Nonperturbative input for heavy quark physics from electric field correlators

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Heavy quark systems as probe of quark-gluon plasma

- Heavy quark systems provide some very important probes of the QGP medium.
- Heavy quark energy loss, collectivity: nonperturbative nature of the plasma.
- Suppression of quarkonia yield: point to the deconfining nature of the plasma.
- The large mass of the quark offers many theoretical simplifications.
- However, precise theoretical calculations difficult due to the nonperturbative nature of the plasma.
- Lattice QCD can provide nonperturbative inputs.
- I will discuss how the study of thermal electric field correlators are useful in theoretical calculations of heavy quark thermalization and quarkonia yield.

Heavy quarks in QGP

- For moderate momenta heavy quarks in a plasma, with $M \gg T$, a Langevin framework can be used.

$$\frac{dp_i}{dt} = \xi_i(t) - \eta_D(p) p_i, \qquad \langle \xi_i(t)\, \xi_j(t') \rangle = \kappa_Q^{ij}\, \delta(t - t')$$

Svetitsky '88; Mustafa et al., '97; Moore & Teaney '05; Rapp & van Hees '05

- For low momenta, just one coefficient κ_Q.
  Standard nonrelativistic relations:

$$\eta_D = \frac{\kappa_Q}{2\, M\, T} \sim \tau_Q^R$$

- In the static ($M \to \infty$) limit κ_Q can be obtained from thermal color electric field correlators.

J. Casalderrey-Solana & D. Teaney, PRD 74 (2006) 085012;
S. Caron-Huot, M. Laine & G. Moore, JHEP 04 (2009) 53

$\mathcal{O}\left(\frac{1}{M}\right)$ expansion

- The heavy quark lagrangian can be expanded in a series of $\mathcal{O}\left(\frac{1}{M}\right)$.

$$\mathcal{L} = \Psi_+^\dagger \left[iD_0 - M + \frac{D^2 + c_b g\sigma \cdot \mathbf{B}}{2M} \right] \Psi_+ \; + \; \Psi_-^\dagger \left[iD_0 + M - \frac{D^2 + c_b g\sigma \cdot \mathbf{B}}{2M} \right] \Psi_-$$
$$+ \frac{ic_e}{2M} \left[\Psi_+^\dagger \, g\sigma \cdot \mathbf{E}\Psi_- \; - \; \Psi_-^\dagger \, g\sigma \cdot \mathbf{E}\Psi_+ \right] \; + \; \mathcal{O}\left(\frac{1}{M}\right)2$$

$\Psi_\pm = \frac{1 \pm \gamma^0}{2}\Psi$ are quark and antiquark field operators.

- From this, the leading order ("Static") Hamiltonian is

$$H \; = \int d^3x \Psi_+^\dagger \left[-gA_0 + M \right] \Psi_+ + \Psi_-^\dagger \left[-gA_0 - M \right] \Psi_- \; + \mathcal{O}\left(\frac{1}{M}\right)$$

- Quark current and force:

$$J^i = \frac{-i}{M} \left[\Psi_+^\dagger D^i \Psi_+ \; - \; \Psi_-^\dagger D^i \Psi_- \right] + \mathcal{O}\left(\frac{1}{M^2}\right)$$
$$M \frac{dJ^i}{dt} = -\Psi_+^\dagger g E^i \Psi_+ \; + \; \Psi_-^\dagger g E^i \Psi_- \; + \mathcal{O}\left(\frac{1}{M}\right)$$

κ_Q from EE correlator

- In the leading order in $1/M$, then, κ can be obtained from the force-force correlator:

$$\kappa_Q = \frac{\frac{1}{3}\sum_i \int_{t,x} \frac{1}{2}\left\langle\left\{\left(\Psi_-^\dagger gE^i\Psi_- - \Psi_+^\dagger gE^i\Psi_+\right)(t,x), \left(\Psi_-^\dagger gE^i\Psi_- - \Psi_+^\dagger gE^i\Psi_+\right)(0,0)\right.\right.}{\int_{t,x}\left\langle\left(\Psi_+^\dagger\Psi_+ + \Psi_-^\dagger\Psi_-\right)(t,x)\left(\Psi_+^\dagger\Psi_+ + \Psi_-^\dagger\Psi_-\right)(0,0)\right\rangle}$$

- In the static theory the quark propagators give only a phase factor, leading to a thermal EE correlator.
- κ_Q has been calculated to NLO in HTL PT: huge NLO correction, astronomically large temperature required for LO to dominate.

$$\frac{\kappa_E}{T^3} = \frac{2\,g^4}{27\,\pi}\left[N_c\left(\ln\frac{2T}{m_D}+\xi\right) + \frac{N_f}{2}\left(\ln\frac{4T}{m_D}+\xi\right) + \frac{N_c\,m_D}{T}C + \mathcal{O}(g^2)\right]$$

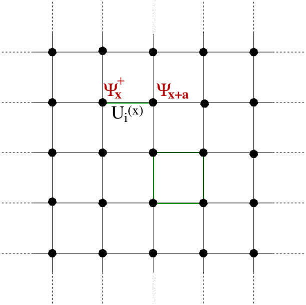
Caron-Huot & Moore, PRL 100 (2008) 052301

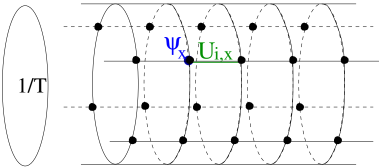
- We want to calculate κ_Q using lattice QCD. On the lattice we can only evaluate Euclidean time correlators like $\langle F(\tau)F(0)\rangle$.

κ_Q from lattice

The connection to the expression above is well-known: we calculate the spectral function

$$G(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho(\omega) \frac{\cosh \omega(\tau - 1/2T)}{\sinh \omega/2T}, \qquad \kappa_E = \lim_{\omega \to 0} \frac{2T}{\omega} \rho(\omega)$$





κ_Q from lattice

- Static lagrangian on lattice:

$$\mathcal{L}_{\rm stat} = \Psi_+^\dagger(x)\,\Big(\Psi_+(x) - U_0^\dagger(x-\hat{0})\,\Psi_+(x-\hat{0})\Big) + \Psi_-^\dagger(x)\,\Big(U_0(x)\,\Psi_-(x+\hat{0}) - U_0^\dagger(x-\hat{0})\,\Psi_+(x-\hat{0})\Big)$$

- Quark propagators:

$$G_+(x,y) = \theta(x_0-y_0)\,\delta^3(\vec{x}-\vec{y})\;U_0^\dagger(x_0-\hat{0})\,U_0^\dagger(x_0-2\hat{0})\ldots.\,U_0^\dagger(y_0+\hat{0})\,U_0^\dagger(y_0)$$
$$G_-(x,y) = \theta(y_0-x_0)\,\delta^3(\vec{x}-\vec{y})\;U_0(x_0)\,U_0(x_0-\hat{0})\ldots.\,U_0(y_0+2\hat{0})\,U_0(y_0+\hat{0})$$

- The *EE* correlator then takes the form

$$G(\tau) = \frac{-\frac{1}{3}\sum_i\sum_x \langle \Re\,{\rm Tr}\;\{U(\beta,\tau;x)\,gE_i(\tau,x)\,U(\tau,0;x)\,gE_i(0,x)\}\rangle}{\sum_x \langle \Re\,{\rm Tr}\;U(\beta,0;x)\rangle}$$

S. Caron-Huot, M. Laine & G. Moore, JHEP 04 (2009) 53

Steps of lattice calculation

- On lattice with spacing a ($T = 1/N_\tau a$) calculate $G(\tau; a, T)$ using numerical Monte Carlo methods.
  We used Multilevel for reduction of operator noise.
  Gradient flow has also been used by other groups.
- Make sure finite volume effects are under control (take $V \to \infty$).
  We checked that finite volume effects are negligble for $LT \gtrsim 3$.
- Regularize $G(\tau; a, T)$.
  $G(\tau)$ as defined does not require any infinite renormalization.
  Finite renormalization $Z_E(a)$ for smooth continuum limit.
  Gradient flow also leads to a nonperturbative renormalization.
- Take $a \to 0$, keeping $T = 1/N_\tau a$ fixed, to get $G(\tau, T)$.
- Analyze to get κ.

Lattice studies

- We studied $G_{EE}(\tau)$ for the gluonic plasma in the temperature range $T_c < T \lesssim 3.5\,T_c$.

  Banerjee, Datta, Gavai, Majumdar, 2012, NP A1038(2023)122721

- A discretization using $E_i = [D_i\,,\,D_0]$ turns out to be convenient for calculating the continuum correlator.
- We used a perturbative $Z_E(a)$.

  Christensen & Laine, PLB 755 (2016) 316

- For noise reduction multilevel technique was used. There have also been other calculations using Multilevel.

  Francis et al. PRD (2015); Brambilla, et al., PRD (2020)

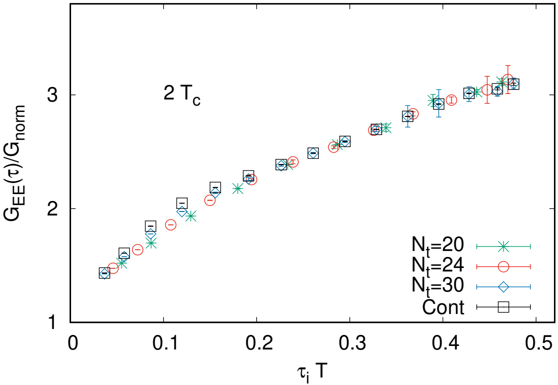
- Gradient flow has also been used for the gluon plasma calculation.

  Altenkort et al.,PRD103(2021)014511; Brambilla et al,PRD107(2023)054508

- Results for theory with thermal quarks have also been obtained recently.

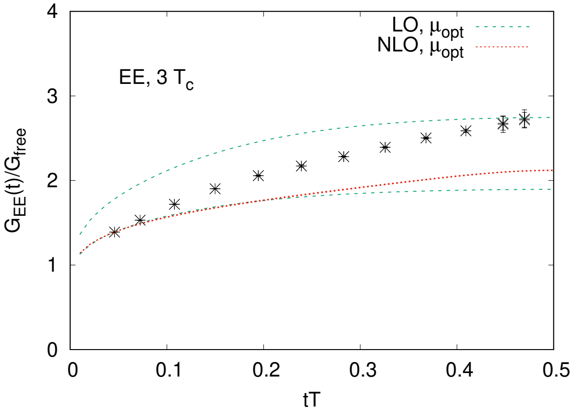
  Bollweg et al., JHEP09(2025) 180; Altenkort et al., PRL 130(2023)231902

$G_{EE}(\tau)$ from lattice



- $\mu_{\rm opt}$ is the optimized scale for NLO.
- Here the correlators are normalized using the leading order behavior, $G_{\rm norm}(\tau) = \frac{G_{LO}(\tau)}{g^2 C_f}$

$G_{EE}(\tau)$ from lattice



- μ_{opt} is the optimized scale for NLO.
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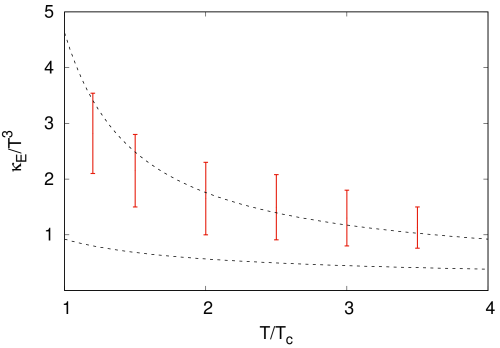
$\rho_{EE}(\omega)$ and κ_E

- $G_{EE}(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho_{EE}(\omega) \frac{\cosh \omega(\tau - 1/2T)}{\sinh \omega/2T}$
- κ_E can be extracted from the infrared behavior of $\rho_{EE}(\omega)$: $\rho_{IR} \underset{\omega \to 0}{\approx} \frac{\kappa_E \omega}{2T}$
- For $\omega \gg T$ we expect $\rho_{UV} \approx \frac{g^2(\mu) C_f \omega^3}{6\pi}$
- To extract $\rho_{EE}(\omega)$, we use these limiting behaviors.
  We use a linear regularization with these two limiting behaviors.
  We also use various model spectral functions with these limiting behaviors.
- Other methods of inversion, e.g., Brackus-Gilbert and Maximum entropy methods, have been used by other groups.
  Altenkort et al. (2021)
- Very good agreement between the different calculations for the gluon plasma.

κ_E from lattice



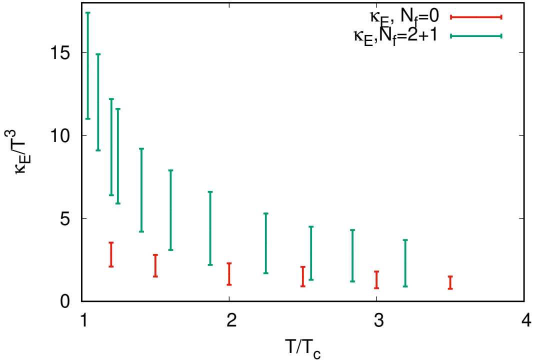
Banerjee, Datta, Gavai, Majumdar (2023)

- The error is dominated by uncertainty in the extraction of $\rho_{EE}(\omega)$.
- These values agree very well with the results of other groups, where available.
- Consistent with NLO pert. theory.

κ_E for QCD with $N_f = 2+1$

κ_E has also been calculated recently in a theory with 2+1 flavors of thermal quarks.

HotQCD: Bollweg et al., JHEP09(2025) 180; Altenkort et al., PRL 130(2023)231902



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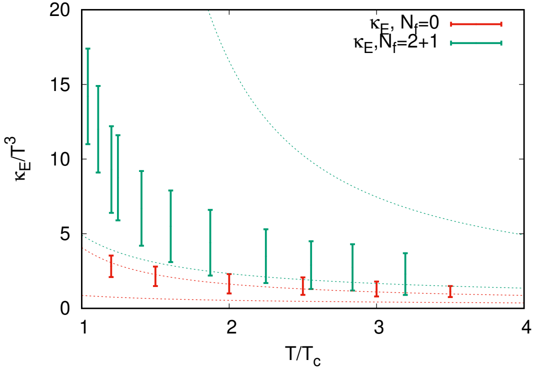

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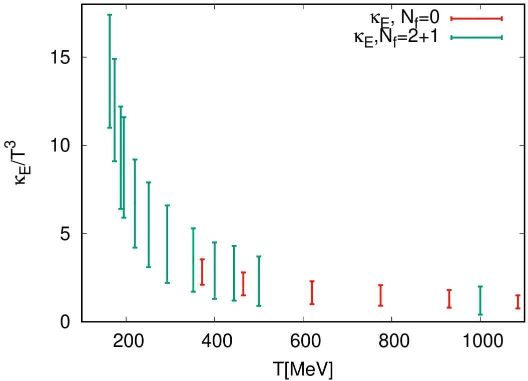
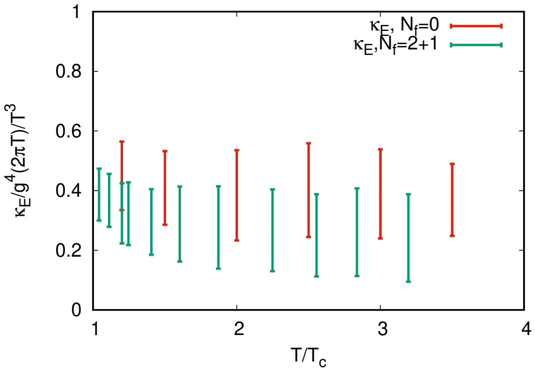


- Much of the N_f dependence can be attributed to the difference in T_c ($T_c^{N_f=0} \approx 310$ MeV).

Parametrization of κ_E

An interesting observation: (Altenkort et al., PRD 109 (2024) 114506) results for κ_E for T near T_c can be parametrized as $c\, g^4(2\pi T)\, T^3$, with similar c for both the gluonic and the 2+1 flavor theory.



- Not perturbative: in PT the coefficient c increases with N_f.
- Also in this temperature range g^5 term dominates in NLO PT.

$1/M$ correction

- The correction to static limit is also known: at $\mathcal{O}\left(\frac{1}{M}\right)$

$$\kappa_Q \approx \kappa_E + \frac{2}{3}\langle v^2\rangle \kappa_B , \qquad \langle \gamma v^2\rangle = \frac{3T}{M_{\rm kin}}$$

Bouttefeux & Laine, JHEP 12 ('20) 150

- κ_B is similar to κ_E, for the $B-B$ correlator.
  The $B-B$ correlator has an anomalous dimension, making the renormalization more difficult.
- We need to calulate $c_B(\mu)\, G_{BB}(\mu, T; \tau)$ where G_{BB} is the correlator renormalized at scale μ, and c_B is a Wilson coefficient. The μ dependence cancels in the product.
- κ_B has been calculated for gluonic plasma in the temperature range 1-3 T_c.

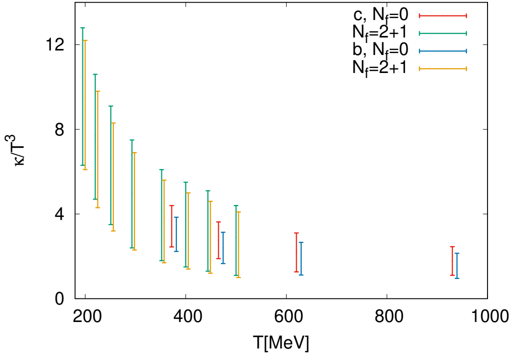
  Banerjee, Datta & Laine, JHEP 08 (2022) 128

  Brambilla et al., PRD107(2023)074503, Altenkort et al., PRD 109(2024)114506

  Full QCD results are also available.

  Bollweg et al., JHEP09(2025) 180

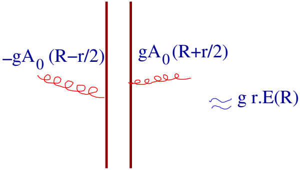
κ_c, κ_b

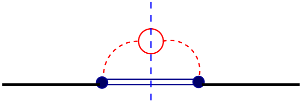


- Correction to the static limit is $\lesssim$ 10% for bottom near T_c for the gluonic theory, rising to $\approx$ 15% at 2 T_c. For the three flavor theory it is $<$ 10% for $T < 1.6 T_c$.
- Even for charm the $1/M$ correction is $\approx$ 25% near T_c for the quenched and $<$ 20% for the 3-flavor theory.

Quarkonia: color dipole in QGP

- The EE correlator, in a different color configuration, is important for the study of quarkonia yield in QGP.
- For $r \ll 1/T$, e.g., for Υ, interaction of $\bar{Q}Q$ with medium: color dipole.





- The decay width for bottomonia in the plasma due to the $r.E$ interaction:

$$\Gamma = \frac{g^2}{3N} \sum_f |\langle f|r|\psi\rangle|^2 \; n_B(k_0) \; \int_{\vec{k}} \rho^{EE}(k_0, \vec{k})\big|_{k_0=E_f-E_\psi}$$

Brambilla, Ghiglieri, Vairo, Petreczky, PRD78(2008) 014017

where ρ^{EE} is the spectralfunction for a correlator connecting electric fields with an adjoint gauge link.

EE correlators for quarkonia

- Let us expand in r and further, write the $\bar{Q}Q$ in $|S\rangle$, $|O\rangle$ basis (pNRQCD).

$$\mathcal{L}_{pNR} = \mathrm{Tr}\left[S^\dagger\left(i\partial_0 - V_s(r)\right)S + O^\dagger\left(iD_0 - V_o(r)\right)O\right] + \mathrm{Tr}\left[S^\dagger\vec{r}.g\vec{E}O + h.c. + \frac{1}{2}O^\dagger\{\vec{r}.g\vec{E}, O\}\right] + \mathcal{O}\left(\frac{1}{M}\right)$$

Pineda & Soto (1998); Brambilla, et al., RMP 77 (2005) 1423

- The interaction term $S^\dagger\vec{r}.g\vec{E}O$ leads to medium correction captured by

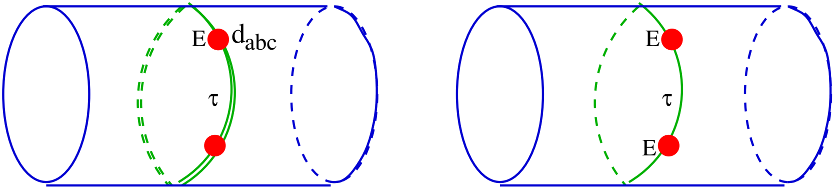
$$G(t) = \frac{1}{3}\sum_i \langle T\, E^{a,i}(t,\vec{0})\, W^{ab}(t,0)\, E^{b,i}(0,\vec{0})\rangle.$$

- In the Markovian limit, the effect of this term can be captured by a transport coefficient, κ_A.
- Similarly, the effect of the $O^\dagger\{\vec{r}.g\vec{E}, O\}$ term on the medium evolution of quarkonia is captured by a different EE correlator, $G_{\rm oct}(\tau)$.

Brambilla, Escobedo, Soto, Vairo, PRD 96(2017) 034021.

$G_{\rm oct}(\tau)$

Let us first consider the $O^\dagger \{r.gE\,,\,O\}$ term, and the associated Euclidean correlator $G_{\rm oct}(\tau)$.



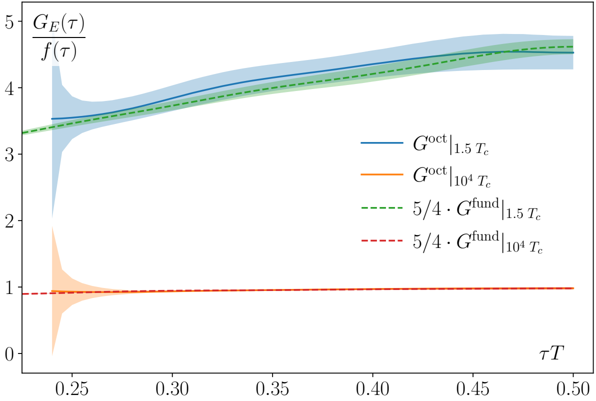
This has the same LO structure as $G(\tau)$, modulo color factor:

$$G_{\rm oct}(\tau) \;=\; g^2\,\frac{N^2-4}{N}\;G_{\rm norm}(\tau)$$

We can analyze it similar to $G(\tau)$ by dividing it by $L_A(\tau)$. Then the correlator does not require any infinite renormalization, and calculating continuum correlators is straightforward.

$G_{\text{oct}}(\tau)$

Structure of $G_{\text{oct}}(\tau)$, which calculates a correction of the thermal octet potential, is simple: it has a similar structure as $G_{\text{fund}}(\tau)$, and shows a color scaling, $\Rightarrow \kappa_{\text{oct}} \approx \frac{5}{4}\kappa_{\text{fund}}$

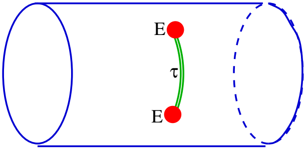


Brambilla, Datta, Janer, Leino, Mayer-Steudte, Petreczky, Vairo, PRD 112(2025) 074509 (arXiv:2505.16603)

Quarkonia in pNRQCD

- The most interesting correlator for quarkonia is $G(\tau)$, associated with the singlet-octet transition.

$G(\tau) \; = \; -\frac{1}{3} \sum_i \langle E^i_a(\tau) \, U^A_{ab}(\tau, 0) \, E^i_b(0) \rangle.$



- This can be analytically continued to the real time correlator $G(t)$.

$$G(\tau) \; = \; \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \rho(\omega) \, \frac{e^{-\omega \, \tau}}{1 \, - \, e^{-\omega/T}}, \qquad \rho(\omega) \; = \; \rho_{\rm odd}(\omega) \; + \; \rho_{\rm even}(\omega)$$

B. Scheihing-Hitschfeld & X. Yao, PRD108 (2023) 054024
N. Brambilla, et al., PRD97(2018)074009, JHEP08(2025)219 and in progress

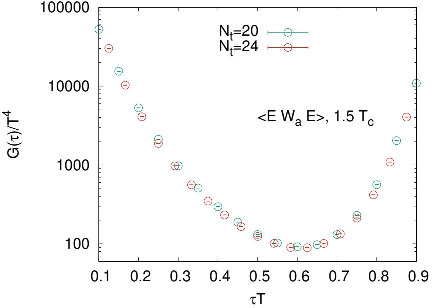
- In the Markovian approximation, the quarkonia singlet-to-octet transition quantified by a transport coefficient κ:

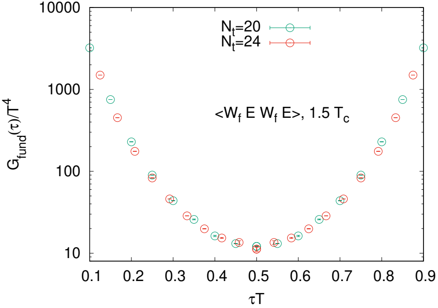
$$\kappa \; = \; \lim_{\omega \to 0} \frac{T}{N \, \omega} \, \rho(\omega)$$

Nonperturbative comparison

$$\left.\begin{matrix} G(\tau)|_{LO} \\ G_{\rm fund}(\tau)|_{LO} \end{matrix}\right\} = g^2 \left\{\begin{matrix} N^2-1 \\ C_f \end{matrix}\right\} G_{\rm norm}(\tau)$$

They differ in NLO. In particular, $\langle E W_A E \rangle$ does not obey $\tau \to \beta - \tau$ symmetry.



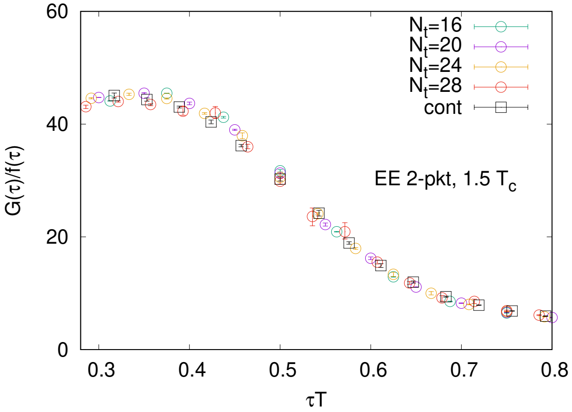


Brambilla, Datta, Janer, Leino, Mayer-Steudte, Petreczky, Vairo,
PRD 112(2025) 074509 (arXiv:2505.16603)

Calculational details

- We calculated $G(\tau)$ at 1.5 T_c and 10^4 T_c in SU(3) pure gauge theory.
- We have done the calculation using both the gradient flow and a multilevel calculation.
- $G_r(\tau; a) \; = \; Z_E^2(a) \; e^{\delta m(a) \, \tau} \; G_{\rm bare}(\tau; a); \qquad a \; = \; \frac{1}{N_t \; T}$
- We estimate the mass divergence from L_a, using the calculations of Gupta, Huebner, Kaczmarek, PRD77(2008) 034503
  $L_r^A(T) \; = \; e^{\delta m(a)/T} \, L^a_{\rm bare}(T; a)$
- The remaining finite renormalization is fixed by gradient flow. For the multilevel, we use LO tadpole normalization, and keep an overall normlization as a fit parameter.

$G(\tau)$: Continuum results



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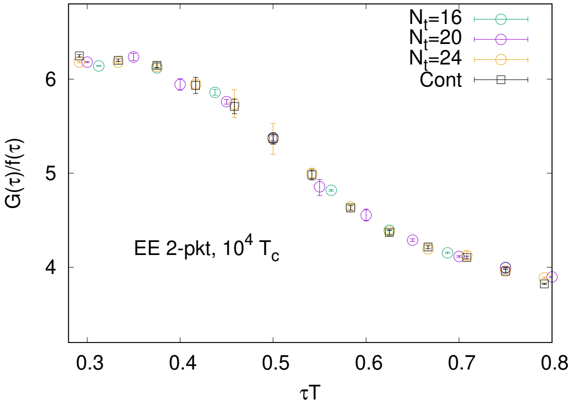
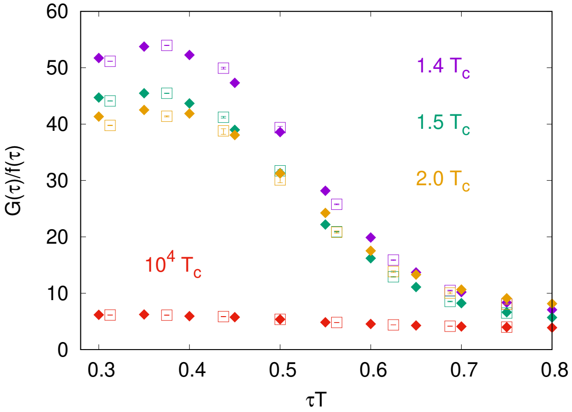
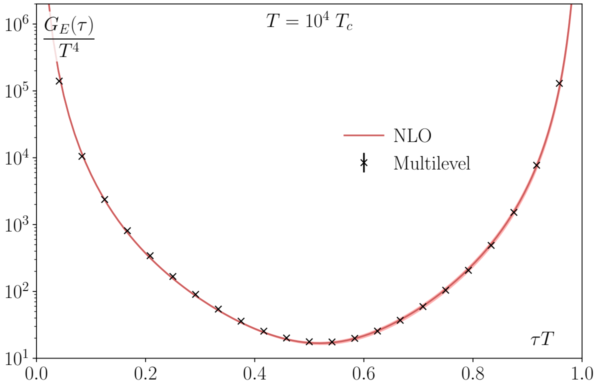


T dependence of $G(\tau)$



Datta, Banerjee, Brambilla, Janer, Leino, Mayer-Steudte, Petreczky, Singh, Vairo, JSPC 4 (2025) 100156 (2506.22594)

High temperature

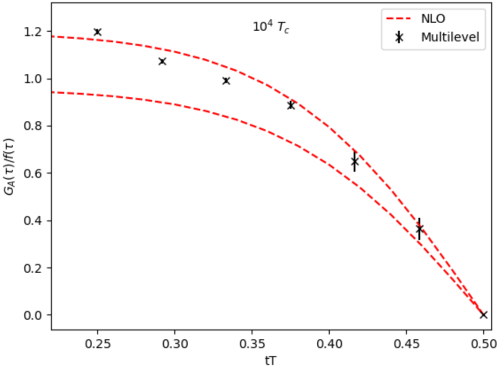


At very high temperatures the nonperturbative correlator is close to the NLO correlator. But at temperatures a few T_c they are different.

Brambilla, Datta, Janer, Leino, Mayer-Steudte, Petreczky, Vairo, PRD(2505.16603)
Pert.th.: Brambilla, Panayiotou, Sappi, Vairo, JHEP (2505.16604)

The asymmetric part of the correlator

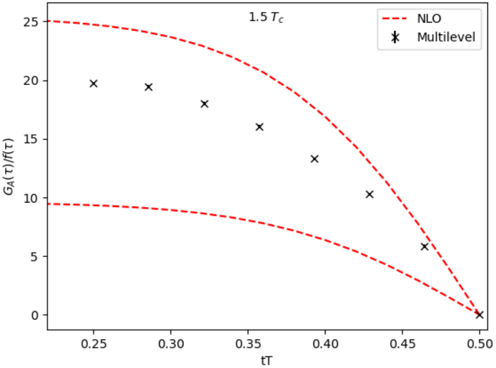
Zoom in the asymmetric part: $G_{asym}(\tau) = G(\tau) - G(\beta - \tau)$



$G_{asym}(\tau)$ is in good agreement with perturbation theory at $10^4\, T_c$; even at 1.5 T_c, but the perturbative errorband is too large there.

The asymmetric part of the correlator

Zoom in the asymmetric part: $G_{asym}(\tau) \; = \; G(\tau) \; - \; G(\beta - \tau)$



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Analysis of the correlator at 1.5 T_c

- The Euclidean correlator can be connected to the real time correlator through the spectral function.

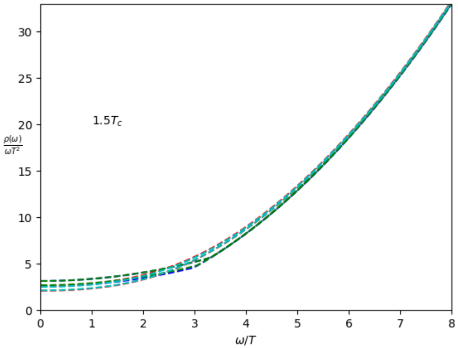
$$G(\tau) = \left\{ \int_0^\infty \frac{d\omega}{\pi} \rho_a(\omega; T) \frac{\cosh \omega(\tau - 1/2T)}{\sinh \omega/2T} + \int_0^\infty \frac{d\omega}{\pi} \rho_s(\omega; T) \frac{\sinh \omega(\tau - 1/2T)}{\sinh \omega/2T} \right\}$$

- A direct extraction of $\rho_a(\omega; T), \rho_s(\omega; T)$ from $G(\tau)$ is very difficult.
- We are interested in the small ω behavior of $\rho_a(\omega; T)$.
- Take perturbative form for $\rho_s(\omega; T)$.
-

$$\rho_a(\omega; T) \sim \begin{cases} \rho_{NLO}(\omega, T) & \omega \gg T \\ \frac{\kappa_A N \omega}{T} & \omega \to 0 \end{cases}$$

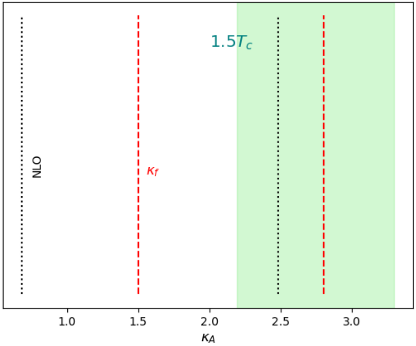
$\rho_a(\omega; T)$ at 1.5 T_c

- We use different models of the spectral function consistent with these limits to extract $\rho(\omega)$ from $G(\tau)$.



- Better control over the intermediate ω regime needed if we want results at $\omega > 0$, for non-Markovian studies.

κ_A at 1.5 T_c



PRELIMINARY

- Within uncertainties, κ_A is consistent with our results for heavy quark diffusion coefficient at same temperature.
- It is also consistent with the NLO perturbative results.

Summary

- Nonperturbative information about heavy quark systems in QGP can be obtained from a study of electric field corrlators.
- The heavy quark diffusion coefficient, in the leading order in $1/M$, can be obtained from $G_{EE}(\tau)$.
- Studied for both gluonic plasma and for full QCD in the temperature range of interest for HIC experiments.
- The $\mathcal{O}\left(\frac{1}{M}\right)$ correction to the diffusion coefficient has also been calculated from a study of the BB correlators.
- The decay of quarkonia in QGP can also be calculated from thermal correlators of electric fields connected by adjoint Wilson lines.
- Preliminary results for κ_A agree, within uncertainties, with the κ_E obtained for heavy quark diffusion.
- We also studied the correlators $G_{\rm oct}(\tau)$, relevant for the octet-octet transition in the plasma. $\kappa_{\rm oct} \approx \frac{5}{4}\kappa_E$.

EXTRA SLIDE: Model details

- $\rho_{EE}^{IR}(\omega) \equiv \frac{\kappa\, N\, \omega}{T}$, $\qquad \rho_{UV}(\omega) \equiv \frac{g^2(\mu)\,(N^2-1)\,\omega^3}{6\pi}$
- $\mu = \max\left[2\,\omega\, e^{(6\pi^2-149)/66}\,,\; 4\pi T\, e^{-\gamma-1/22},\right]$ allowed to vary.
- Different models for $\rho_a(\omega; T)$ satisfying UV and IR limits:

$$(1)\;\; \max\left[\rho_{EE}^{IR}(\omega), \rho_{UV}(\omega)\right] \qquad (2)\sqrt{(\rho_{EE}^{IR}(\omega))^2 \,+\, (\rho_{UV}(\omega))^2}$$

$$(3,4)\,(1 + d \sin\,\pi y)\;\; \rho_{1,2}(\omega), \qquad y \;=\; \frac{\log\left(1+\frac{\omega}{\pi T}\right)}{1\,+\,\log\left(1+\frac{\omega}{\pi T}\right)}$$

- $\rho_s(\omega; T) = \frac{g^2(\mu)}{16\pi^2}\,(N^2-1)\,\omega^3\, sign(\omega)$
- $$G(\tau) = Z\left\{\int_0^\infty \frac{d\omega}{\pi}\rho_a(\omega; T)\cosh\omega(\tau - 1/2T) \,+\, c_s\int_0^\infty \frac{d\omega}{\pi}\rho_s(\omega; T)\sinh\omega(\tau - 1/2T$$

- Z is an overall normalization factor.
- c_s is an extra factor multiplying $\rho_s(\omega; T)$. We experimented with fixing c_s=1 and also keeping it as a fit parameter: we got results very close to 1.