.

This picture extends to situations where S is stacky and S' is a stack of the form $K(A,1)$ for an abelian group sheaf A.

Introduction

Two particular cases arise in algebraic geometry.

- When S has a structure of Dolbeault stack then every point supports an abelian group gerb that is the source for the Higgs field.
- When S is a de Rham stack with logarithmic singularities along a divisor D, then the residues are actions of abelian groups and we may view this again as $S' = K(A, 1) \hookrightarrow S$ for A a formally completed commutative group scheme.

The study of secondary classes for such substacks provides a look at a "stacky" aspect of the relationship between Higgs fields and transport for connections.

The topological picture

Suppose $S = (S^1)^k$ is a torus with basepoint denoted s. Suppose we are given a smooth fibration $X \to S$, which we assume is relatively 1-connected. By monodromy $\pi_1(S, s) \cong \mathbb{Z}^k$ acts on the homotopy groups $\pi_i(X_s)$ and cohomology groups $H^i(X_s, \mathbb{C})$ of the fiber.

The fibration contains further information. A Postnikov tower for the fiber relative to the base has structure involving cohomology classes for the successive fibrations, and these classes may have Leray components that are higher cohomology classes on the base with coefficients in cohomology of the fiber.

The topological picture

One way the secondary classes may be detected is by cup products. Supposing that the fundamental group acts trivially on the cohomology groups of the fiber, we might ask whether the fibration is a product.

In the case of a product, the Künneth formula says

$$H^*(X) \cong H^*(X_s) \otimes H^*(S),$$

and this is compatible with cup-product. So, for example, given a class $\eta \in H^2(X_s)$ such that $\eta \cup \eta = 0$ in $H^4(X_s)$, the lift to a class $\eta \in H^2(X)$ (assuming X_s is 1-connected) will also satisfy $\eta \cup \eta = 0$.

The topological picture

In general, a fibration X/S having trivial fundamental group action on the cohomology of the fibers, is not necessarily a product. Suppose for example $H^1(X_s) = 0$ and $H^3(X_s) = 0$ and also suppose we have a basepoint section.

A class $\eta \in H^0(S, H^2(X_s))$ lifts to a unique class in $H^2(X)$ vanishing on the basepoint section. Suppose $\eta \cup \eta = 0$ in $H^0(S, H^4(X_s))$.

The *secondary class* in this case is the cup-product that goes a step lower in the Leray filtration, which here means

$$\eta \cup \eta \in H^2(S, H^2(X_s)).$$

The de Rham formal groupoid

Suppose now $X \xrightarrow{f} S$ is a projective smooth morphism of smooth complex varieties.

Recall that the *de Rham formal groupoid* S_{DR} is the groupoid-object in the category of formal schemes, whose object-object is S and whose morphism-object

$$\mathcal{F}_{DR}(S) \hookrightarrow S \times S$$

is the formal completion of the diagonal.

Think of the morphism elements of $\mathcal{F}_{DR}(S)$ as being links glueing together infinitesimally nearby points of S.

The de Rham formal groupoid

This represents a stack that may be written in simplfied form as

$$S_{DR}(Y) := S(Y^{\mathrm{red}}).$$

In the direction of a more concrete expression, it corresponds to the Lie algebroid struction (Lie-Rinehart algebra) on the tangent bundle TS. There is a morphism $S \to S_{DR}$ and

$$S \times_{S_{DR}} S = \mathcal{F}_{DR}(S).$$

We similarly have a formal groupoid X_{DR} with a morphism $X_{DR} \to S_{DR}$. The pullback $X_{DR} \times_{S_{DR}} S$ is the relative de Rham stack $X_{DR/S}$ whose fiber on any point $s \in S$ is the de Rham stack $(X_s)_{DR}$ of the fiber.

Gauss-Manin

A bundle over S_{DR} is the same thing as a bundle over S together with a flat connection. The stacky viewpoint extends this naturally to higher homotopy by considering higher stacks over S_{DR}.

The relative cohomology and homotopy invariants for $X_{DR/S}$ over S are the usual de Rham cohomology and de Rham homotopy type of X/S, and the fact that these come from relative invariants for X_{DR} over S_{DR} encapsulates the Gauss-Manin connection on the de Rham invariants of X/S.

The connection defined in this way includes the higher homotopy coherence structures, giving the higher Gauss-Manin connection on rational homotopy types that was first investigated by Navarro-Aznar.

The Higgs field

We now turn to the Higgs picture corresponding to the above discussion. Deformation to the normal cone of $\mathcal{F}_{DR}(S)$ and $\mathcal{F}_{DR}(X)$ yield formal groupoids $\mathcal{F}_{Dol}(S)$ and $\mathcal{F}_{Dol}(X)$. These are associated to Lie algebroids whose underlying bundles are again TS and TX, but where the Lie algebroid structure is trivial (zero bracket and zero action on functions).

Thus, $\mathcal{F}_{Dol}(X)$ is just the commutative formal groupoid whose morphism object is the completion of the tangent bundle TX along the zero-section, viewed as lying over the diagonal $\Delta_X \subset X \times X$. Same for S.

The infinitesimal links become infinitesimal loops based at the points of S individually.

The Higgs field

Locally given a trivialization of the tangent bundle $TS \cong \mathcal{O}_S^k$ we get

$$S_{Dol} \cong S \times K(\widehat{A}^k, 1)$$

where $\widehat{A}^k$ is the formal completion of $\mathbb{A}^k$ at 0. In the general case we may write this formula as $S_{Dol} \cong K(\widehat{TS}/S, 1)$.

A Higgs bundle on S is a sheaf on S_{Dol} that pulls back to a locally free sheaf on S. In other words, for a bundle on S the data of a descent to the stack S_{Dol} is the same thing as a Higgs field, or equivalently an action of the Lie algebroid TS (having its trivial structure).

The Higgs field

Suppose $f : X \to S$ is a smooth projective morphism.

We may again form the relative version $f_{Dol/S} : X_{Dol/S} \to S$ and

$$\mathbf{R}^i(f_{Dol/S})_*\mathcal{O}_{X_{Dol/S}} \cong \mathbf{R}^i f_*(\bigoplus \Omega^j_{X/S}) =: H^i_{Dol}(X/S).$$

is the relative Dolbeault cohomology over S.

Recall that this may be expressed using the Hodge filtration:

$$H^i_{Dol}(X/S) = \bigoplus_p Gr^p_F(H^i_{DR}(X/S)) = \bigoplus_{p+q=i} H^q(\Omega^p_{X/S}/S).$$

The Higgs field

One can define a relative Dolbeault homotopy type too (it works best under the hypothesis that f is relatively 1-connected).

The cohomology and homotopy types relative to S then obtain actions of the formal completed tangent bundle $\widehat{TS}$ and hence, indeed, infinitesimal actions of TS.

The action of TS on the Dolbeault cohomology is a Higgs field that is the well-known *Kodaira-Spencer map* on the associated-graded of the Hodge filtration.

Secondary Kodaira-Spencer Higgs fields

Suppose the classical Higgs field giving the action of TS on cohomology is trivial. We then get *secondary Kodaira-Spencer Higgs fields* that express the homotopy coherence relations of the action of TS on the homotopy type.

One way of approaching these structures is to consider relative nonabelian cohomology, with a first case being cohomology with coefficients in the schematic 2-sphere $S^2_{\mathbb{C}}$. This is a 1-connected geometric very presentable homotopy type given as a fibration over $K(\mathbb{G}_a, 2)$ with fiber $K(\mathbb{G}_a, 3)$ and structural map for the Postnikov tower being a nonzero map

$$H^4(K(\mathbb{G}_a, 2), \mathcal{O}) \cong \mathcal{O} \to \mathcal{O}.$$

Secondary Kodaira-Spencer Higgs fields

Given $f : X \to S$ we can form the 3-stack

$$Hom(X_{Dol/S}/S, S^2_{\mathbb{C}}) \to S.$$

In the fiber over $s \in S$, a point of $Hom((X_s)_{Dol}, S^2_{\mathbb{C}})$ is basically a cohomology class $\eta \in H^2_{Dol}(X_s)$ such that $\eta \cup \eta = 0$.

We may then look at the secondary class just as we discussed in the topological case: the value of $\eta \cup \eta$ goes a step down in the Leray filtration.

If the fibers have $h^1 = h^3 = 0$, we get back to a degree two class on the base with coefficients in $H^2_{Dol}(X_s)$.

Secondary Kodaira-Spencer Higgs fields

One obtains an explicit example. Suppose $S = Z$ is a smooth projective surface such that $h^{2,0}(S) \neq 0$.

Let X be obtained by blowing up the diagonal in $Z \times S$, with f being its second projection back to S. Here we distinguish the space Z of the fiber with the space S of the base, and we work locally on S. The fiber X_s is the blow-up of Z at s.

Choose the class $\eta = m[E] + n[H]$ where E is the exceptional divisor and H is a hyperplane class on Z not meeting the diagonal over our local patch in S. One may choose m, n (possibly complex) such that $\eta \cup \eta = 0$ in $H^4(X_s)$.

Secondary Kodaira-Spencer Higgs fields

Theorem

The secondary class is a map

$$\bigwedge^2 (T_s S) \longrightarrow H^2(X_s, \mathcal{O}_{X_s})$$

whose Poincaré dual is the evaluation map $H^0(X_s, K_{X_s}) \to \bigwedge^2 (T_s^* S)$. *In particular, under our hypothesis, it is nonzero.*

With Ludmil Katzarkov and Tony Pantev, we looked at a notion of *variation of nonabelian mixed Hodge structure*. The secondary Kodaira-Spencer Higgs fields provide the variational data for the Dolbeault realization.

Understanding the weight and purity properties is more subtle and calls for further exploration.

Logarithmic connections with nilpotent residues

We now turn to the second picture. The proofs in this part are in progress. A longer-range goal will be to fill in the details all the way to the formalized proof level (MALINCA project).

As motivation, we recall a few basic observations.

Suppose S is a disk with divisor $D = \{0\}$ and coordinate z. If (V, ∇) is a vector bundle with logarithmic connection

$$\nabla : V \to V \otimes \Omega^1_S(\log D)$$

the residue is an endomorphism $N : V_0 \to V_0$.

Assume that N is nilpotent.

Logarithmic connections with nilpotent residues

Use (V_0, N) to create a model logarithmic connection on the sheaf

$$M(V_0, N) := V_0 \otimes_{\mathbb{C}} \mathcal{O}_S$$

with connection defined by

$$\nabla_N := d + N\frac{dz}{z}.$$

Then, there is a unique isomorphism $V \cong M(V_0, N)$ respecting the logarithmic connections and inducing the identity on V_0.

Another way of looking at this construction is to let $\underline{N}$ denote the unique ∇-invariant section of $End(V)$ inducing N on V_0. Then the connection $\nabla^f := \nabla - \underline{N}$ is flat. It induces a trivialization $V \cong V_0 \otimes \mathcal{O}_S$ which is the isomorphism considered above.

Logarithmic connections with nilpotent residues

This works in the same way in higher dimesions, suppose $S = \Delta^k$ is a polydisk. The residue then induces a k-tuple $N = (N_1, \ldots, N_k)$ of commuting endomorphisms of V_0. Again, suppose that they are nilpotent.

We again have a model bundle $M(V_0, N) := V_0 \otimes \mathcal{O}_S$ with connection

$$\nabla_M := d + N_1 \frac{dz_1}{z_1} + \ldots + N_k \frac{dz_k}{z_k}.$$

Recall that this construction constitutes the first step in the classical *nilpotent orbit theorem* for variations of Hodge structure. Of course it is also basic to Deligne's LNM on regular singularities.

Logarithmic connections on perfect complexes

From the above discussion it follows that logarithmic connections with nilpotent residues satisfy strictness, so they form an abelian category.

This leads to the second preliminary observation. If $\mathcal{V}$ is a perfect complex on S with logarithmic connection with respect to a normal crossings divisor $D \subset S$, suppose that the residues act on the fibers of $\mathcal{V}$ nilpotently. More precisely, suppose that for every $s \in D$, the action of the residue on the cohomology spaces of the fiber $\mathcal{V}_s$ are nilpotent.

Then $\mathcal{V}$ locally splits as a direct sum of locally free sheaves.

This was noted in the arXiv version of an article with Jaya Iyer.

The logarithmic de Rham formal groupoid

Suppose S is a smooth complex variety or complex analytic space, with a divisor $D \subset S$ with simple normal crossings. We obtain the formal groupoid $S_{DR,\log D}$ whose associated Lie algebroid $\mathcal{L}$ is the sheaf of tangent vector fields that are tangent to D. Let $[S_{DR,\log D}]$ denote the corresponding *logarithmic de Rham stack* that has appeared in a recent preprint by Barz.

There is a morphism $S \to [S_{DR,\log D}]$ and the fiber product

$$S \times_{[S_{DR,\log D}]} S \to S \times S$$

is identified with the morphism object of the formal groupoid $S_{DR,\log D}$.

Consider an n-stack $T \to S_{DR,\log D}$ such that the pullback

$$T_S := T \times_{[S_{DR,\log D}]} S \to S$$

is a geometric n-stack that is relatively 1-connected.

Residues

Suppose D_I is a smooth stratum of the natural stratification of D. The Lie algebroid $\mathcal{L}_I := \mathcal{L}|_{D_I}$ is an extension

$$0 \to \mathcal{R}_I \to \mathcal{L}_I \to \mathcal{F}_{DR}(D_I) \to 0.$$

The subsheaf $\mathcal{R}_I$ consists of the tangent vector fields that vanish along D_I.
The stack T_{S_I} (restriction of T_S to S_I) comes with an action of $\mathcal{R}_I$.

Recall that we have relative homotopy sheaves $\pi_i(T_S/S)$ over S that don't depend, in the present situation (relative 1-connectedness), on the choice of local basepoint sections. The geometricity hypothesis implies that these are vector sheaves over S, and the restrictions to S_I are the $\pi_i(T_{S_I}/S_I)$.

Secondary residue classes for semistable maps

We fall into a situation very similar to the higher Kodaira-Spencer classes. At a point $s \in D_I$, the space of residual directions $\mathcal{R}_{I,s}$ is a commutative unipotent group scheme acting (at least formally) on the homotopy type T_s over s.

There will similarly be secondary residue classes for the action of the residual directions.

A typical situation in which this arises is when we have a semistable family $f : X \to S$. We recall that the *semistable reduction theorem for higher dimensional base and higher dimensional fibers* was shown only fairly recently, by Adiprasito-Liu-Temkin following ideas of Abramovich-Karu. It says that general morphisms have semistable models, after ramified modifications of the base and birational modifications of the total space.

Secondary residue classes for semistable maps

Therefore, it makes sense to study the case of semistable fibrations as a model for the local topology of general fibrations, of course with the understanding that this will require appropriate birational and ramified modification.

Suppose $f : X \to S$ is a projective and generically smooth semistable morphism. Let $D \subset X$ denote the singular divisor. We have a relative formal groupoid $X_{DR,\log D/S} \to S$ that comes equiped with a descent to a formal groupoid $X_{DR,\log D} \to S_{DR,\log P}$ where P is the divisor in S.

We may consider the relative schematic homotopy type of $X_{DR,\log D/S}/S$. Suppose that the general smooth fiber is 1-connected, and assume an appropriate corresponding hypothesis for the singular fibers.

Secondary residue classes for semistable maps

This homotopy type may be viewed as the universal morphism

$$\begin{array}{ccc} X_{DR,\log D/S} & \to & T \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

such that T is relatively geometry and very presentable over S. Let us say that we truncate to n-stacks so T is universal for maps to n-stacks of this form.

Recall P is the divisor in S with its strata P_I.

Main Observation

If $s \in P_I$ then the residue $\mathcal{R}_{I,s} \cong \mathbb{C}^k$ acts on the homotopy type T_s. Here k is the codimension of the stratum P_I.

Central fiber

Suppose $\{s\}$ is a 0-dimensional stratum of P. Then the fiber X_s is a union of strata that we'll denote here X_J (rather than D_J).

These strata have the restricted Lie algebroids $\mathcal{L}_I$ whose associated stacks glue together (via a gluing for a cover by closed subschemes) to give the stack

$$X_{DR,\log D/S} \times_S \{s\}.$$

The Lie algebroids all have $\mathcal{R}_{I,s}$ as abelian subspace.

The fiber homotopy type T_s is the representing schematic homotopy type of the glued stack $X_{DR,\log D/S} \times_S \{s\}$. The nonabelian shape functor that it represents may be defined as the limit of a diagram with the functors of nonabelian shape for the Lie algebroid pieces $\mathcal{L}_I$.

Nearby points

The link around the divisor D at the point $s \in D_I$ is a small torus $(S^1)^k \subset S - D$ located near to s. It may be described as the boundary of a small k-dimensional polydisk transverse to D_I at s. Let $\eta \in (S^1)^k$ be a *nearby point*.

The fiber X_η is a smooth complex variety, and we obtain a fibration of smooth manifolds over $(S^1)^k$ with X_η as fiber. This is the homotopical situation considered in the beginning.

We may schematize to get a schematic homotopy type $X_\eta \otimes \mathbb{C}$, having a homotopy coherent action of $\mathbb{Z}^k = \pi_1((S^1)^k, \eta)$.

Comparison

Using our general assumption that X/S is relatively 1-connected, we know that $X_\eta \otimes \mathbb{C}$ is isomorphic to the de Rham homotopy type T_η described previously. We now have the same homotopy type with two different actions of an abelian group, either $\mathcal{R}_{I,s} \cong \mathbb{C}^k$, or else $\mathbb{Z}^k$ that complexifies also to an action of $\mathbb{C}^k$.

Question

Can these be compared?

We describe next a method.

Comparison

Hypothesis

The action of $\mathcal{R}_I$ on the homotopy sheaves of T_{S_I}/S_I is nilpotent.

This is equivalent to saying that the action on the cohomology sheaves is nilpotent.

Proposition

Under the Hypothesis, the cohomology sheaves and homotopy sheaves of T_S/S are locally free.

Comparison

Specialize now to the case that S is a polydisk and D is the union of the coordinate hyperplanes. Over the origin $0 \in S$ we obtain the fiber T_0, with action of a vector space $\mathcal{R}_0$ (in this case it is the full tangent space to S at 0). The fiber is an n-truncated and 1-connected complex homotopy type.

We can create a model $M(T_0, \mathcal{R}_0) \to S$ along the lines of the construction described above, having also an action of the formal groupoid $S_{DR,\log D}$. The central fiber of $M(T_0, \mathcal{R}_0)$ is T_0.

Comparison

Proposition

There is a unique isomorphism $T_S/S \cong M(T_0, \mathcal{R}_0)$ *compatible with the formal groupoid actions and being the identity on* T_0.

Corollary

For any $s \in S$ *the complex homotopy type* T_s *is isomorphic to* T_0. *The monodromy action of* $\pi_1(S - D, s) \otimes_{\mathbb{Z}} \mathbb{C}$ *on* T_s *is the exponential of the action of* $\mathcal{R}_0$ *on* T_0 *via this identification. In particular, the secondary deformation invariants of* T_0 *are the same as the secondary monodromy invariants of the nearby fiber* T_s.

Further directions

It will be interesting to use the Hodge filtration to compare the secondary deformation invariants and the limiting secondary Kodaira-Spencer classes.

I don't currently have any examples for secondary residues in the case of a semistable family, it would be good to look for those.

The discussion has been in the relatively 1-connected case, but of course it will be important to consider these structures when there is a fundamental group of the fibers, and in particular for higher stacks over the moduli spaces of local systems on X/S.

Some references (very non-exhaustive)

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